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A DISCRETE LIMIT THEOREM ON THE COMPLEX PLANE FOR THE PERIODIC ZETA-FUNCTION

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Abstract. In the paper a discrete limit theorem on the complex plane for the periodic zeta-function is proved. An explicit form of the limit measure is given.

Key words and phrases: limit theorem, periodic zeta-function, weak convergence of probability measures.

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Let $\mathfrak{A} = \{a_m : m \in \mathbb{Z}\}$ be a periodic sequence of complex numbers, with period $k > 0$, and let $s = \sigma + it$ denote a complex variable. The periodic zeta-function $\zeta(s;\mathfrak{A})$ is defined, for $\sigma > 1$, by

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

If $\mathfrak{A} = \{1\}$ and $k = 1$, then the function $\zeta(s;\mathfrak{A})$ becomes the Riemann zeta-function. Let $0 < \alpha \leqslant 1$. The Hurwitz zeta-function $\zeta(s,\alpha)$, is defined, for $\sigma > 1$, by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

Since $\zeta(s,\alpha)$ is regular in the whole complex plane $\mathbb{C}$, except for a simple pole at $s = 1$ with residue 1, the equality

$$\zeta(s;\mathfrak{A}) = \frac{1}{k^s} \sum_{q=1}^{k} a_q \zeta\big(s, \frac{q}{k}\big), \quad \sigma > 1,$$

shows that the function $\zeta(s;\mathfrak{A})$ is analytically continuable to the whole complex plane, except, possibly, for a simple pole at $s=1$ with residue

$$a=\frac{1}{k}\sum_{m=1}^{k}a_m.$$

If $a=0$, then $\zeta(s;\mathfrak{A})$ is an entire function.

In [2] and [3] limit theorems in the sense of weak convergence of probability measures for the function $\zeta(s;\mathfrak{A})$ have been obtained. Denote by $\operatorname{meas}\{A\}$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$, and let, for $T>0$,

$$\nu_T(\dots)=\frac{1}{T}\operatorname{meas}\{\tau\in[0;T]\ :\ \dots\},$$

where in place of dots a condition satisfied by τ is to be written. Let $\mathcal{B}(S)$ stand by the class of Borel sets of the space S. For a region G on the complex plane, denote by $H(G)$ and $M(G)$ the spaces of analytic and meromorphic functions, respectively, equipped with the topology of uniform convergence on compacta. Let $D=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ and $D_1=\{s\in\mathbb{C}:\sigma>\frac{1}{2}\}$. Then in [2] and [3] it was proved that the probability measures

$$\nu_T\big(\zeta(s+i\tau;\mathfrak{A})\in A\big),\quad A\in\mathcal{B}(H(D)),$$
$$\nu_T\big(\zeta(s+i\tau;\mathfrak{A})\in A\big),\quad A\in\mathcal{B}(H(D_1)),$$

if $a=0$, and

$$\nu_T\big(\zeta(s+i\tau;\mathfrak{A})\in A\big),\quad A\in\mathcal{B}(M(D_1)),$$

if $a\neq 0$, converge weakly to some explicitly given probability measures on corresponding spaces as $T\to\infty$.

The aim of this note is to obtain a discrete limit theorem on the complex plane for the function $\zeta(s;\mathfrak{A})$. Let $h>0$ be a fixed number such that $\exp\{\frac{2\pi k}{h}\}$ is irrational for all $k\in\mathbb{Z}\setminus\{0\}$. Let, for $N\in\mathbb{N}\cup\{0\}$,

$$\mu_N(\dots)=\frac{1}{N+1}\#\{0\leqslant m\leqslant N\ :\dots\},$$

where in place of dots a condition satisfied by m is to be written. Define

$$\Omega=\prod_p\gamma_p,$$

where $\gamma_p=\gamma=\{s\in\mathbb{C}:|s|=1\}$ for all primes p. Then Ω is a compact topological Abelian group, therefore, on $(\Omega,\mathcal{B}(\Omega))$ the probability Haar measure

m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define a complex -valued random element $\zeta(s, \omega; \mathfrak{A})$ by

$$\zeta(s, \omega; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m \ \omega(m)}{m^s}, \quad \sigma > \frac{1}{2}.$$

Here

$$\omega(m) = \sum_{p^\alpha \| m} \omega^\alpha(p),$$

where $p^\alpha \parallel m$ means that $p^\alpha \mid m$, but $p^{\alpha+1} \nmid m$.

THEOREM. *Let $\sigma > \frac{1}{2}$. Then the probability measure*

$$P_N(A) \stackrel{def}{=} \mu_N\big(\zeta(\sigma + imh; \mathfrak{A}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution of the random variable $\zeta(s, \omega; \mathfrak{A})$ as $N \to \infty$.

We give a short proof of the theorem. Let $\sigma_1 > \frac{1}{2}$ be fixed, $v(m, n) = \exp\{-\big(\frac{m}{n}\big)^{\sigma_1}\}$ and

$$\zeta_{M,n}(s; \mathfrak{A}) = \sum_{m=1}^{M} \frac{a_m \ v(m, n)}{m^s}, \quad \zeta_{M,n}(s, \omega; \mathfrak{A}) = \sum_{m=1}^{M} \frac{a_m \ \omega(m) v(m, n)}{m^s}.$$

Define two probability measures

$$P_{N,M,n}(A) = \mu_N\big(\zeta_{M,n}(\sigma + imh; \mathfrak{A}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\widehat{P}_{N,M,n}(A) = \mu_N\big(\zeta_{M,n}(\sigma + imh, \omega; \mathfrak{A}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

LEMMA 1. *The probability measures $P_{N,M,n}$ and $\widehat{P}_{N,M,n}$ both converges weakly to the same probability measure $P_{M,n}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$.*

PROOF. The lemma is proved similarly to, for example, Theorems 5.2.1 and 5.2.2 of [4], where the case of the space of analytic functions is considered.

Now define

$$\zeta_n(s; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m \ v(m, n)}{m^s}$$

and

$$\zeta_n(s,\omega;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m\ \omega(m)v(m,n)}{m^s}.$$

Then it is not difficult to see that

$$\zeta_n(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m\ a_n(m)}{m^s},$$

where

$$a_n(m) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{\Gamma\big(\frac{s}{\sigma_1}\big) n^s\ \mathrm{d}s}{\sigma_1\ m^s} = \mathrm{O}_n(m^{-\sigma_1}).$$

Therefore, the series for $\zeta_n(s;\mathfrak{A})$ as well as for $\zeta_n(s,\omega;\mathfrak{A})$ converges absolutely for $\sigma > \frac{1}{2}$.

Since the function $\zeta(s;\mathfrak{A})$ is of finite order and, for $\sigma > \frac{1}{2}$,

$$\int\limits_0^T \big|\zeta(\sigma+it;\mathfrak{A})\big|^2 \mathrm{d}t = \mathrm{O}(T), \quad T \to \infty,$$

the contour integration shows that, for $\sigma > \frac{1}{2}$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \big|\zeta(s+imh;\mathfrak{A}) - \zeta_n(\sigma+imh;\mathfrak{A})\big| = 0. \tag{1}$$

An application of the classical Birkhoff–Khinchine ergodic theory allows to prove that, for $\sigma > \frac{1}{2}$,

$$\int\limits_0^T \big|\zeta(\sigma+it,\omega;\mathfrak{A})\big|^2 \mathrm{d}t = \mathrm{O}(T), \quad T \to \infty,$$

for almost all $\omega \in \Omega$. From this, similarly to (1),we deduce that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \big|\zeta(s+imh,\omega;\mathfrak{A}) - \zeta_n(\sigma+imh,\omega;\mathfrak{A})\big| = 0. \tag{2}$$

Now define else one pair of probability measures

$$P_{N,n}(A) = \mu_N\big(\zeta_n(\sigma+imh;\mathfrak{A}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\widehat{P}_{N,n}(A) = \mu_N\big(\zeta_n(\sigma+imh,\omega;\mathfrak{A}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

LEMMA 2. *Let $\sigma > \frac{1}{2}$. The probability measures $P_{N,n}$ and $\widehat{P}_{N,n}$ both converges weakly to the same probability measure P_n on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$.*

PROOF. We use Lemma 1, the tightness and hence the relative compactness of the family of probability measures $\{P_{M,n}\}$ for fixed n. Since

$$\lim_{M\to\infty} \zeta_{M,n}(s;\mathfrak{A}) = \zeta_n(s;\mathfrak{A})$$

uniformly on compact subsets of the half-plane $\{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$, we have that

$$\lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left|\zeta_n(s+imh;\mathfrak{A}) - \zeta_{M,n}(\sigma+imh;\mathfrak{A})\right| = 0.$$

This, the weak convergence of the measure $P_{N,M,n}$ to $P_{M,n}$ and the relatively compactness of $\{P_{M,n}\}$ together with Theorem 4.2 of [1] give the assertion of lemma.

Define

$$\widehat{P}_N(A) = \mu_N\big(\zeta(\sigma+imh,\omega;\mathfrak{A}) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

LEMMA 3. *Let $\sigma > \frac{1}{2}$. The probability measures P_N and $\widehat{P}_N$ both converges weakly to the same probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$.*

PROOF. We argue similarly to the proof of Lemma 2. By Lemma 2, the measure $P_{N,n}$ converges weakly to P_n. We prove the tightness of the family $\{P_n\}$, and from this we obtain its relative compactness. Using this, (1) and Theorem 4.2 of [1], we prove the weak convergence of P_n to some measure P. Now, using Lemma 2, relation (2) and repeating the above arguments for the measures $\widehat{P}_{N,n}$ and $\widehat{P}_N$, we find that the measure $\widehat{P}_N$ also converges weakly to P as $N \to \infty$.

PROOF OF THEOREM. It remains to show that the limit measure P in Lemma 3 coincides with the distribution of the random variable $\zeta(\sigma,\omega;\mathfrak{A})$.

Let $A \in \mathcal{B}(\mathbb{C})$ be a continuity set of the measure P. Then by Lemma 3, for $\sigma > \frac{1}{2}$,

$$\lim_{N\to\infty} \mu_N\big(\zeta(\sigma+imh,\omega;\mathfrak{A}) \in A\big) = P(A). \tag{3}$$

Now we fix the set A, and on $(\Omega, \mathcal{B}(\Omega), m_H)$ define the random variable θ by

$$\theta(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(\sigma,\omega;\mathfrak{A}) \in A, \\ 0 & \text{if} \quad \zeta(\sigma,\omega;\mathfrak{A}) \notin A. \end{cases}$$

Then

$$\mathbb{E}(\theta) = \int_{\Omega} \theta \, \mathrm{d}m_H = m_H(\omega \in \Omega : \zeta(\sigma, \omega; \mathfrak{A}) \in A) = (A),$$

where P_ζ is the distribution of $\zeta(\sigma, \omega; \mathfrak{A})$. This together with the Birkhoff–Khinchine theorem implies that

$$\lim_{N \to \infty} \mu_N\big(\zeta(\sigma + imh, \omega; \mathfrak{A}) \in A\big) = P_\zeta(A).$$

Hence and from (3) we have that

$$P(A) = P_\zeta(A)$$

for all continuity sets A of P. Therefore, $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(\mathbb{C})$. The theorem is proved.

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