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ZEROS AND EXTREME VALUES OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE

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Abstract. Denote by t_n the distinct positive ordinates of zeros of the Riemann zeta-function $\zeta(s)$ on the critical line in ascending order. We prove $\sum_{t_n\leqslant T}|\zeta'(\frac{1}{2}+it_n)|\ll T(\log T)^{9/4}$, and $\sum_{t_n\leqslant T}\max_{t_n<t<t_{n+1}}|\zeta(\frac{1}{2}+it)|\ll T(\log T)^{5/4}$. These estimates support special cases of conjectures on the vertical zero-distribution of $\zeta(s)$ (Montgomery's pair correlation conjecture, predictions from random matrix theory). The first estimate above is from the previous work [41] of the authors on simple zeros (in this note we shall give a slightly new proof of this result by means of Hardy's Z-function), the second one on extreme values is new (and follows by a similar argument).

Key words and phrases: extreme values, random matrix theory, Riemann zeta-function, simple zeros.

Mathematics Subject Classification: 11M06.

1. The main actor: the zeta-function

§ 1.1. Zeta and the primes

The Riemann zeta-function is given by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},$$

where the product is taken over all prime numbers. Both, the Dirichlet series and the Euler product converge absolutely for $\mathrm{Re}\, s>1$ and uniformly in each compact subset of this half-plane. The identity between the Dirichlet series and the Euler product was discovered by Euler in 1737 and can be regarded as the analytic version of the unique prime factorization of integers. The Euler product gives a first glance on the intimate connection between the zeta-function and the distribution of prime numbers. A first immediate consequence is Euler's proof of the infinitude of the primes. Assuming that there were only finitely many primes, the product is finite, and therefore convergent throughout the whole complex plane, contradicting the fact that the Dirichlet series defining $\zeta(s)$ reduces to the divergent harmonic series as $s \to 1+$. Hence, there exist infinitely many prime numbers.

This fact is well known since Euclid's elementary proof, but the analytic access gives more information on the prime number distribution. It was the young Gauss who conjectured in 1791 for the number $\pi(x)$ of primes $p \leqslant x$ the asymptotic formula

$$\pi(x) \sim \int_2^x \frac{\mathrm{d}u}{\log u};$$

Gauss' conjecture states that, in first approximation, the number of primes $\leqslant x$ is asymptotically $\frac{x}{\log x}$.

§ 1.2. Moving to the left

Riemann was the first to investigate the zeta-function as a function of a complex variable. In his only one but outstanding paper [37] on number theory from 1859 he outlined how Gauss' conjecture could be proved by means of $\zeta(s)$. At Riemann's time the theory of functions was not developed so far, however, the open questions concerning the zeta-function pushed the research in this field quickly forward. We briefly sketch the basic facts concerning $\zeta(s)$.

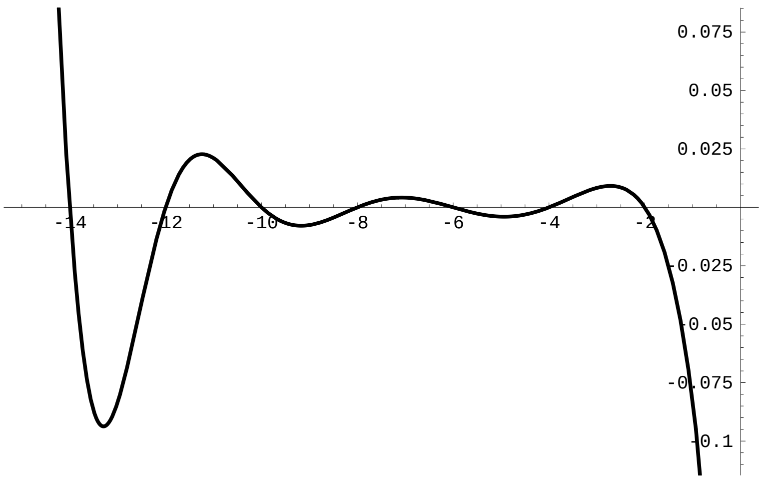

Figure 1: Ups and downs: $\zeta(s)$ for $-15 \leqslant s \leqslant 0$.

First of all, by partial summation,

$$\zeta(s) = \sum_{n\leqslant N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{[u]-u}{u^{s+1}}\mathrm{d}u;$$

here $[u]$ denotes the maximal integer less than or equal to u. This gives an analytic continuation for $\zeta(s)$ to the half-plane $\operatorname{Re} s > 0$, except for a simple pole at $s = 1$ with residue 1 (corresponding to the harmonic series). This process can be continued to the left half-plane and shows that $\zeta(s)$ is analytic throughout the whole complex plane except for $s = 1$. Riemann found the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1}$$

where $\Gamma(s)$ denotes Euler's Gamma-function. In view of the Euler product it is easily seen that $\zeta(s)$ has no zeros in the half-plane $\operatorname{Re} s > 1$. It follows from the functional equation and from basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\operatorname{Re} s < 0$ exactly at the so-called trivial zeros $s = -2n$ with $n \in \mathbb{N}$ (see Figure 1). All other zeros of $\zeta(s)$ are said to be nontrivial, and we denote them by $\varrho = \beta + i\gamma$.

§ 1.3. Where are the complex zeros?

Obviously, the nontrivial zeros have to lie inside the so-called critical strip $0 \leqslant \operatorname{Re} s \leqslant 1$, and it is not too difficult to prove that is none on the real axis. The functional equation (1), in addition with the reflection principle $\zeta(\overline{s}) = \overline{\zeta(s)}$, show some symmetries of $\zeta(s)$. The nontrivial zeros of $\zeta(s)$ have to be distributed symmetrically with respect to the real axis and to the vertical line $\operatorname{Re} s = \frac{1}{2}$. It was Riemann's ingenious contribution to number theory to point out how the distribution of these nontrivial zeros is linked to the distribution of prime numbers. Riemann conjectured for the number $N(T)$ of nontrivial zeros $\varrho = \beta + i\gamma$ with $0 < \gamma \leqslant T$ (counted according multiplicities) an asymptotic formula. This was proved by von Mangoldt who found more precisely

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi \mathrm{e}} + \mathrm{O}(\log T). \tag{2}$$

Riemann worked with $\zeta(\frac{1}{2} + it)$ and wrote

> "... *und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen ....*"

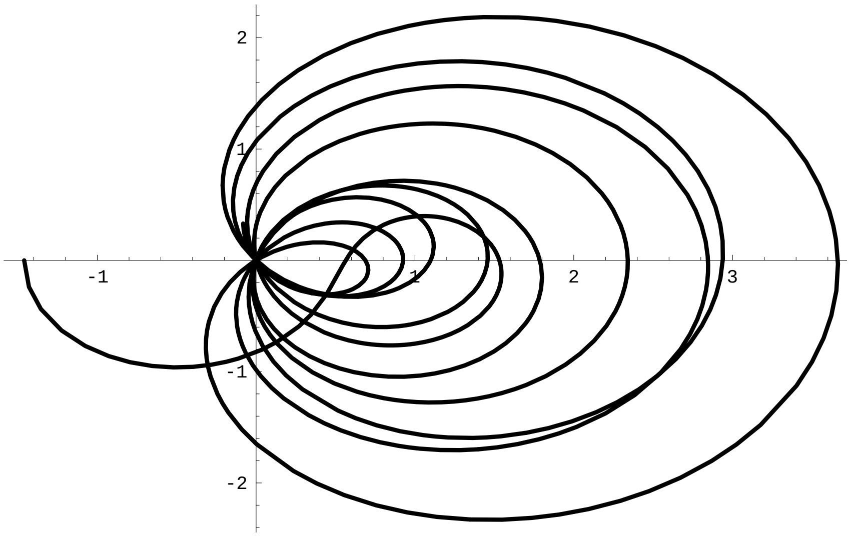

Figure 2: Bird's eye view: $\zeta(0.5 + it)$ for $t \in [0, 50]$.

That means that *very likely* all roots t of $\zeta(\frac{1}{2} + it)$ are real, i. e., all nontrivial zeros lie on the so-called critical line $\operatorname{Re} s = \frac{1}{2}$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

Riemann's hypothesis. $\zeta(s) \neq 0$ *for* $\operatorname{Re} s > \frac{1}{2}$.

In support of his conjecture, Riemann calculated some zeros; the first one with positive imaginary part is $\varrho = \frac{1}{2} + i\, 14.134\ldots$ (see Figure 2).

§ 1.4. Feedback on the prime number distribution

Riemann's ideas led to the first proof of Gauss' conjecture, the celebrated prime number theorem, by Hadamard and de la Vallée-Poussin (independently) in 1896. Their proofs rely in the main part on contour integration and an appropriate zero-free region. The *exact* explicit formula states

$$\Psi(x) := \sum_{p^k \leqslant x} \log p = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} - \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right) - \log(2\pi). \tag{3}$$

The right hand side above is not absolutely convergent. If $\zeta(s)$ would have only finitely many zeros, the right hand side would be a continuous function of x, contradicting the jumps of $\Psi(x)$ for prime powers x. The explicit formula indicates the impact of the Riemann hypothesis on the prime number distribution. One can deduce from (3) that, for fixed $\theta \in [\frac{1}{2}, 1)$,

$$\pi(x) - \int_2^x \frac{\mathrm{d}u}{\log u} \ll x^{\theta+\varepsilon} \quad \iff \quad \zeta(s) \neq 0 \quad \text{for} \quad \operatorname{Re} s > \theta;$$

here and in the sequel ε stands for an arbitrary small positive constant. With regard to known zeros of $\zeta(s)$ on the critical line it turns out that an error term in the prime number theorem with $\theta < \frac{1}{2}$ is impossible. Thus, the Riemann hypothesis states that the prime numbers are *as uniformly distributed as possible*!

§ 1.5. Hardy's Z-function

For our investigations on the behaviour of the zeta-function on the critical line, it is very convenient to replace $\zeta(\frac{1}{2} + it)$ by a real-valued function.

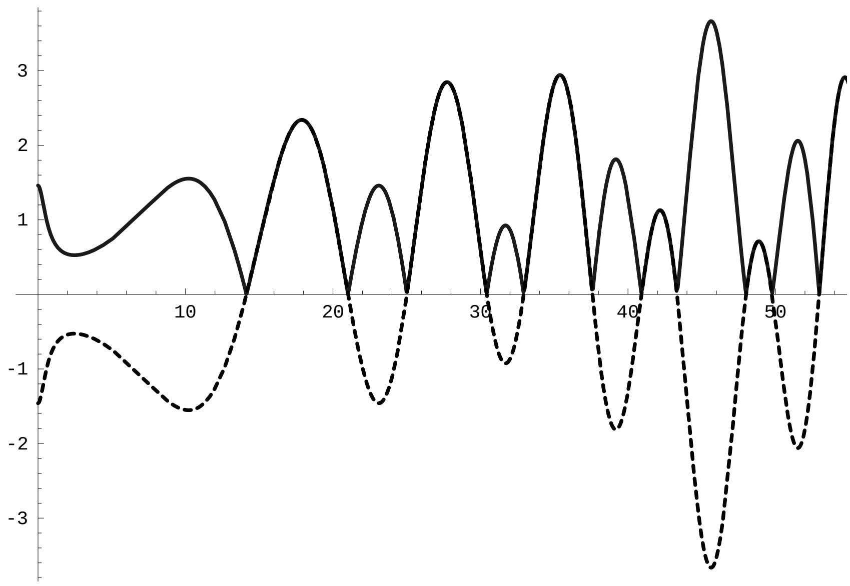

Figure 3: Graphs of the modulus of the Riemann zeta-function (solid) on the critical line $\operatorname{Re} s = \frac{1}{2}$ and of Hardy's Z-function (dashed). You can see the first zeros of the zeta-function.

Hardy's Z-function is defined by

$$Z(t) = \exp(i\vartheta(t))\zeta\left(\frac{1}{2} + it\right) \tag{4}$$

where

$$\vartheta(t) := \pi^{-\frac{it}{2}} \frac{\Gamma(\frac{1}{4} + \frac{it}{2})}{|\Gamma(\frac{1}{4} + \frac{it}{2})|}.$$

It is easily seen that $Z(t)$ is a differentiable function which is real for real t, and that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| = |Z(t)|; \tag{5}$$

these basic facts follow immediately from the functional equation. Consequently, the zeros and extrema of $Z(t)$ correspond to the zeros and extrema of the Riemann zeta-function on the critical line, respectively (see Figure 3).

The function $Z(t)$ has a negative local maximum at $t = 2.4757\ldots$, and this is the only known negative local maximum in the range $t \geqslant 0$; a positive local minimum is not known. The occurrence of a negative local maximum, besides the one at $t = 2.4757\ldots$, or a positive local minimum of $Z(t)$, would disprove Riemann's hypothesis. This follows from the fact that *if the Riemann's hypothesis is true, the graph of the logarithmic derivative $\frac{Z'}{Z}(t)$ is monotonically decreasing between the zeros of $Z(t)$ for $t \geqslant 1000$.* The proof of this proposition is not difficult and can be found in [7].

§ 1.6. Lehmer's phenomenon

The Riemann-Siegel formula (discovered by Riemann, rediscovered by Siegel while studying Riemann's unpublished papers) provides a very good approximation of the zeta-function on the critical line. In terms of Hardy's Z-function,

$$Z(t) = 2 \sum_{n \leqslant \sqrt{t/(2\pi)}} \frac{\cos(\vartheta(t) - t \log n)}{n^{\frac{1}{2}}} + \mathrm{O}\left(t^{-\frac{1}{4}}\right),$$

valid for $t \geqslant 1$. The Riemann-Siegel formula is the basis of all high precision computations of the zeta-function on the critical line. [1]

Lehmer [28] detected that the zeta-function occasionally has two very close zeros on the critical line; for instance the zeros at $t = 7005.0629\ldots$ and $t = 7005.1006\ldots$. So the graph of $Z(t)$ sometimes barely crosses the t-axis (see Figure 4).

In view of our observation relating the graph of $\frac{Z'}{Z}(t)$ with Riemann's hypothesis from the previous section, $Z(t)$ has exactly one critical point between successive zeros for sufficiently large t. Hence, Lehmer's observation, in the literature called Lehmer's phenomenon, is a near-counterexample to the Riemann hypothesis.

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[1] A very nice animated plot of $Z(t)$ can be found on Pugh's webpage http://www.math.ubc.ca/ pugh/RiemannZeta/RiemannZetaLong.html.

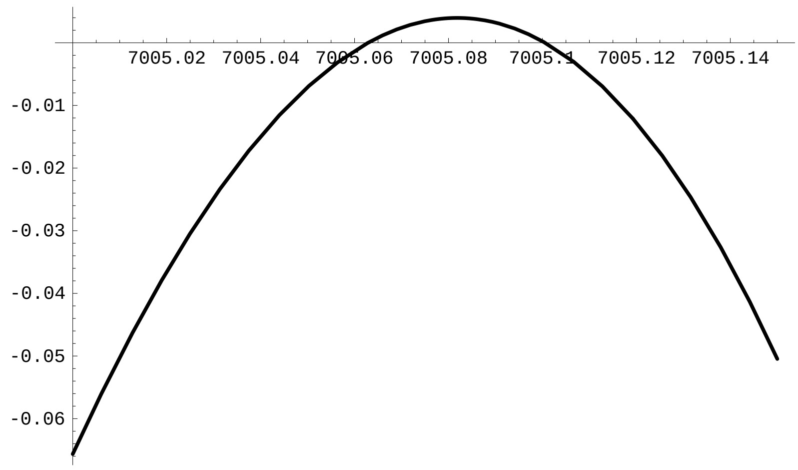

Figure 4: Lehmer's phenomenon.

2. Zeros on the critical line

§ 2.1. Observations on the multiplicities

Many computations were done to find a counterexample to the Riemann hypothesis. Van de Lune, te Riele & Winter [30] localized the first 1 500 000 001 zeros, all lying without exception on the critical line; moreover they all are simple! The Riemann-von Mangoldt formula (2) implies for the multiplicity $m(\varrho)$ of a zero ϱ the estimate

$$m(\varrho) \ll \log |\gamma|.$$

It is widely believed that multiple zeros do not exist, or at least there should not exist too many of them.

Essential simplicity hypothesis. *All or at least almost all zeros of the zeta-function are simple:* $\zeta'(\varrho) \neq 0$.

This conjecture has arithmetical consequences. Cramér showed, assuming the Riemann hypothesis,

$$\frac{1}{\log X} \int_1^X \left(\frac{\Psi(x) - x}{x} \right)^2 \mathrm{d}x \sim \sum_{\varrho} \left| \frac{m(\varrho)}{\varrho} \right|^2,$$

where the sum is taken over distinct zeros. The right-hand side is minimal if all zeros are simple. Going further, Goldston, Gonek and Montgomery [12] observed interesting relations between the essential simplicity hypothesis, mean-values of the logarithmic derivative of $\zeta(s)$, the error term in the prime number theorem, and Montgomery's pair correlation conjecture (see §4.1).

§ 2.2. What is known? – Hardy, Selberg & co

Hardy [18] showed that infinitely many zeros lie on the critical line, and Selberg [38] was the first to prove that a positive proportion of all zeros lies exactly on $\mathrm{Re}\, s = \frac{1}{2}$. Let $N_0(T)$ denote the number of zeros ϱ of $\zeta(s)$ on the critical line with imaginary part $0 < \gamma \leqslant T$. The idea to use mollifiers to dampen the oscillations of $|\zeta(\frac{1}{2} + it)|$ led Selberg to prove

$$\liminf_{T\to\infty} \frac{N_0(T+H) - N_0(T)}{N(T+H) - N(T)} > 0,$$

as long as $H \geqslant T^{1/2+\varepsilon}$. Karatsuba [26] improved this result to $H \geqslant T^{27/82+\varepsilon}$ by some technical refinements. However, the localized zeros are not necessarily simple. By an ingenious new method, working with mollifiers of finite length, Levinson [29] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, and as Heath-Brown [21] and Selberg (unpublished) discovered, they are all simple. By optimizing the technique Levinson himself and others improved the proportion slightly, but more recognizable is Conrey's idea in introducing Kloosterman sums. So Conrey [3] was able to choose a longer mollifier to show that more than two fifths of the zeros are simple and on the critical line. The use of longer mollifiers leads to larger proportions. Farmer [8] observed that if it is possible to take mollifiers of infinite length, then almost all zeros lie on the critical line and are simple. In [39], Steuding found a new approach (combining ideas and methods of Atkinson, Jutila and Motohashi) to treat short intervals. Let $N_1(T)$ denote the number of simple zeros $\varrho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ on the critical line with $0 < \gamma \leqslant T$. In [39] it was shown that

$$N_1(T+H) - N_1(T) \gg N(T+H) - N(T) \gg H \log T \tag{6}$$

whenever $T^{0.552} \leqslant H \leqslant T$.

§ 2.3. Short series over simple zeros

Now we study discrete moments of the first derivative of the zeta-function at simple zeros on the critical line.

Denote the positive roots of the function $\zeta(\frac{1}{2} + it)$ in ascending order by t_n (counting multiplicities); notice that the Riemann hypothesis is equivalent to $t_n = \gamma_n$ for all $n \in \mathbb{N}$.

THEOREM 1. *Let $T^{\frac{1}{2}+\varepsilon} \leqslant H \leqslant T$. Then, for sufficiently large T,*

$$\sum_{T<t_n\leqslant T+H} \left|\zeta'\left(\frac{1}{2}+it_n\right)\right| \ll H(\log T)^{\frac{9}{4}}.$$

Note that only simple zeros give a non-zero contribution to the discrete moment under observation; in view of (6) the number of the relevant simple zeros counted is $N_1(T+H) - N_1(T) \asymp H\log T$ if $H \geqslant T^{0.552}$ with respect to (6). However, we believe that these asymptotics hold in a wider range.

We present here a slightly different argument than the one we gave in [41] (more precisely, instead of applying Garaev's lemma [9], we argue with Hardy's Z-function).

PROOF. Differentiation of (4) gives

$$Z'(t) = i\exp(i\vartheta(t))\left\{\vartheta'(t)\zeta\left(\frac{1}{2}+it\right) + \zeta'\left(\frac{1}{2}+it\right)\right\}. \tag{7}$$

It follows that

$$|Z'(t_n)| = \left|\zeta'\left(\frac{1}{2}+it_n\right)\right|. \tag{8}$$

Thus, if and only if the first derivative $Z'(t)$ does not vanish in the ordinate t_n of a zero of the zeta-function on the critical line, the zero $\frac{1}{2}+it_n$ is simple. Now let λ_n be the least positive real number for which $|Z(t)|$ is maximal in the interval (t_n, t_{n+1}). Then

$$-Z'(t_n) = \int_{t_n}^{\lambda_n} Z''(t)\mathrm{d}t.$$

Summing up over n we get

$$\sum_{T<t_n\leqslant T+H} |Z'(t_n)| \leqslant \int_{t_{N(T)}}^{t_{N(T+H)+1}} |Z''(t)|\mathrm{d}t. \tag{9}$$

Here we have to replace the upper limit of the integral by T. The gap of consecutive zeros of the Riemann zeta-function on the critical line cannot be too large. Hardy and Littlewood [19] proved

$$t_{n+1} - t_n \ll t_n^{\frac{1}{4}+\varepsilon}. \tag{10}$$

This gives in (9)

$$\sum_{T<t_n\leqslant T+H}|Z'(t_n)|\leqslant\int\limits_T^{T+H}|Z''(t)|\mathrm{d}t+\mathrm{O}\Bigg(\int\limits_0^{\mathrm{O}(T^{\frac14+\varepsilon})}|Z''(t+T+H)|\mathrm{d}t\Bigg).$$

By the Phragmén-Lindelöf principle one easily gets the estimate $Z''(t)\ll t^{1/2+\varepsilon}$ as $t\to\infty$. This rather rough estimate leads to

$$\sum_{T<t_n\leqslant T+H}|Z'(t_n)|\leqslant\int\limits_T^{T+H}|Z''(t)|\mathrm{d}t+\mathrm{O}\Big(T^{\frac34+\varepsilon}\Big).\qquad(11)$$

Now we have to replace $Z''(t)$ in terms of the zeta-function. Differentiation of (7) gives

$$\begin{aligned}Z''(t) &= \exp(i\vartheta(t))\Bigg\{(i\vartheta''(t)-\vartheta'(t)^2)\zeta\Big(\frac12+it\Big)+\\ &\quad -2\vartheta'(t)\zeta'\Big(\frac12+it\Big)-\zeta''\Big(\frac12+it\Big)\Bigg\}.\end{aligned}$$

By Stirling's formula, we get

$$Z''(t)\ll(\log t)^2\zeta\Big(\frac12+it\Big)+(\log t)\zeta'\Big(\frac12+it\Big)+\zeta''\Big(\frac12+it\Big).\qquad(12)$$

Ramachandra [35] proved, for $H\geqslant T^{1/2+\varepsilon}$,

$$\int\limits_T^{T+H}\left|\zeta^{(m)}\Big(\frac12+it\Big)\right|\mathrm{d}t\ll H(\log T)^{m+1/4},\qquad(13)$$

valid for $m=0,1,\ldots$. In view of (12) this gives for the integral in (11) the upper bound $H(\log T)^{9/4}$. Taking into account (8), this proves the theorem.

§ 2.4. Negative moments: Garaev's theorem

Recently, Garaev [9] showed that the series

$$\sum_{\substack{\varrho\\ \zeta'(\varrho)\neq0}}|\varrho\zeta'(\varrho)|^{-1}$$

is divergent; this remarkable result was before only known subject to the truth of the Riemann hypothesis (see [42], page 374). Actually, Garaev proved a stronger estimate, namely

$$\sum_{\substack{t_n<T\\ \zeta'(\varrho)\neq 0}} \left|\varrho\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1} \gg (\log T)^{\frac{1}{2}}.$$

Such estimates measure how far the zeros of the zeta-function are from being simple.

From the previous theorem we can deduce a slight improvement of Garaev's theorem; the same improvement was found by Garaev [10] (independently).

Corollary 2. *For sufficiently large T,*

$$\sum_{\substack{t_n\leqslant T\\ \zeta'(\frac{1}{2}+it_n)\neq 0}} \left|\gamma\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1} \gg (\log T)^{\frac{3}{4}}.$$

Proof. The Cauchy-Schwarz inequality yields

$$\begin{aligned} &N_1(T+H)-N_1(T)\\ &\quad=\sum_{\substack{T<t_n\leqslant T+H\\ \zeta'(\frac{1}{2}+it_n)\neq 0}} \left|\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-\frac{1}{2}}\left|\zeta'\left(\frac{1}{2}+it_n\right)\right|^{\frac{1}{2}}\\ &\quad\leqslant\Bigg(\sum_{\substack{T<t_n\leqslant T+H\\ \zeta'(\frac{1}{2}+it_n)\neq 0}} \left|\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1}\Bigg)^{\frac{1}{2}}\Bigg(\sum_{\substack{T<t_n\leqslant T+H\\ \zeta'(\frac{1}{2}+it_n)\neq 0}} \left|\zeta'\left(\frac{1}{2}+it_n\right)\right|\Bigg)^{\frac{1}{2}}. \end{aligned}$$

The estimate (6) together with Theorem 1 imply

$$\sum_{\substack{T<t_n\leqslant T+H\\ \zeta'(\frac{1}{2}+it_n)\neq 0}} \left|\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1} \gg H(\log T)^{-\frac{1}{4}},$$

however, contrary to the related estimate of Theorem 1, this only holds for $T^{0.552}\leqslant H\leqslant T$; any extension of the range for H relies on improvements upon (6). Setting $H=T$ yields

$$\sum_{\substack{T<t_n\leqslant 2T\\ \zeta'(\frac{1}{2}+it_n)\neq 0}} \left|t_n\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1} \gg (\log T)^{-\frac{1}{4}}. \tag{14}$$

Let c be a positive constant less than $(\log 2)^{-1}$. Using (14) with $2^{-k}T$ instead of T and summing up over all positive integers $k \leqslant c\log T$, we obtain

$$\sum_{T^{1-c\log 2}<t_n\leqslant T\,\zeta'(\frac{1}{2}+it_n)\neq 0} \left|t_n\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1}$$
$$\geqslant \sum_{k\leqslant c\log T}\ \sum_{2^{-k}T<t_n\leqslant 2^{1-k}T\,\zeta'(\frac{1}{2}+it_n)\neq 0} \left|t_n\zeta'\left(\frac{1}{2}+it_n\right)\right|^{-1}$$
$$\gg \sum_{k\leqslant c\log T} (\log(2^{-k}T))^{-\frac{1}{4}}.$$

For each term $k \leqslant c\log T$ we have $\log(2^{-k}T) \ll \log T$. This proves the corollary.

3. Extreme values

§ 3.1. What is known about extreme values?

Balasubramanian and Ramachandra [1] proved that

$$\max_{T\leqslant t\leqslant T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right| \geqslant \exp\left(\frac{3}{4}\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{2}}\right)$$

for any H satisfying $(\log T)^{1+\varepsilon} \leqslant H \leqslant T$ with sufficiently large T and any positive ε. However, such extreme values can only appear quite rarely. What can be said about extreme values on average?

Denote the maxima between consecutive t_n's by

$$M_n := \max\left\{\left|\zeta\left(\frac{1}{2}+it\right)\right| \,:\, t_n < t < t_{n+1}\right\}.$$

Assuming the truth of Riemann's hypothesis, Conrey and Ghosh [4] proved

$$\frac{1}{N(T)}\sum_{t_n\leqslant T} M_n^2 = \left(\frac{\mathrm{e}^2-5}{2}+\mathrm{o}(1)\right)\log T.$$

This result is surprising because it implies that the discrete average value of $|\zeta(\frac{1}{2}+it)|^2$ at its critical values is only a constant factor larger than the continuous average of $|\zeta(\frac{1}{2}+it)|^2$ which is $\log t$ as it follows from the classical asymptotic formula

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \sim \log T.$$

Moreover, Conrey [2] proved, also conditional on the Riemann hypothesis,

$$\sum_{t_n \leqslant T} M_n^4 \asymp T(\log T)^5$$

with explicit implicit constants. It is expected that, for fixed $k \geqslant 0$,

$$\sum_{t_n \leqslant T} M_n^{2k} \sim c_k T(\log T)^{k^2+1},$$

where c_k is a positive constant.

§ 3.2. The first moment

Odd moments are in general harder to tackle than even moments. We shall prove an upper estimate for the first moment which is of the expected size (but we cannot obtain comparable unconditional lower bounds).

THEOREM 3. *Let $T^{1/2+\varepsilon} \leqslant H \leqslant T$. Then, as $T \to \infty$,*

$$\sum_{T < t_n \leqslant T+H} M_n \ll H(\log T)^{\frac{5}{4}}.$$

PROOF. In view of (5) we may consider the extreme values of Hardy's Z-function in place of those of the Riemann zeta-function on the critical line. Recall that we defined in §2.3 the real numbers λ_n by the property $M_n = |Z(\lambda_n)|$. It follows that

$$Z(\lambda_n) = \int_{t_n}^{\lambda_n} Z'(t)\mathrm{d}t \quad \text{and} \quad -Z(\lambda_n) = \int_{\lambda_n}^{t_{n+1}} Z'(t)\mathrm{d}t.$$

Under assumption of the Riemann hypothesis, the numbers λ_n coincide with the zeros of the derivative $Z'(t)$ for $t > t_1$. Consequently, $Z'(t)$ changes its sign at $t = \lambda_n$ and we deduce from the above

$$2|Z(\lambda_n)| = \int_{t_n}^{t_{n+1}} |Z'(t)|\mathrm{d}t.$$

Thus,

$$\sum_{T < t_n \leqslant T+H} M_n = \frac{1}{2} \int_{t_{N(T)}}^{t_{N(T+H)+1}} |Z'(t)|\mathrm{d}t.$$

In view of (10) we get by the same argument as in the proof of Theorem 1

$$\sum_{T<t_n\leqslant T+H} M_n = \frac{1}{2}\int_T^{T+H} |Z'(t)|\mathrm{d}t + \mathrm{O}\Big(T^{\frac{3}{4}+\varepsilon}\Big). \tag{15}$$

Since we are not interested in a result conditional on the truth of the Riemann hypothesis, we observe that without assumption of any unproved hypothesis the argument above yields the right hand side as an upper bound for the discrete moment under observation. It should be noted that then not necessarily all extreme values are incorporated.

Similarly as in §2.3, we deduce from (7) that, for sufficiently large t,

$$Z'(t) \ll \log t \cdot \left|\zeta\left(\frac{1}{2}+it\right)\right| + \left|\zeta'\left(\frac{1}{2}+it\right)\right|.$$

Taking into account Ramachandra's estimate for the first moment of the derivatives $\zeta^{(m)}(s)$ of the Riemann zeta-function on the critical line (13), we deduce from (15) the upper bound of the theorem.

§ 3.4. Related estimates due to Ivić and Ramachandra

In [25] Ivić asked for an unconditional estimate of the discrete second moment of the values of $|\zeta(\frac{1}{2}+i\gamma)|$ taken over all ordinates γ of non-trivial zeros. If the Riemann hypothesis is true, this quantity is identically zero. But what upper bounds can be obtained without assuming any unproved hypothesis? Studying the argument of the zeta-function on the critical line, Ivić derived an interesting estimate. His result was slightly improved by Ramachandra [36] who proved by a different method, for any $0<k<2$ and $T^{1/2+\varepsilon}\leqslant H\leqslant T$,

$$\sum_{T\leqslant H\leqslant T+H} M(\gamma)^{2k} \ll H(\log T)^{1+k^2}\log\log T,$$

where $M(\gamma)$ is the maximum of $|\zeta(s)|$ in the rectangle

$$1/2-\frac{c_1}{\log T}\leqslant \mathrm{Re}\, s\leqslant 2\ , \quad |\mathrm{Im}\, s-\gamma|\leqslant \frac{c_2\log\log T}{\log T},$$

where $c_1, c_2>0$ are certain fixed constants. Since Ramachandra incorporates the value-distribution of $\zeta(s)$ in a small neighbourhood off the critical line, we cannot directly compare Theorem 3 with his estimate for $k=\frac{1}{2}$. Nevertheless, it is interesting that the exponent $\frac{5}{4}$ appears in both results.

§ 3.5. Negative moments

In analogy to §2.4 we shall briefly study the impact of Theorem 3 on the related negative moment. Since there are at least as many distinct zeros as simple zeros, the Cauchy-Schwarz inequality yields

$$T\log T \ll \sum_{T<t_n\leqslant 2T} M_n^{-\frac{1}{2}} M_n^{\frac{1}{2}} \leqslant \Big(\sum_{T<t_n\leqslant 2T} M_n^{-1}\Big)^{\frac{1}{2}} \Big(\sum_{T<t_n\leqslant 2T} M_n\Big)^{\frac{1}{2}}.$$

Thus it follows from Theorem 3 that

$$\sum_{T<t_n\leqslant 2T} M_n^{-1} \gg T(\log T)^{\frac{3}{4}}.$$

By the same arguments as in §2.4 we get

Corollary 4. *For sufficiently large T,*

$$\sum_{t_n\leqslant T} (t_n M_n)^{-1} \gg (\log T)^{\frac{7}{4}}.$$

4. Widely believed but yet unproved

§ 4.1. Vertical zero-distribution: the pair correlation

Assuming the truth of the Riemann hypothesis, Montgomery [31] studied the distribution of consecutive zeros $\frac{1}{2}+i\gamma, \frac{1}{2}+i\gamma'$ of the Riemann zeta-function and conjectured

Montgomery's pair correlation conjecture. *For fixed α, β satisfying $0<\alpha<\beta$,*

$$\lim_{T\to\infty} \frac{1}{N(T)} \# \Big\{ 0<\gamma,\gamma'<T \ : \ \alpha \leqslant \frac{(\gamma-\gamma')\log T}{2\pi} \leqslant \beta \Big\}$$
$$= \int_\alpha^\beta \Big(1-\Big(\frac{\sin \pi u}{\pi u}\Big)^2\Big) \mathrm{d}u.$$

The function on the right side of Montgomery's conjectural formula is the pair correlation function of the eigenvalues of large random Hermitean matrices, or more specifically of the Gaussian Unitary Ensemble. Indeed, by the work of Odlyzko [33] it turned out that the pair correlation and the nearest neighbour spacing for the zeros of $\zeta(s)$ were amazingly close to those for the Gaussian Unitary Ensemble. However, there is more evidence for the pair correlation conjecture than numerical data. Many results from random matrix theory were found which perfectly fit to certain results on the value-distribution of the Riemann zeta-function. For example, Keating & Snaith [27] showed that certain random matrix ensembles have in a sense the same value-distribution as the zeta-function on the critical line predicted by Selberg's limit law.

§ 4.2. Predictions from random matrix theory

It is a long standing conjecture that for $k > 0$, there exists a constant $C(k)$ such that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \mathrm{d}t \sim C(k)\, T(\log T)^{k^2}, \tag{16}$$

as $T \to \infty$. It is not known whether this conjecture is related to Riemann's hypothesis or not. By the work of Ramachandra [34] it is known that a lower estimate of the predicted size holds subject to the truth of Riemann's hypothesis. The asymptotic formula (16) is known to be true only in the trivial case $k = 0$, and in the cases $k = 1$ with $C(1) = 1$ and $k = 2$ with $C(2) = \frac{1}{(2\pi^2)}$ by the classical results of Hardy and Littlewood [20] and Ingham [24], respectively. (16) is a bit unsatisfactory as long as we do not know what values $C(k)$ to take. Recently, some new insights were found by random matrix theory.

Random matrix theory may be used for doing *good* predictions. Extending a conjecture of Conrey & Ghosh [6], Keating & Snaith [27] conjectured that the asymptotic formula (16) holds with

$$C(k) = \frac{G^2(k+1)}{G(2k+1)} \prod_p \left(1 - \frac{1}{p^2}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m},$$

where $G(z)$ is Barnes G-functions, defined by

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{1}{2}(z(z+1)+\gamma z^2)\right) \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n \exp\left(-z+\frac{z^2}{n}\right),$$

and γ is Euler's constant. Note that in the above definition of the numbers $C(k)$, one must take an appropriate limit if $k = 0$ or $k = -1$.

Besides (16), Hall [15] conjectured that

$$\int_0^T |Z(t)|^{2h-2k} |Z'(t)|^{2k} \mathrm{d}t \sim c(h,k) T(\log T)^{2h+k^2}$$

for some constants $c(h,k)$. This would lead via (15) to the asymptotic formula

$$\sum_{\gamma_n\leqslant T} M_n \sim \frac{c(\frac{1}{2},\frac{1}{2})}{2} T(\log T)^{\frac{5}{4}},$$

conditional to the truth of the Riemann hypothesis and Hall's conjecture.

§ 4.3 Discrete moments

Assuming the truth of Riemann's hypothesis and, additionally, that all nontrivial zeros of the zeta-function are simple, Hughes, Keating and O'Connell [22, 23] conjectured for fixed $k > -3/2$ the asymptotic formula

$$\frac{1}{N(T)} \sum_{0<\gamma\leqslant T} \left|\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a(k) \left(\log \frac{T}{2\pi}\right)^{k(k+2)}. \tag{17}$$

This conjecture fits perfectly to a conjecture of Gonek [14] on negative moments ($k=-1$). However, (17) is only known to be true in the trivial case $k=0$ and the case $k=1$, settled by a theorem of Gonek [13] under assumption of the truth of Riemann's hypothesis. Ng [32] proved upper and lower bounds of the predicted size, also under assumption of the Riemann hypothesis. Unconditional upper bounds of the correct size were given by Garaev [9] (implicitly) and the authors [41] (independently) for $k=\frac{1}{2}$, and by Garunkštis and the first author [11] for $k=2$.

§ 4.4 Gaps between consecutive zeros

Hall [15], [16] investigated discrete moments of the extreme values of the zeta-function between consecutive zeros, i. e., under the restriction $t_{n+1}-t_n \leqslant \frac{2\pi\theta}{\log T}$, where θ is a small positive constant. His unconditional upper estimates might be seen as positive evidence to Montgomery's famous but yet unproved pair correlation conjecture [31] relating to small gaps between zeros ($\theta \to 0$). The first author [40] gave a different (perhaps simpler) approach to Hall's estimates (which yields slightly weaker implicit constants) and extended the line of these investigation also to simple zeros on the critical line.

Hall also started to investigate how large gaps between consecutive zeros on the critical line can be and obtained very interesting bounds for the limit superior of the suitably normalized difference between such consecutive zeros (with respect to (2)). Following his argument we can prove a related result for extreme values. If consecutive zeros leave a large gap one can also expect a large gap between consecutive extrema on the critical line. In this direction we can prove unconditionally

$$\limsup_{n\to\infty} \frac{\lambda_{n+1}-\lambda_n}{\frac{2\pi}{\log\lambda_n}} \geqslant \left(\frac{297}{68}\right)^{\frac{1}{4}} = 1.44564\ldots\ .$$

In analogy to the corresponding question for the ordinates of nontrivial zeros we expect that the limit superior above is $+\infty$.

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Rymano dzeta funkcijos ekstremumai ir nuliai kritinėje tiesėje

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Pažymėkime t_n Rymano dzeta funkcijos $\zeta(s)$ nulių kritinėje tiesėje skirtingas teigiamas ordinates, surikiuotas didėjančia tvarka. Straipsnyje pateikiamas naujas įverčio

$$\sum_{t_n \leqslant T} \left| \zeta'\left(\frac{1}{2} + it_n\right) \right| \ll T(\log T)^{\frac{9}{4}},$$

gauto autorių [41] darbe, įrodymas (Hardžio Z-funkcijos terminais). Be to, remiantis panašiais argumentais, gaunamas naujas įvertis

$$\sum_{t_n \leqslant T} \max_{t_n < t < t_{n+1}} \left| \zeta\left(\frac{1}{2} + it\right) \right| \ll T(\log T)^{\frac{5}{4}}.$$

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