THE JOINT UNIVERSALITY OF ZETA-FUNCTIONS
ATTACHED TO CERTAIN CUSP FORMS
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Abstract. In the paper a joint universality theorem for zeta-functions of normalized eigenforms is proved. For this some orthogonality conditions (A)–(C) are used.
Key words: cusp form, eigenform, limit theorem, probability measure, random element, support of random element, universality, weak convergence.

1. Introduction

The notion of the universality appears in various fields of mathematics. A general definition of the universality is given in [6]. Let $X$ and $Y$ be topological spaces and $T_j : X \rightarrow Y$, $j \in I$, be continuous mappings. Then an element $x \in X$ is called universal with respect to the family $\{T_j, j \in I\}$ if the set $\{T_jx : j \in I\}$ is dense in $Y$.

Many universal objects of various nature are known, see an excellent survey paper [6]. We recall a result of Birkhoff [4]. He proved that there exists an entire function $f$ such that for every entire function $g$ there exists a sequence of complex numbers $\{a_n\}$ such that

$$f(z + a_n) \xrightarrow{n \rightarrow \infty} g(z)$$

locally uniformly in the complex plane $\mathbb{C}$. In the latter theorem universal translates appear. However, the most known universal objects, as translates in the Birkhoff theorem, are not explicitly given: they are either constructed in some countable inductive process, or their existence is based on the Baire category theorem. Only in 1975 S. M. Voronin [26] found a concrete universal object, the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, which enjoys some kind of Birkhoff translates universality. To give the modern statement of Voronin’s theorem we use the notation

$$\nu_T(\ldots) = T^{-1}\text{meas}\{\tau \in [0, T] : \ldots\}$$

for $T > 0$, where $\text{meas}\{A\}$ denotes the Lebesgue measure of the set $A$, and in place of dots we write a condition satisfied by $\tau$. Now the Voronin theorem can be
stated as follows, see Chapter 6 of [15].

Let $K$ be a compact subset of the strip $\{ s = \sigma + it \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \}$ with connected complement. Let $f(s)$ be a non-vanishing continuous function on $K$ which is analytic in the interior of $K$. Then for any $\varepsilon > 0$

$$\liminf_{T \to \infty} \nu_T \left( \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right) > 0.$$  

The Voronin theorem was generalized and improved in [1], [5], [8]–[12], [14]–[19], [21], [23]–[26]. By the way, there exists the Linnik–Ibragimov conjecture that all Dirichlet series are universal in the above sense (of course, the case of absolutely convergent series for all $s$ is excluded). An important role because of their application to zero-distribution problems is played by joint universality theorems for Dirichlet series. The joint universality for Dirichlet $L$-functions was proved in [1], [2], [5], [28]. A theorem of such kind for Dirichlet series with multiplicative coefficients was obtained in [13], while for generalized Euler products it was proved in [19]. The joint universality of Lerch zeta-functions was given in [22].

Let $F(z)$ be a holomorphic cusp form of weight $\kappa$ for the full modular group $SL(2, \mathbb{Z})$, and we assume that $F(z)$ is a normalized eigenform. Then $F(z)$ has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}, \quad c(1) = 1.$$ 

E. Hecke [7] introduced and studied the zeta-function $\varphi(s, F)$, for $\sigma > \frac{\kappa+1}{2}$, given by absolutely convergent Dirichlet series with coefficients $c(m)$:

$$\varphi(s, F) = \sum_{m=1}^{\infty} c(m)m^{-s}.$$ 

The function $\varphi(s, F)$ is analytically continuable to an entire function. Let $D = \{ s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2} \}$. Then the univerality of $\varphi(s, F)$ is contained in the following theorem [23]:

Let $F(z)$ be a normalized eigenform of weight $\kappa$ for $SL(2, \mathbb{Z})$. Let $K$ be a compact subset of $D$ with connected complement, and let $f(s)$ be a non-vanishing continuous function on $K$ which is analytic in the interior of $K$. Then for any $\varepsilon > 0$

$$\liminf_{T \to \infty} \nu_T \left( \sup_{s \in K} |\varphi(s + i\tau, F) - f(s)| < \varepsilon \right) > 0.$$  

It is the purpose of the present paper to obtain the joint universality of zeta-functions attached to normalized eigenforms. Let $F_j(z)$ be a holomorphic cusp form of weight $\kappa_j$ for the full modular group $SL(2, \mathbb{Z})$, and we assume that $F_j(z)$ is a normalized eigenforms, $j = 1, \ldots, n$, $n \geq 2$. Consider the zeta-functions

$$\varphi(s, F_j) = \sum_{m=1}^{\infty} c_j(m)m^{-s}, \quad j = 1, \ldots, n,$$
and their analytic continuation. Here \( c_j(m) \) denote the coefficients of Fourier series expansion for \( F_j(z) \).

To obtain the joint universality of the functions \( \varphi(s, F_j) \), \( j = 1, \ldots, n \), we will use some restrictions. In the case of Dirichlet \( L \)-functions the orthogonality of Dirichlet characters is employed. In our case we will apply the following condition. Let \( c_{jp} = c_j(p)p^{-\frac{\sigma}{2}} \) where \( p \) denotes a prime number. Suppose that there exist functions \( \rho_1(p), \ldots, \rho_n(p) \) and real numbers \( \theta_1, \ldots, \theta_n \) such that

\[
\sum_{p \leq x} |\rho_j(p)|^2 = \frac{\theta_j x}{\log x} (1 + o(1)), \quad x \to \infty, \quad j = 1, \ldots, n
\]  

(A)

and numbers \( \lambda_{jk}, k, j = 1, \ldots, n \), satisfying

\[
c_{jp} = \lambda_{jk}\rho_k(p) + b_{jk}(p), \quad j, k = 1, \ldots, n,
\]  

(B)

where

\[
\sum_p \frac{|b_{jk}(p)|}{p^\sigma} < \infty, \quad j, k = 1, \ldots, n,
\]  

for \( \sigma > \frac{1}{2} \), and

\[
\begin{vmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{n1} & \lambda_{n2} & \ldots & \lambda_{nn}
\end{vmatrix} \neq 0.
\]  

(C)

Let \( D_j = \{ s \in \mathbb{C} : \frac{k_j}{2} < \sigma < \frac{k_j + 1}{2}, \quad |t| < N \}, \quad j = 1, \ldots, n \).

Theorem. Suppose that the cusp forms \( F_1(z), \ldots, F_n(z) \) satisfy the above conditions. Let \( K_j \) be a compact set of the strip \( D_j \) with connected complement, and let \( f_j(s) \) be a non-vanishing continuous function on \( K_j \) which is analytic in the interior of \( K_j \), \( j = 1, \ldots, n \). Then for any \( \epsilon > 0 \)

\[
\liminf_{T \to \infty} \nu_T \left( \sup_{1 \leq j \leq n} \sup_{s \in K_j} |\varphi(s + it, F_j) - f_j(s)| < \epsilon \right) > 0.
\]

2. A joint limit theorem for the functions \( \varphi(s, F_j) \)

For any region \( G \) on \( \mathbb{C} \), by \( H(G) \) denote the space of analytic on \( G \) functions equipped with the topology of uniform convergence on compacta. Let \( N > 0 \),

\[
D_{j,N} = \{ s \in \mathbb{C} : \frac{k_j}{2} < \sigma < \frac{k_j + 1}{2}, \quad |t| < N \}, \quad j = 1, \ldots, n,
\]

and

\[
H_n = H_n(D_{1,1}, \ldots, D_{n,1}) = H(D_{1,1}) \times \cdots \times H(D_{n,1}).
\]

Denote by \( \mathcal{B}(S) \) the class of Borel sets of the space \( S \), and define the probability
measure
\[ \nu_T(A) = \nu_T(\{\varphi(s_1 + i\tau, F_1), \ldots, \varphi(s_n + i\tau, F_n) \in A\}), \quad A \in \mathcal{B}(H_n). \]

For the proof of the theorem we need a limit theorem for \( P_T \) as \( T \to \infty \) in the sense of the weak convergence of probability measures with explicit form of the limit measure. Let \( \gamma \) stand for the unit circle on \( \mathbb{C} \), and
\[
\Omega = \prod_p \gamma_p,
\]
where \( \gamma_p = \gamma \) for all primes \( p \). The infinitesimaltorus \( \Omega \) is a compact Abelian topological group, therefore on \( (\Omega, \mathcal{B}(\Omega)) \) the probability Haar measure \( m_H \) exists.

This gives a probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \). Let \( \omega(p) \) denote the projection of \( \omega \in \Omega \) onto the coordinate space \( \gamma_p \).

Since \( F_j(z) \) is a normalized eigenform, the zeta-function \( \varphi(s, F_j) \), for \( \sigma > \frac{n+1}{2} \), has the Euler product expansion
\[
\varphi(s, F_j) = \prod_p \left( 1 - \frac{\alpha_j(p)}{p^s} \right)^{-\frac{1}{2}} \left( 1 - \frac{\beta_j(p)}{p^s} \right)^{-\frac{1}{2}},
\]
with
\[
\alpha_j(p) = \alpha_j(p) + \beta_j(p), \quad j = 1, \ldots, n.
\]

On the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \) define the \( H_j \)-valued random element \( \varphi(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n) \) by the formula
\[
\varphi(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n) = (\varphi(s_1, \omega; F_1), \ldots, \varphi(s_n, \omega; F_n)), \tag{1}
\]
where
\[
\varphi(s_j, \omega; F_j) = \prod_p \left( 1 - \frac{\alpha_j(p)\omega(p)}{p^{s_j}} \right)^{-\frac{1}{2}} \left( 1 - \frac{\beta_j(p)\omega(p)}{p^{s_j}} \right)^{-\frac{1}{2}},
\]
for \( s_j \in D_{j,N}, \quad j = 1, \ldots, n. \)

Let \( P_\varphi \) be the distribution of the random element \( \varphi(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n) \), i.e.
\[
P_\varphi(A) = m_H(\omega \in \Omega : \varphi(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n) \in A), \quad A \in \mathcal{B}(H_n).
\]

**Lemma 1.** The probability measure \( P_T \) converges weakly to the measure \( P_\varphi \) as \( T \to \infty \).

**Proof.** Let \( D_{j,0} = \{ s \in \mathbb{C} : \sigma > \frac{n}{2} \} \), \( j = 1, \ldots, n, \)
\[
H_{n,0} = H(D_{1,0}) \times \ldots \times H(D_{n,0}),
\]
and
\[
P_{T,0}(A) = \nu_T(\{\varphi(s_1 + i\tau, F_1), \ldots, \varphi(s_n + i\tau, F_n) \in A\}), \quad A \in \mathcal{B}(H_{n,0}).
\]
Denote by \( \varphi_0(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n) \) the \( H_{n,0} \)-valued random element defined by formula (1) with \( s_j \in D_{j,0}, \quad j = 1, \ldots, n. \) Then in [20] it was proved that the probability measure \( P_{T,0} \) converges weakly to the distribution of the random element \( \varphi_0 \) as \( T \to \infty \). Since the function \( h : H_{n,0} \to H_n \) defined by the coordinatewise restriction is continuous, hence, applying a property of the weak convergence of probability measures [3], Theorem 5.1, we obtain the lemma.
3. Supports of certain random elements

Let $G_1, \ldots, G_n$ be simply connected regions on $\mathbb{C}$, and $H(G_1, \ldots, G_n) = H(G_1) \times \ldots \times H(G_n)$. In this section we will consider supports of $H(G_1, \ldots, G_n)$-valued random elements.

**Lemma 2.** Let $X$ and $Y$ be two independent $H(G_1, \ldots, G_n)$-valued random elements with distributions $P$ and $Q$, respectively. Then the distribution of the sum $X + Y$ is the convolution $P \ast Q$ of $P$ and $Q$.

**Proof.** Suppose that $X$ and $Y$ are defined on $(\Omega_0, \mathcal{F}, \mathbb{P})$, and let $A \in B(H(G_1, \ldots, G_n))$. From the independence of $X$ and $Y$ we have that the distribution $\mathbb{P}(X + Y \in A)$ of $X + Y$ is equal to the product $P \times Q$ of $P$ and $Q$. However, denoting by $I_A$ the indicator function of the set $A$, by the Fubini theorem we find

$$
(P \times Q)((x, y) : x + y \in A), \quad x, y \in H(G_1, \ldots, G_n).
$$

Denote by $S_X$ the support of the random element $X$.

**Lemma 3.** Let $X$ and $Y$ be two independent $H(G_1, \ldots, G_n)$-valued random elements. Then the support $S_{X+Y}$ of the sum is the closure of the set

$$
S = \{ f \in H(G_1, \ldots, G_n) : f = x + y \text{ with } x \in S_X, y \in S_Y \}.
$$

**Proof.** Let $\{K_{jm}\}$ be a sequence of compact subsets of $G_j$ such that

$$
G_j = \bigcup_{m=1}^{\infty} K_{jm},
$$

$K_{jm} \subset K_{j,m+1}$, and if $K_j$ is a compact and $K_j \subset G_j$, then $K_j \subset K_{jm}$ for some $m$, $j = 1, \ldots, n$. For $f_j, g_j \in H(G_j)$ we put

$$
\rho_j(f_j, g_j) = \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} 2^{-m} \frac{\rho_{jm}(f_j, g_j)}{1 + \rho_{jm}(f_j, g_j)},
$$
where
\[ \rho_{jm}(f_j, g_j) = \sup_{s \in K_{jm}} |f_j(s) - g_j(s)|, \quad j = 1, \ldots, n. \]

Then \( \rho_j \) is a metric on \( H(G_j) \) which induces its topology, \( j = 1, \ldots, n \). For \( f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_n) \in H(G_1, \ldots, G_n) \) we put
\[ \rho(f, g) = \max_{1 \leq j \leq n} \rho_j(f_j, g_j). \]

Then \( \rho \) is a metric on \( H(G_1, \ldots, G_n) \) which induces its topology.

Now we begin the proof of the lemma. Let \( x \in S_X, y \in S_Y \), and \( f = x + y \). We take an arbitrary positive number \( \delta \), and put
\[ A = \{ g \in H(G_1, \ldots, G_n) : \rho(f, g) < \delta \}. \]

Moreover, let \( P \) and \( Q \) be the distribution of random elements \( X \) and \( Y \), respectively. Then we have
\[
\begin{align*}
(P \times Q)(A) &= \int_{H(G_1, \ldots, G_n)} P(A - g)Q(dg) \\
&> \int_{\{ g : \rho(x, g) < \frac{\delta}{2} \}} P(A - g)Q(dg) \\
&\geq P(\{ g : \rho(x, g) < \frac{\delta}{2} \}) \int_{\{ g : \rho(x, g) < \frac{\delta}{2} \}} Q(dg) \\
&= P\left( \{ g : \rho(x, g) < \frac{\delta}{2} \} \right) \cdot Q\left( \{ g : \rho(y, g) < \frac{\delta}{2} \} \right) > 0,
\end{align*}
\]

since by the definition of the support each multiplier is positive. This and Lemma 2 show that \( S \subseteq S_{X+Y} \). But the set \( S_{X+Y} \) is closed, hence \( \overline{S} \subseteq S_{X+Y} \).

It remains to show that \( \overline{S} \supseteq S_{X+Y} \). Suppose that there exists a point \( f \) such that \( f \in S_{X+Y} \) but \( f \notin \overline{S} \). Since \( f \in S_{X+Y} \), for any \( \delta > 0 \) we have
\[
\begin{align*}
(P \times Q)(A) &= \int_{H(G_1, \ldots, G_n)} P(A - g)Q(dg) > 0.
\end{align*}
\]

The later inequality is possible only if there exists a point \( u \in S_Y \) such that \( P(A - u) > 0 \). Therefore there exists a point \( v \in S_X \) in the sphere \( \{ g : \rho(f - u, g) < \delta \} \). Then \( \rho(f, u + v) < \delta \) and \( f' = u + v \in \overline{S} \). Moreover, if \( \delta \to 0 \), then \( f' \to f \). Thus \( f \in \overline{S} \), and this contradiction proves that \( \overline{S} \supseteq S_{X+Y} \).

Now let \( \{ A_m \} \) be a sequence of sets on \( H(G_1, \ldots, G_n) \). By \( \text{Lim}A_m \) denote a set of such \( f \in H(G_1, \ldots, G_n) \) that every neighbourhood of \( f \) contains at least one point belonging to almost all sets \( A_m \).
Lemma 4. Let $P_n$ and $P$ be probability measures on $(H(G_1, \ldots, G_n), B(H(G_1, \ldots, G_n)))$ and let $P_n$ converge weakly to $P$ as $n \to \infty$. Then $S_P \subseteq \lim S_{P_n}$.

Proof. Let $f \in S_P$, and, for $\varepsilon > 0$, $A_\varepsilon = \{ g : \rho(f, g) < \varepsilon \}$. For a fixed $f$ the boundaries of the spheres $\rho(f, g) < \varepsilon$ do not intersect for different $\varepsilon$. Therefore we can choose $\varepsilon$ that the $A_\varepsilon$ should be the continuity set of $P$. Then the properties of the weak convergence yield

$$
\lim_{n \to \infty} P_n(A_\varepsilon) = P(A_\varepsilon) > 0.
$$

So, we have $P_n(A_\varepsilon) > 0$ for $n > n_0(f, \varepsilon)$. For these $n > n_0$ the distance of $f$ from $S_{P_n}$ does not exceed $\varepsilon$. Hence, since $\varepsilon$ is an arbitrary number, we find that $f \in \lim S_{P_n}$. Therefore $S_P \subseteq \lim S_{P_n}$.

Lemma 5. Let $\{X_n\}$ be a sequence of independent $H(G_1, \ldots, G_n)$-valued random elements such that the series

$$
\sum_{m=1}^\infty X_m
$$

converges almost surely. Then the support of the sum of the later series is the closure of the set of all $f \in H(G_1, \ldots, G_n)$ which may be written as a convergent series

$$
f = \sum_{m=1}^\infty f_m, \quad f_m \in S_{X_m}.
$$

Proof. Let $\{X_n\}$ be given on $(\Omega_0, \mathcal{F}, \mathbb{P})$, and

$$
X = \sum_{m=1}^\infty X_m = L_n + R_n,
$$

where

$$
L_n = \sum_{m=1}^n X_m, \quad R_n = \sum_{m=n+1}^\infty X_m.
$$

Since the series of the lemma converges almost surely, for any $\varepsilon > 0$

$$
\mathbb{P}(\omega \in \Omega_0 : \rho(R_n, \mathbb{Q}) \geq \varepsilon) \to 0, \quad n \to \infty.
$$

(2)

Let

$$
P_n(A) = \mathbb{P}(L_n \in A), \quad P(A) = \mathbb{P}(X \in A), \quad A \in B(H(G_1, \ldots, G_n)).
$$

Then the above relations imply the weak convergence of $P_n$ to $P$ as $n \to \infty$. Therefore in view of Lemma 4

$$
S_X \subseteq \lim S_{L_n}.
$$

(3)
Now let $f_0 \in \text{Lim} \mathcal{S}_L$, and, for any $\delta > 0$,

$$A_\delta = \{ f : \rho(f, f_0) < \delta \}.$$

Then there exists $n_1$ such that for $n > n_1$

$$\mathbb{P}(L_n \in A_\epsilon) = P_n(A_\epsilon) > 0. \quad (4)$$

Define $B_\delta = \{ f : \rho(f, 0) < \delta \}$. Then by (2) for $n > n_2$

$$\mathbb{P}(R_n \in B_\delta) > 0. \quad (5)$$

Let $Q_n(A) = \mathbb{P}(R_n \in A), A \in \mathcal{B}(H(G_1, \ldots, G_n))$. Then we have that $P = P_n * Q_n$. Hence and from (4), (5) and Lemma 2 we obtain

$$P(A_{2\delta}) = \int_{H(G_1, \ldots, G_n)} P_n(A_{2\delta} - g)Q_n(dg) \geq \int_{B_\delta} P_n(A_{2\delta} - g)Q_n(dg) \geq P_n(A_\delta) \int_{B_\delta} Q_n(dg) = P_n(A_\delta)Q_n(B_\delta) > 0$$

for $n \geq n_3 = \max(n_1, n_2)$. This means that $f_0 \in \mathcal{S}_X$. Therefore $\mathcal{S}_X \supseteq \text{Lim} \mathcal{S}_L$.

This and (3) imply

$$\mathcal{S}_X = \text{Lim} \mathcal{S}_L. \quad (6)$$

Since $X_1, \ldots, X_n$ are independent, by Lemma 3 we have that $\mathcal{S}_L$ is the closure of the set of all $\underline{f} \in H(G_1, \ldots, G_n)$ which can be written as a sum

$$\underline{f} = \sum_{m=1}^{n} \underline{f}_m, \quad \underline{f}_m \in \mathcal{S}_m.$$

Now if $\underline{f} \in \mathcal{S}_X$, then there exists a sequence $\{ \underline{g}_n : \underline{g}_n \in \mathcal{S}_L \}$ and $\lim_{n \to \infty} \underline{g}_n = \underline{f}$. This together with (6) yield the assertion of the lemma.

4. The denseness of one set of series

In this section we will consider convergent series in $H(G_1, \ldots, G_n)$.

**Lemma 6.** Let $\{ \underline{f}_m \} = \{(f_{1m}, \ldots, f_{nm})\}$ be a sequence in $H(G_1, \ldots, G_n)$ which satisfies:
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a) If \( \mu_1, \ldots, \mu_n \) are complex measures on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) with compact supports contained in \(G_1, \ldots, G_n\), respectively, such that

\[
\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} f_{jm} \, d\mu_j \right| < \infty,
\]

then

\[
\int_{\mathbb{C}} s^r \, d\mu_j(s) = 0
\]

for \( j = 1, \ldots, n, \ r = 0, 1, 2, \ldots \);

b) The series

\[
\sum_{m=1}^{\infty} f_m
\]

converges in \( H(G_1, \ldots, G_n) \);

c) For any compacts \( K_1 \subseteq G_1, \ldots, K_n \subseteq G_n \)

\[
\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K_j} |f_{jm}(s)|^2 < \infty.
\]

Then the set of all convergent series

\[
\sum_{m=1}^{\infty} a_m f_m
\]

with \( a_m \in \gamma \) is dense in \( H(G_1, \ldots, G_n) \).

Proof. The lemma for \( G_1 = \ldots = G_n \) was proved in [1], Lemma 5.2.9, and in the case \( n = 1 \) it can be found in [15], Theorem 6.3.10.

Let \( K_j \) be a compact subset of \( G_j, j = 1, \ldots, n \). We choose a simply connected region \( U_j \) such that \( K_j \subseteq U_j, \overline{U}_j \) is a compact subset of \( G_j \) and the boundary \( \partial U_j \) of \( U_j, j = 1, \ldots, n \), is an analytic simple closed curve. Consider the Hardy space \( H^2(U_j) \), see for definition [15], Section 6.3, which is a Hilbert space, \( j = 1, \ldots, n \).

Now let

\[
H^2(U_1, \ldots, U_n) = H^2(U_1) \times \cdots \times H^2(U_n).
\]

Define for \( f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_n) \in H^2(U_1, \ldots, U_n) \) the inner product by

\[
(f, g) = \sum_{j=1}^{n} (f_j, g_j),
\]

where \((f_j, g_j)\) is the inner product on \( H^2(U_j), j = 1, \ldots, n \). Thus we have that \( H^2(U_1, \ldots, U_n) \) is a Hilbert space again.
From the proof of Theorem 6.3.10 from [15] we have
\[ \|f_m\| = (f_m, f_m) = \sum_{j=1}^{n} (f_j, f_j) = \sum_{j=1}^{n} \|f_j\|^2 = B \sum_{j=1}^{n} \sup_{s \in \mathcal{U}_j} |f_j(s)|^2. \]
Therefore in view of the condition c) and of choice of \( U_j \)
\[ \sum_{m=1}^{\infty} \|f_m\|^2 < \infty. \]
Now let \( g \in H^2(U_1, \ldots, U_n) \) be such that
\[ \sum_{m=1}^{\infty} |f_m, g| < \infty. \] (7)
Using the formula for the inner product in \( H^2(U_j) \), see [15], Section 6.3, we obtain that there exists a complex measure \( \mu_j \) on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) with support contained in \( \partial U_j \), \( j = 1, \ldots, n \), such that
\[ (f, g) = \sum_{j=1}^{n} \int_{\mathbb{C}} f_{jm} d\mu_j. \]
Hence by (7)
\[ \sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} f_{jm} d\mu_j \right| < \infty, \]
and therefore by the condition a) of the lemma
\[ \int_{\mathbb{C}} s^r d\mu_j = 0 \]
for \( j = 1, \ldots, n, \ r = 0, 1, 2, \ldots \). This means that \( g_j \) is orthogonal to all the polynomials, \( j = 1, \ldots, n \). Since the polynomials are dense in the topology of \( H^2(U_j) \), Theorem 6.3.9 of [15], hence we obtain that \( g_j = 0, \ j = 1, \ldots, n \), and \( g = 0 \). Consequently,
\[ \sum_{m=1}^{\infty} \|f_m, g\| = \infty \]
for \( \emptyset \neq g \in H^2(U_1, \ldots, U_n) \). Therefore by Theorem 6.1.16 form [15] we obtain that the set of all convergent series in \( H^2(U_1, \ldots, U_n) \)
\[ \sum_{m=1}^{\infty} \alpha_m f_m \]
with \( \alpha_m \in \gamma \) is dense in \( H^2(U_1, \ldots, U_n) \). Let \( f = (f_1, \ldots, f_n) \in H(G_1, \ldots, G_n) \) and \( \varepsilon > 0 \). Since the convergence in the \( H^2(U_j) \) topology implies the uniform convergence on compact subsets of \( U_j, j = 1, \ldots, n \), we deduce that there exists a sequence \( \{\alpha_m, \alpha_m \in \gamma\} \) such that the series

\[
\sum_{m=1}^{\infty} \alpha_m f_{jm}
\]

converges uniformly on \( K_j \) for all \( j = 1, \ldots, n \), and

\[
\sum_{j=1}^{n} \sup_{s \in K_j} \left| \sum_{m=1}^{\infty} \alpha_m f_{jm}(s) - f_j(s) \right| < \frac{\varepsilon}{4}.
\]

Hence there exists a natural number \( M \) such that

\[
\sum_{j=1}^{n} \sup_{s \in K_j} \left| \sum_{m=M+1}^{\infty} f_{jm}(s) \right| < \frac{\varepsilon}{2}, \tag{8}
\]

and in view of the condition b)

\[
\sum_{j=1}^{n} \sup_{s \in K_j} \left| \sum_{m=1}^{M} \alpha_m f_{jm}(s) - f_j(s) \right| < \frac{\varepsilon}{2}, \tag{9}
\]

Let

\[
a_m = \begin{cases} \alpha_m, & 1 \leq m \leq M, \\ 1, & m > M. \end{cases}
\]

Then (8) and (9) yield

\[
\sum_{j=1}^{n} \sup_{s \in K_j} \left| \sum_{m=1}^{\infty} a_m f_{jm}(s) - f_j(s) \right| < \varepsilon,
\]

and the lemma is proved.

\section{5. The support of the random element \( \varphi \)}

Let, for \( |z| < 1 \),

\[
\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots,
\]

and define

\[
\mathcal{f}_p(s_1, \ldots, s_n, a_p) = (f_{1p}(s_1, a_p), \ldots, f_{np}(s_n, a_p))
\]

\[
= \left( - \log \left( 1 - \frac{\alpha_1(p)a_p}{p^{s_1}} \right) - \log \left( 1 - \frac{\beta_1(p)a_p}{p^{s_1}} \right), \ldots, 
- \log \left( 1 - \frac{\alpha_n(p)a_p}{p^{s_n}} \right) - \log \left( 1 - \frac{\beta_n(p)a_p}{p^{s_n}} \right) \right),
\]

\( s_j \in D_j, \quad j = 1, \ldots, n, a_p \in \gamma. \)
Lemma 7. The set of all convergent series
\[ \sum_p \tilde{f}_p(s_1, \ldots, s_n, a_p) \]
is dense in $H_n$.

Proof. Let
\[ \tilde{f}_p = \tilde{f}_p(s_1, \ldots, s_n) = \left( -\log \left( \frac{1 - \alpha_1(p)}{p^{s_1}} \right) - \log \left( \frac{1 - \beta_1(p)}{p^{s_1}} \right), \ldots, -\log \left( \frac{1 - \alpha_n(p)}{p^{s_n}} \right) - \log \left( \frac{1 - \beta_n(p)}{p^{s_n}} \right) \right). \]
We take $p_0 > 0$ and define
\[ \hat{f}_p = \hat{f}_p(s_1, \ldots, s_n) = \begin{cases} \tilde{f}_p(s_1, \ldots, s_n) & \text{if } p > p_0, \\ 0 & \text{if } p \leq p_0. \end{cases} \]
The products for random elements $\varphi(s_j, \omega, F_j)$ converge uniformly on compact subsets of $D_{j,N}$, $j = 1, \ldots, n$, for almost all $\omega \in \Omega$. Therefore there exists a sequence \{\hat{a}_p, \tilde{a}_p \in \gamma\} such that the series
\[ \sum_p \hat{a}_p \hat{f}_p \]
converges in $H_n$, the details can be found in [23], p. 349. We will prove that the set of all convergent series
\[ \sum_p a_p \hat{f}_p, \quad a_p \in \gamma, \tag{10} \]
is dense in $H_n$. Denoting $g_p = (g_{1p}, \ldots, g_{np}) = \hat{a}_p \hat{f}_p$, we see that for this it suffices to show the denseness of the set of all convergent series
\[ \sum_p a_p g_p(p), \quad a_p \in \gamma. \tag{11} \]

By choice of $\tilde{a}_p$, the series
\[ \sum_p g_p \]
converges in $H_n$. Since
\[ |\alpha_j(p)| \leq p^{(\kappa_j - 1)/2}, \quad |\beta_j(p)| \leq p^{(\kappa_j - 1)/2}, \quad j = 1, \ldots, n, \]
we have that
\[ \tilde{f}_p(s_1, \ldots, s_n) = \left( \frac{\alpha_1(p) + \beta_1(p)}{p^{s_1}} + r_{1p}(s_1), \ldots, \frac{\alpha_n(p) + \beta_n(p)}{p^{s_n}} + r_{np}(s_n) \right) \]
\[ = \left( \frac{c_1(p)}{p^{s_1}} + r_{1p}(s_1), \ldots, \frac{c_n(p)}{p^{s_n}} + r_{np}(s_n) \right) \tag{12} \]
with
\[ r_{jp}(s_j) = B_p^{\kappa_j - 2s_j - 1}, \quad j = 1, \ldots, n. \]

Let \( K_j \) be an arbitrary compact subset of \( D_{j,N}, j = 1, \ldots, n \). Then (12) implies
\[ \sum_p \sum_{j=1}^{n} \sup_{s \in K_j} |g_{jp}(s)|^2 < \infty. \]

To apply Lemma 6 for the set of the series (11) it remains to check the condition a) of the latter lemma. Let \( \mu_j \) be a complex measure on \( (\mathbb{C}, B(\mathbb{C})) \) with compact support contained in \( D_{j,N}, j = 1, \ldots, n \), such that
\[ \sum_p \left| \sum_{j=1}^{n} \int_{\mathbb{C}} g_{jp}(s) d\mu_j(s) \right| < \infty. \]  
(13)

We put \( h_{jp}(s_j) = \hat{a}_p c_j(p)p^{-s_j}, j = 1, \ldots, n \). Then in view of (12)
\[ \sum_p \sum_{j=1}^{n} \sup_{s \in K_j} |g_{jp}(s) - h_{jp}(s)| < \infty. \]
Combining this with (13), we obtain
\[ \sum_p \left| \sum_{j=1}^{n} \int_{\mathbb{C}} h_{jp}(s) d\mu_j(s) \right| < \infty, \]
and, by the definition of \( h_{jp} \),
\[ \sum_p \left| \sum_{j=1}^{n} c_j(p) \int_{\mathbb{C}} p^{-s} d\mu_j(s) \right| < \infty. \]  
(14)

Now let \( D_N = \{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < N \} \), and let
\[ h_j(s) = s - \frac{\kappa_j - 1}{2}, \quad j = 1, \ldots, n. \]
Then we have that
\[ \mu_j h_j^{-1}(A) = \mu_j(h_j^{-1}A), \quad A \in B(\mathbb{C}), \]
is a complex measure on \( (\mathbb{C}, B(\mathbb{C})) \) with compact support contained in \( D_N, j = 1, \ldots, n \). Now (14) can be rewritten in the form
\[ \sum_p \left| \sum_{j=1}^{n} c_{jp} \int_{\mathbb{C}} p^{-s} d\mu_j h_j^{-1}(s) \right| < \infty. \]  
(15)
Using the condition (B) on $c_{jp}$, we deduce from (15)
\[ \sum_p |\rho_k(p)| \sum_{j=1}^n \lambda_{jk} \int_{\mathbb{C}} p^{-s} d\mu_j h_j^{-1}(s) \mid < \infty, \quad k = 1, \ldots, n. \] (16)

Taking
\[ \tilde{\mu}_k(A) = \sum_{j=1}^n \lambda_{jk} \mu_j h_j^{-1}(A), \quad A \in \mathcal{B}(\mathbb{C}), \]
\[ \eta_k(z) = \int_{\mathbb{C}} e^{-sz} d\tilde{\mu}_k(z), \quad z \in \mathbb{C}, \quad k = 1, \ldots, n, \]
we write (16) as
\[ \sum_p |\rho_k(p)||\eta_k(\log p)| < \infty, \quad k = 1, \ldots, n. \]

Since $\rho_k(p), k = 1, \ldots, n,$ satisfies the condition (A), similarly as in [23], Lemma 6, we derive that
\[ \eta_k(z) \equiv 0, \quad k = 1, \ldots, n. \]

This, the definition of $\tilde{\mu}_k$ and the condition (C) show that
\[ \int_{\mathbb{C}} e^{-sz} d\mu_j h_j^{-1}(s) \equiv 0, \quad j = 1, \ldots, n. \]

Hence by the differentiation we find that
\[ \int_{\mathbb{C}} s^r d\mu_j h_j^{-1}(s) = 0, \quad j = 1, \ldots, n, \quad r = 0, 1, 2, \ldots, \]
and together with definition of $h_j$ this implies
\[ \int_{\mathbb{C}} s^r d\mu_j(s) = 0, \quad j = 1, \ldots, n, \quad r = 0, 1, 2, \ldots. \]

Thus all conditions of Lemma 6 are satisfied, and we have the denseness of all convergent series (11) and therefore that of series (10).

It remains the final part of the proof. Let $\xi(s_1, \ldots, s_n) = (x_{01}(s_1), \ldots, x_{0n}(s_n)) \in H_n$, $K_j$ be a compact subset of $D_{j,N}$, $j = 1, \ldots, n$, and $\varepsilon > 0$. We choose $p_0$ to satisfy
\[ \sum_{j=1}^n \sup_{s_j \in K_j} \sum_{p>p_0} \sum_{l=2}^\infty \frac{|\alpha_j(p)|^l + |\beta_j(p)|^l}{lp^{l\sigma}} < \frac{\varepsilon}{4}. \] (17)
The joint universality of zeta-functions attached to certain cusp forms

The denseness of the set of all convergent series (10) shows that there exists a sequence \( \{\tilde{a}_p, \tilde{a}_p \in \gamma \} \) such that

\[
\sum_{j=1}^{n} \sup_{s_j \in K_j} \sum_{p \leq p_0} \left| \tilde{x}_{0j}(s_j) - \sum_{p \leq p_0} \tilde{f}_{jp}(s_j) - \sum_{p > p_0} \tilde{a}_p \tilde{f}_{jp}(s_j) \right| < \frac{\varepsilon}{2}.
\]

We take

\[
a_p = \begin{cases} 1 & \text{if } p \leq p_0, \\ \tilde{a}_p & \text{if } p > p_0. \end{cases}
\]

Then this and (17) yield

\[
\sum_{j=1}^{n} \sup_{s_j \in K_j} \left| \tilde{x}_{0j}(s_j) - \sum_{p \leq p_0} f_{jp}(s_j, a_p) \right|
\leq \sum_{j=1}^{n} \sup_{s_j \in K_j} \left| \tilde{x}_{0j}(s_j) - \sum_{p \leq p_0} \tilde{f}_{jp}(s_j) - \sum_{p > p_0} a_p \tilde{f}_{jp}(s_j) \right|
+ \sum_{j=1}^{n} \sup_{s_j \in K_j} \left| \sum_{p \geq p_0} \tilde{a}_p \tilde{f}_{jp}(s_j) - \sum_{p > p_0} f_{jp}(s_j, \tilde{a}_p) \right| < \frac{\varepsilon}{2}
+ 2 \sum_{j=1}^{n} \sup_{s_j \in K_j} \left( \sum_{p \geq p_0} \sum_{l=2}^{\infty} \frac{|\alpha_l(p)|l + |\beta_l(p)|l}{lp^{l\sigma_j}} \right) < \varepsilon.
\]

This shows that

\[
\rho \left( \tilde{x}_0(s_1, \ldots, s_n), \sum_{p} f_p(s_1, \ldots, s_n, a_p) \right) < \varepsilon,
\]

and the lemma is proved.

Let

\[ S_{j,N} = \{ f \in H(D_{j,N}) : f(s) \not\equiv 0 \text{ for any } s \in D_{j,N} \text{ or } f(s) \equiv 0 \}, \quad j = 1, \ldots, n, \]

and

\[ S_n = S_{1,N} \times \ldots \times S_{n,N}. \]

**Lemma 8.** The support of the random element \( \varphi(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n) \) is the set \( S_n \).

**Proof.** By the definition \( \{\omega(p)\} \) is a sequence of independent random variables on the probability space \((\Omega, \mathcal{B}(\Omega), m_H)\), and the support of each \( \omega(p) \) is the unit circle \( \gamma \). Therefore

\[
\left\{ \log \left( 1 - \frac{\alpha_1(p)\omega(p)}{p^{s_1}} \right)^{-1} + \log \left( 1 - \frac{\beta_1(p)\omega(p)}{p^{s_1}} \right)^{-1}, \ldots, \log \left( 1 - \frac{\alpha_n(p)\omega(p)}{p^{s_n}} \right)^{-1} + \log \left( 1 - \frac{\beta_n(p)\omega(p)}{p^{s_n}} \right)^{-1} \right\}
\]
is a sequence of independent $H_n$-valued random elements, and the set
\[
\left\{ F_p = (f_{1p}(s_1), \ldots, f_{np}(s_n)) \in H_n : f_{jp}(s_j) = f_{jp}(a, p) \right\}
\]
\[
= -\log \left( 1 - \frac{\alpha_j(p)a}{p^{s_j}} \right) - \log \left( 1 - \frac{\beta_j(p)a}{p^{s_j}} \right), \quad j = 1, \ldots, n, \quad a \in \gamma
\]
is the support of each element. By Lemma 5 the support of $H_n$-valued random element
\[
\{ \sum_p f_{1p}(s_1, \omega(p)), \ldots, \sum_p f_{np}(s_n, \omega(p)) \}
\]
is the closure of the set of all convergent series
\[
\sum_p F_p(s_1, \ldots, s_n; a_p), \quad a_p \in \gamma.
\]
By Lemma 7 the latter set is dense in $H_n$. Hence the support of the random element (18) is $H_n$. Using the Hurwitz theorem hence we deduce as in [23] that the support of the random element $\varphi(s_1, \ldots, s_n, \omega; F_1, \ldots, F_n)$ is the set $S_n$.

6. Proof of the theorem

The proof of the theorem runs in the same way as in the case $n = 1$ [23]. First we suppose that the functions $f_j(s)$ have non-vanishing analytic continuation to $H(D_j, N), j = 1, \ldots, n$. Denote by $G$ the set of all $(g_1, \ldots, g_n) \in H_n$ such that
\[
\sup_{1 \leq j \leq n} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{4}
\]
with $\varepsilon > 0$. Since $G$ is an open set, Lemma 1 together with properties of the weak convergence and Lemma 8 yield
\[
\liminf_{T \to \infty} \nu_T \left( \sup_{1 \leq j \leq n} \sup_{s \in K_j} |\varphi(s + i\tau, F_j) - f_j(s)| < \frac{\varepsilon}{4} \right) \geq P(\varphi(G) > 0.
\]
Now let the functions $f_j(s), j = 1, \ldots, n$, be the same as in the statement of the theorem. By the Mergelyan theorem [29] there exists polynomials $p_j(s), p_j(s) \neq 0$ on $K_j$, such that
\[
\sup_{1 \leq j \leq n} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{4}
\]
Moreover, since $p_j(s) \neq 0$ on $K_j$, by the Mergelyan theorem again there exists polynomials $g_j(s), j = 1, \ldots, n$, such that
\[
\sup_{1 \leq j \leq n} \sup_{s \in K_j} |p_j(s) - e^{g_j(s)}| < \frac{\varepsilon}{4}.
\]
This and (19) imply
\[ \sup_{1 \leq j \leq n} \sup_{s \in K_j} |f_j(s) - e^{\gamma_j(s)}| < \frac{\varepsilon}{2}. \]  

(20)

However, \( e^{\gamma_j(s)} \neq 0 \) for all \( s \). Therefore by the first case
\[ \lim \inf_{T \to \infty} \nu_T \left( \sup_{1 \leq j \leq n} \sup_{s \in K_j} |\varphi(s + it, F_j) - e^{\gamma_j(s)}| < \frac{\varepsilon}{2} \right) > 0. \]

Hence and from (20) we have the assertion of the theorem.

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Kai kurį parabolinių formų dzeta funkcijų rinkinio universalumas

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Straipsnyje irodytas normuotų eigenformų dzeta funkcijų rinkinio universalumas. Irodymui naudojamos ortogonalumo salygos (A)–(C).

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