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# THE JOINT LIMIT THEOREM FOR DISTRIBUTIONS OF INTEGER VALUED ADDITIVE FUNCTIONS

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**Abstract.** The necessary and sufficient conditions for the weak convergence of $\nu_x(n \leqslant x, f_x(n) < u)$ and $\mu_x(n \leqslant x, f_x(n) < u)$ to the same limit law are obtained. Here $\nu_x(n \leqslant x, f_x(n) < u)$ is the distribution of a set of strongly additive functions $f_x$ with respect to the usual frequency on the positive integers, and $\mu_x(n \leqslant x, f_x(n) < u)$ is the distribution of the same set with respect to the logarithmic frequency. The case when $f_x(p) \in \{0, 1\}$ for every prime $p$ is considered.

*Key words and phrases:* additive functions, frequency, logarithmic frequency, weak convergence.

*Mathematics Subject Classification:* 11K65.

## 1. Introduction

Let $f_x, x \geqslant 3$, be a set of strongly additive functions (s.a.f.), $\nu_x(\mathcal{A})$ denote the usual frequency of the subset $\mathcal{A}$ of positive integers, i. e.,

$$\nu_x(\mathcal{A}) = \frac{1}{[x]} \sum_{\substack{n \leqslant x \\ n \in \mathcal{A}}} 1.$$

Further let more $\mu_x(\mathcal{A})$ denote the logarithmic frequency of the same set $\mathcal{A}$, i. e.,

$$\mu_x(\mathcal{A}) = \left( \sum_{n \leqslant x} \frac{1}{n} \right)^{-1} \sum_{\substack{n \leqslant x \\ n \in \mathcal{A}}} \frac{1}{n}.$$

The limiting behavior of the distribution functions (d.f.)

$$\nu_x(n \leqslant x, f_x(n) - \alpha_1(x) < u)$$

and

$$\mu_x(n \leqslant x, f_x(n) - \alpha_2(x) < u)$$

with some centering functions $\alpha_1(x), \alpha_2(x)$ is the central problem of probabilistic number theory. Many classical results on the weak convergence of d.f. $\nu_x(n \leqslant x, f_x(n) - \alpha_1(x) < u)$ we can found in [1], [2], [3]. Limits of d.f. $\mu_x(n \leqslant x, f_x(n) - \alpha_2(x) < u)$ were investigated in [4], [5], [6], [11].

The case when strongly additive functions $f_x$ depend on $x$ in arbitrary way and $f_x(p) \in \{0, 1\}$ for every prime number $p$ was investigated by author of this paper in [8]–[10]. In [8] and [9], the necessary and sufficient conditions for the weak convergence of $\nu_x(n \leqslant x, f_x(n) < u)$ and $\mu_x(n \leqslant x, f_x(n) < u)$ were found. In [10] it was proved that d.f. $F$ for which

$$\nu_x(n \leqslant x, f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u)$$

and

$$\mu_x(n \leqslant x, f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u)$$

has the characteristic function

$$\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)} \prod_{p\in Q} \left(1 + \frac{\mathrm{e}^{it} - 1}{p}\right),$$

where $\lambda$ is some positive constant and $Q$ is a subset of prime numbers such that the series

$$\sum_{p\in Q} \frac{1}{p}$$

converges.

The main result of this paper is closely related to the last assertion.

THEOREM. *Let $f_x$ be a set of s.a.f., $x \geqslant 3$, and $f_x(p) \in \{0,1\}$ for every prime number $p$. The distributions $\nu_x(n \leqslant x, f_x(n) < u)$ and $\mu_x(n \leqslant x, f_x(n) < u)$ converge weakly to the same d.f. if and only if*

$$\lim_{y\to\infty} \limsup_{x\to\infty} \sum_{y<p\leqslant x}^{*} \frac{1}{p} = \lim_{y\to\infty} \liminf_{x\to\infty} \sum_{y<p\leqslant x}^{*} \frac{1}{p} = \lambda \ , \tag{1}$$

$$\lim_{x\to\infty} f_x(p) = \begin{cases} 1 & \text{if} \quad p \in Q, \\ 0 & \text{if} \quad p \notin Q, \end{cases} \tag{2}$$

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p\leqslant x}^{*} \frac{\log p}{p} = 0 \tag{3}$$

*for some positive constant $\lambda$ and some subset of prime numbers $Q$ such that*

$$\sum_{p\in Q} \frac{1}{p} < \infty.$$

*Here and in the sequel the star over the sign of the sum means that the summation is taken over all primes $p$ for which $f_x(p) = 1$.*

## 2. The proof of the theorem. Sufficiency

Suppose that the conditions (1), (2) and (3) are satisfied. In this part we will prove that

$$\nu_x(n \leqslant x, f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u) \tag{4}$$

and

$$\mu_x(n \leqslant x, f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u) \tag{5}$$

for some d.f. $F(u)$. Let $\xi_{x,p}$, $x \geqslant 3$, $p \leqslant x$, be a triangular array of independent in each row random variables for which

$$\mathbb{P}\left(\xi_{x,p} = f_x(p)\right) = \frac{1}{p},$$
$$\mathbb{P}\left(\xi_{x,p} = 0\right) = 1 - \frac{1}{p}.$$

Let $\varphi_x(t)$, $t \in \mathbb{R}$, be a characteristic function of the sum

$$\sum_{p\leqslant x} \xi_{x,p}.$$

It is evident that for every fixed $y \geqslant 2$

$$\varphi_x(t) = \prod_{p \leqslant x} \left(1 + \frac{e^{itf_x(p)} - 1}{p}\right) = \varphi_{x,y,1}(t)\varphi_{x,y,2}(t), \tag{6}$$

where

$$\varphi_{x,y,1}(t) = \prod_{p \leqslant y}{}^{*} \left(1 + \frac{e^{it} - 1}{p}\right),$$
$$\varphi_{x,y,2}(t) = \prod_{y < p \leqslant x}{}^{*} \left(1 + \frac{e^{it} - 1}{p}\right).$$

Here the star over the sign of product means that the product is expanded over primes for which $f_x(p) = 1$. According to the condition (2),

$$f_x(p) = \begin{cases} 1 & \text{if} \quad p \in Q, \\ 0 & \text{if} \quad p \notin Q, \end{cases} \tag{7}$$

for sufficiently large $x$ and all primes $p \leqslant y$. Hence, for such large $x$,

$$\begin{aligned} &\left|\varphi_{x,y,1}(t) - \prod_{p \in Q} \left(1 + \frac{e^{it} - 1}{p}\right)\right| \\ &\quad = \left|\prod_{\substack{p \leqslant y \\ p \in Q}} \left(1 + \frac{e^{it} - 1}{p}\right) - \prod_{p \in Q} \left(1 + \frac{e^{it} - 1}{p}\right)\right| \\ &\quad = \left|\sum_{\substack{p > y \\ p \in Q}} \frac{e^{it} - 1}{p} \prod_{\substack{q < p \\ q \in Q}} \left(1 + \frac{e^{it} - 1}{q}\right) \prod_{\substack{q > p \\ q \in Q}} 1\right| \\ &\quad \leqslant 2 \sum_{\substack{p > y \\ p \in Q}} \frac{1}{p}. \end{aligned}$$

Since the series

$$\sum_{p \in Q} \frac{1}{p}$$

converges, it follows from the obtained estimate that, for sufficiently large $x$,

$$\varphi_{x,y,1} = \prod_{p \in Q} \left(1 + \frac{e^{it} - 1}{p}\right) + \Delta_1(y), \tag{8}$$

where

$$\lim_{y\to\infty} \Delta_1(y) = 0.$$

On the other hand, the condition (1) implies

$$\begin{aligned}\varphi_{x,y,2}(t) &= \exp\Big\{ {\sum_{y<p\leqslant x}}^{*} \log\Big(1+\frac{e^{it}-1}{p}\Big)\Big\}\\ &= \exp\Big\{ {\sum_{y<p\leqslant x}}^{*} \frac{e^{it}-1}{p} + O\Big({\sum_{y<p\leqslant x}}^{*}\frac{1}{p^2}\Big)\Big\}\\ &= \exp\Big\{(e^{it}-1)(\lambda+\Delta(x,y)) + O\Big(\frac{1}{y}{\sum_{p\leqslant x}}^{*}\frac{1}{p}\Big)\Big\},\end{aligned}$$

for every pair $x, y$ $(3 \leqslant y \leqslant x)$ and every $t \in \mathbb{R}$, where

$$\lim_{y\to\infty}\limsup_{x\to\infty} \Delta(x,y) = \lim_{y\to\infty}\liminf_{x\to\infty} \Delta(x,y) = 0.$$

According to the equality (7), for sufficiently large $x$,

$$\begin{aligned}{\sum_{p\leqslant x}}^{*}\frac{1}{p} &= {\sum_{p\leqslant y}}^{*}\frac{1}{p} + {\sum_{y<p\leqslant x}}^{*}\frac{1}{p}\\ &= {\sum_{\substack{p\leqslant y\\ p\in Q}}}^{*}\frac{1}{p} + {\sum_{y<p\leqslant x}}^{*}\frac{1}{p}.\end{aligned}$$

Using the condition (1), we have that

$$\limsup_{x\to\infty} {\sum_{y<p\leqslant x}}^{*}\frac{1}{p} \leqslant {\sum_{p\in Q}}^{*}\frac{1}{p} + \lambda.$$

Thus

$${\sum_{p\leqslant x}}^{*}\frac{1}{p} \ll 1 \tag{9}$$

for all $x \geqslant 3$. This estimate implies

$$\varphi_{x,y,2}(t) = \exp\left\{(e^{it}-1)(\lambda+\Delta(x,y)) + \Delta_2(y)\right\}, \tag{10}$$

where

$$\lim_{y\to\infty} \Delta_2(y) = 0.$$

From equalities (6), (8) and (10) it follows that

$$\begin{aligned}
&\left|\varphi_x(t) - \mathrm{e}^{\lambda(\mathrm{e}^{it}-1)} \prod_{p\in Q}\left(1+\frac{\mathrm{e}^{it}-1}{p}\right)\right| \\
&\quad \leqslant \left|\varphi_{x,y,2}(t) - \mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}\right| + \Delta_1(y) \\
&\quad \leqslant (2|\Delta(x,y)| + \Delta_2(y)) \exp\{2|\Delta(x,y)| + \Delta_2(y)\} + \Delta_1(y)
\end{aligned}$$

for every pair $x, y$ $(3 \leqslant y \leqslant x)$ and every $t \in \mathbb{R}$. Therefore

$$\begin{aligned}
\mathrm{Re}\,(\varphi_x(t)) &= \mathrm{Re}\left(\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)} \prod_{p\in Q}\left(1+\frac{\mathrm{e}^{it}-1}{p}\right)\right) + \widehat{\Delta}(x,y), \\
\mathrm{Im}\,(\varphi_x(t)) &= \mathrm{Im}\left(\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)} \prod_{p\in Q}\left(1+\frac{\mathrm{e}^{it}-1}{p}\right)\right) + \widehat{\Delta}(x,y),
\end{aligned}$$

for above $x, y, t$, where

$$\lim_{y\to\infty} \limsup_{x\to\infty} \widehat{\Delta}(x,y) = \lim_{y\to\infty} \liminf_{x\to\infty} \widehat{\Delta}(x,y) = 0.$$

From the last relations we obtain

$$\lim_{x\to\infty} \varphi_x(t) = \exp\{\lambda(\mathrm{e}^{it}-1)\} \prod_{p\in Q}\left(1+\frac{\mathrm{e}^{it}-1}{p}\right).$$

The limit function is continuous at the point $t = 0$. Therefore, d.f. of the sum

$$\sum_{p\leqslant x} \xi_{x,p}$$

converges weakly to some d.f. with the characteristic function

$$\exp\left\{\lambda(\mathrm{e}^{it}-1)\right\} \prod_{p\in Q}\left(1+\frac{\mathrm{e}^{it}-1}{p}\right). \tag{11}$$

Let $F(u)$ be this d.f.

The Elliott estimate (see Theorem 12.15 in [2])

$$\begin{aligned}
&\varrho\Big(\nu_x\Big(n \leqslant x, f_x(n) < u\Big),\ \mathbb{P}\Big(\sum_{p\leqslant x} \xi_{x,p} < u\Big)\Big) \\
&\qquad \ll \Big(\sum_{\substack{x^\varepsilon < p \leqslant x \\ |f_x(p)|>v}} \frac{1}{p} + \frac{v}{\varepsilon} + \exp\Big\{-\frac{1}{8\varepsilon}\log\frac{1}{\varepsilon}\Big\} + x^{-\frac{1}{15}}\Big)
\end{aligned} \tag{12}$$

holds uniformly for all $v > 0$, $0 < \varepsilon < 1$, $x \geqslant 3$. Here $\varrho(G(u), H(u))$ denotes the Lévy distance between the d.f. $G(u)$ and $H(u)$.

For fixed $\varepsilon \in (0,1)$,

$$\begin{aligned} 0 \ &\leqslant \ \limsup_{x\to\infty} \sum_{\substack{x^\varepsilon < p \leqslant x \\ |f_x(p)| > \varepsilon^2}} \frac{1}{p} = \limsup_{x\to\infty} \sideset{}{^*}\sum_{x^\varepsilon < p \leqslant x} \frac{1}{p} \\ &\leqslant \ \limsup_{x\to\infty} \frac{1}{\varepsilon \log x} \sideset{}{^*}\sum_{p \leqslant x} \frac{\log p}{p}. \end{aligned} \tag{13}$$

The last two estimates and the condition (3) imply the equality

$$\lim_{x\to\infty} \varrho\Big(\nu_x\big(n \leqslant x, f_x(n) < u\big),\ \mathbb{P}\Big(\sum_{p \leqslant x} \xi_{x,p} < u\Big)\Big) = 0.$$

Hence the relation (4) holds for d.f. with the characteristic function (11).

The weak convergence (4) holds if and only if the limit

$$\lim_{x\to\infty} \sideset{}{^*}\sum_{p_1 \leqslant x} \sideset{}{^*}\sum_{\substack{p_2 \leqslant x \\ p_1 \neq p_2}} \cdots \sideset{}{^*}\sum_{\substack{p_l \leqslant x \\ p_l \neq p_1, p_2, \dots p_{l-1} \\ p_1 p_2 \dots p_l \leqslant x}} \frac{1}{p_1 p_2 \dots p_l} = g_l \tag{14}$$

exists for every fixed positive integer $l$ (see, [8]).

Having in mind the condition (3), the inequality (9) and estimate

$$\begin{aligned} &\sideset{}{^*}\sum_{p_1 \leqslant x} \sideset{}{^*}\sum_{\substack{p_2 \leqslant x \\ p_1 \neq p_2}} \cdots \sideset{}{^*}\sum_{\substack{p_l \leqslant x \\ p_l \neq p_1, p_2, \dots p_{l-1} \\ p_1 p_2 \dots p_l \leqslant x}} \frac{\log p_1 p_2 \dots p_l}{p_1 p_2 \dots p_l \log x} \\ &\leqslant \sum_{j=1}^{l} \frac{1}{\log x} \sideset{}{^*}\sum_{p_1 \leqslant x} \sideset{}{^*}\sum_{p_2 \leqslant x} \cdots \sideset{}{^*}\sum_{p_l \leqslant x} \frac{\log p_j}{p_1 p_2 \dots p_l} \\ &= \frac{l}{\log x} \sideset{}{^*}\sum_{p \leqslant x} \frac{\log p}{p} \Big( \sideset{}{^*}\sum_{p \leqslant x} \frac{1}{p} \Big)^{l-1}, \end{aligned}$$

we obtain that

$$\lim_{x\to\infty} \sideset{}{^*}\sum_{p_1 \leqslant x} \sideset{}{^*}\sum_{\substack{p_2 \leqslant x \\ p_1 \neq p_2}} \cdots \sideset{}{^*}\sum_{\substack{p_l \leqslant x \\ p_l \neq p_1, p_2, \dots p_{l-1} \\ p_1 p_2 \dots p_l \leqslant x}} \frac{1}{p_1 p_2 \dots p_l} \Big( 1 - \frac{\log p_1 p_2 \dots p_l}{\log x} \Big) = g_l$$

for every fixed positive integer $l$. Now the second desired relation (5) follows from the weak convergence criteria for the $\mu_x(n \leqslant x,\ f_x(n) < u)$ (see, for example, [9]).

### 3. The proof of the theorem. Necessity

Suppose now that the relations (4) and (5) are satisfied for some d.f. $F(u)$. We will derive from this that the conditions (1)–(3) hold.

Using the weak convergence criteria for $\nu_x(n \leqslant x, f_x(n) < u)$ (see, [8]), we have that the limit (14) exists for every fixed positive number $l$. Moreover, the limit law $F(u)$ has the characteristic function

$$1 + \sum_{l=1}^{\infty} \frac{g_l}{l!}(\mathrm{e}^{it} - 1)^l, \quad t \in \mathbb{R}.$$

Similarly, from the weak convergence criteria for $\mu_x(n \leqslant x, f_x(n) < u)$ (see, [9]) it follows that the limit

$$\lim_{x\to\infty} \sideset{}{^*}\sum_{p_1 \leqslant x} \sideset{}{^*}\sum_{\substack{p_2 \leqslant x \\ p_1 \neq p_2}} \dots \sideset{}{^*}\sum_{\substack{p_l \leqslant x \\ p_l \neq p_1, p_2, \dots p_{l-1} \\ p_1 p_2 \dots p_l \leqslant x}} \frac{1}{p_1 p_2 \dots p_l}\left(1 - \frac{\log p_1 p_2 \dots p_l}{\log x}\right) = h_l$$

also exists for every fixed positive integer $l$. Moreover, in this case the limit law $F(u)$ has the characteristic function

$$1 + \sum_{l=1}^{\infty} \frac{h_l}{l!}(\mathrm{e}^{it} - 1)^l, \quad t \in \mathbb{R}.$$

Since the limit law in (4) and (5) is the same, we have that $g_l = h_l$ for every positive integer $l$. In particular, $g_1 = h_1$. Therefore,

$$\lim_{x\to\infty} \sideset{}{^*}\sum_{p \leqslant x} \frac{\log p}{p} = \lim_{x\to\infty}\left( \sideset{}{^*}\sum_{p \leqslant x} \frac{1}{p} - \sideset{}{^*}\sum_{p \leqslant x} \frac{1}{p}\left(1 - \frac{\log p}{\log x}\right)\right) = g_1 - h_1 = 0,$$

and the condition (3) holds.

From this condition and estimates (12) and (13) it follows that

$$\mathbb{P}\left(\sum_{p \leqslant x} \xi_{x,p} < u\right) \underset{x\to\infty}{\Rightarrow} F(u), \tag{15}$$

where $\xi_{x,p}$, $x \geqslant 3$, $p \leqslant x$, is the triangular array of random variables defined in the first part of our proof, and $F(u)$ is d.f. from relations (4) and (5). According to a result from [10], the characteristic function of d.f. $F(u)$ is of the form

$$\mathrm{e}^{\lambda(\mathrm{e}^{it} - 1)} \prod_{p \in Q}\left(1 + \frac{\mathrm{e}^{it} - 1}{p}\right),$$

where $\lambda$ is some positive constant and $Q$ is some subset of prime numbers such that the series

$$\sum_{p\in Q}\frac{1}{p} \tag{16}$$

converges. Hence the limit law in (15) is of the form

$$\Pi_\lambda + \sum_{p\in Q}\eta_p,$$

where $\Pi_\lambda$ is the Poisson random variable with parameter $\lambda$, and $\eta_p, p\in Q$, are independent random variables for which

$$\mathbb{P}\left(\eta_p = 1\right) = \frac{1}{p},$$
$$\mathbb{P}\left(\eta_p = 0\right) = 1 - \frac{1}{p}.$$

Since

$$\lim_{x\to\infty}\sum_{p\leqslant x}{}^{*}\frac{1}{p} = g_1,$$

we have

$$\sum_{p\leqslant x}{}^{*}\frac{1}{p} \ll 1$$

for all $x \geqslant 3$. The last estimate, the convergence of the series (16) and the classical Rosenthal inequality (see, for example, Chapter III of [7]) imply

$$\begin{aligned}
\varphi_{x,l} &= \mathbb{E}\sum_{p\leqslant x}\xi_{x,p}\Big(\sum_{p\leqslant x}\xi_{x,p} - 1\Big)...\Big(\sum_{p\leqslant x}\xi_{x,p} - l + 1\Big)\\
&\ll_l 1,\\
\varphi_l &= \mathbb{E}\Big(\Pi_\lambda + \sum_{p\in Q}\eta_p\Big)\Big(\Pi_\lambda + \sum_{p\in Q}\eta_p - 1\Big)...\Big(\Pi_\lambda + \sum_{p\in Q}\eta_p - l + 1\Big)\\
&\ll_l 1
\end{aligned}$$

for $x \geqslant 3$, $l \in \mathbb{N}$. From the obtained inequalities and the weak convergence (15) it follows that

$$\lim_{x\to\infty}\varphi_{x,l} = \varphi_l$$

for every fixed positive integer $l$. Hence

$$\lim_{x\to\infty} \sum\nolimits^{*}_{p\leqslant x} \frac{1}{p} = \lambda + \sum_{p\in Q} \frac{1}{p},$$

$$\lim_{x\to\infty} \sum\nolimits^{*}_{p_1\leqslant x} \sum\nolimits^{*}_{p_1<p_2\leqslant x} \frac{1}{p_1p_2} = \frac{\lambda^2}{2!} + \lambda \sum_{p\in Q} \frac{1}{p} + \sum_{p_1\in Q} \sum_{\substack{p_2\in Q\\ p_1<p_2}} \frac{1}{p_1p_2},$$

and, for $l \in \mathbb{N}$,

$$\lim_{x\to\infty} \sum\nolimits^{*}_{p_1\leqslant x} \sum\nolimits^{*}_{p_1<p_2\leqslant x} \dots \sum\nolimits^{*}_{p_{l-1}<p_l\leqslant x} \frac{1}{p_1p_2\dots p_l}$$

$$= \sum_{j=0}^{l} \frac{\lambda^j}{j!} \sum_{p_1\in Q} \sum_{\substack{p_2\in Q\\ p_1<p_2}} \dots \sum_{\substack{p_j\in Q\\ p_{j-1}<p_j\leqslant x}} \frac{1}{p_1p_2\dots p_j}.$$

After some calculations from the last equalities we find that

$$\lim_{x\to\infty} \sum\nolimits^{*}_{p\leqslant x} \frac{1}{p} = \lambda + \sum_{p\in Q} \frac{1}{p} \tag{17}$$

and, for every fixed positive integer $l$,

$$\lim_{x\to\infty} \sum\nolimits^{*}_{p\leqslant x} \frac{1}{p^l} = \sum_{p\in Q} \frac{1}{p^l}. \tag{18}$$

Let $\{\widehat{x}\}$ be an arbitrary unbounded increasing sequence of real numbers. Since $f_x(p) \in \{0, 1\}$, there is an unbounded increasing subsequence $\{\widetilde{x}\}$ such that the limit

$$\lim_{\widetilde{x}\to\infty} f_{\widetilde{x}}(p) = r_p$$

exists for every fixed prime number $p$. Let $\widetilde{Q}$ be the set of prime numbers for which $r_p = 1$. From (18) we have that, for all $3 \leqslant y \leqslant \widetilde{x}$,

$$\sum\nolimits^{*}_{p\leqslant y} \frac{1}{p^2} + \sum\nolimits^{*}_{y<p\leqslant \widetilde{x}} \frac{1}{p^2} = \sum_{p\in Q} \frac{1}{p^2} + \Delta_3(\widetilde{x}),$$

where

$$\lim_{\widetilde{x}\to\infty} \Delta_3(\widetilde{x}) = 0.$$

Since, for $p \in \widetilde{Q}$,

$$\lim_{\widetilde{x}\to\infty} f_{\widetilde{x}}(p) = 1,$$

we obtain that

$$\sum_{\substack{p\in\widetilde{Q}\\ p\leqslant y}}\frac{1}{p^2}+\sum_{y<p\leqslant\widetilde{x}}^{*}\frac{1}{p^2}=\sum_{p\in Q}\frac{1}{p^2}+\Delta_3(\widetilde{x})$$

for every fixed $y$ and sufficiently large $\widetilde{x}$. Therefore, for such $y$ and $\widetilde{x}$,

$$\sum_{\substack{p\in\widetilde{Q}\\ p\leqslant y}}\frac{1}{p^2}=\sum_{p\in Q}\frac{1}{p^2}+\Delta_3(\widetilde{x})+\Delta_4(y),$$

where

$$\lim_{y\to\infty}\Delta_4(y)=0.$$

It follows from the last equality that

$$\sum_{p\in\widetilde{Q}}\frac{1}{p^2}=\sum_{p\in Q}\frac{1}{p^2}.$$

Similarly to the case $l=2$, we can derive from (18) that

$$\sum_{p\in\widetilde{Q}}\frac{1}{p^l}=\sum_{p\in Q}\frac{1}{p^l}$$

for every fixed $l\geqslant 2$. According to the auxiliary lemma (see the next section), the subsets of prime numbers $\widetilde{Q}$ and $Q$ coincide. Hence, for a fixed prime number, every unbounded increasing sequence $\{\widehat{x}\}$ has a subsequence $\{\widetilde{x}\}$ such that

$$\lim_{\widetilde{x}\to\infty}f_{\widetilde{x}}(p)=\begin{cases}1 & \text{if}\quad p\in Q,\\ 0 & \text{if}\quad p\notin Q.\end{cases}$$

This implies the relation (2).

It remains to prove the relation (1) of our theorem. According to the proved relation (2), we have that the equation (7) holds for fixed $y\geqslant 3$ and sufficiently large $x$. Hence, for such large $x$, it follows from (17) that

$$\sum_{\substack{p\in Q\\ p\leqslant y}}+\sum_{y<p\leqslant x}^{*}\frac{1}{p}=\lambda+\sum_{p\in Q}\frac{1}{p}+\Delta_5(x),$$

where

$$\lim_{x\to\infty}\Delta_5(x)=0.$$

Therefore,

$$\limsup_{x\to\infty} \sum_{y<p\leqslant x}^{*} \frac{1}{p} = \liminf_{x\to\infty} \sum_{y<p\leqslant x}^{*} \frac{1}{p} = \lambda + \sum_{p\in Q} \frac{1}{p} - \sum_{\substack{p\in Q\\ p\leqslant y}} \frac{1}{p}.$$

It is easily seen that the last equation and the condition (16) provide the desired relation (1).

## 4. Auxiliary lemma

In this section we prove one lemma which we have used in the proof of necessity of our theorem.

LEMMA. *Let $a_i$ and $b_i$ be two decreasing sequences of real numbers from interval $(0,1)$. If*

$$\sum_{i=1}^{\infty} a_i^l = \sum_{i=1}^{\infty} b_i^l \tag{19}$$

*for every fixed positive integer $l \geqslant 2$, then the sequences $a_i$ and $b_i$ coincide.*

PROOF. It is sufficient to prove that $a_1 = b_1$. Suppose, on the contrary, that $a_1 > b_1$. In this case, $a_i < a_1$ and $b_i < b_1 < a_1$ for every $i \geqslant 2$. The equality (19) shows that

$$1 + \sum_{i=2}^{\infty} \left(\frac{a_i}{a_1}\right)^l = \sum_{i=1}^{\infty} \left(\frac{b_i}{a_1}\right)^l.$$

Hence, for every fixed positive integer $N$,

$$1 + \sum_{i=2}^{N} \left(\frac{a_i}{a_1}\right)^l + \sum_{i=N+1}^{\infty} \left(\frac{a_i}{a_1}\right)^l = \sum_{i=1}^{N} \left(\frac{b_i}{a_1}\right)^l + \sum_{i=N+1}^{\infty} \left(\frac{b_i}{a_1}\right)^l.$$

This gives

$$1 + \sum_{i=2}^{N} \left(\frac{a_i}{a_1}\right)^l + \mathrm{O}\left(\frac{1}{a_1^2} \sum_{i=N+1}^{\infty} {a_i}^2\right) = \sum_{i=1}^{N} \left(\frac{b_i}{a_1}\right)^l + \mathrm{O}\left(\frac{1}{a_1^2} \sum_{i=N+1}^{\infty} {b_i}^2\right).$$

Letting at first $l$ then $N$ to infinity, we obtain from the last equality $1 = 0$. This shows that our premise $a_1 > b_1$ is wrong. Similarly, we can show that inequality $a_1 < b_1$ gives a contradiction, too. Hence it remains only the case $a_1 = b_1$. This completes the proof.

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