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ONE LIMIT THEOREM FOR DISTRIBUTIONS OF ADDITIVE FUNCTIONS

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Abstract. The criteria for the weak convergence of $\nu_x^\alpha(f_x(n) < u)$ to the Poisson limit law is obtained. Here $\nu_x^\alpha(f_x(n) < u)$ denote the distribution function of a set of strongly additive functions f_x with respect to the frequency

$$\nu_x^\alpha(\mathcal{A}_x) = \left\{ \sum_{m \leqslant x} \frac{1}{m^\alpha} \right\}^{-1} \sum_{\substack{m \leqslant x \\ m \in \mathcal{A}_x}} \frac{1}{m^\alpha}, \quad \alpha \in [0, 1].$$

The case when $f_x(p) \in \{0, 1\}$ for every prime p is considered.

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1. Introduction

Suppose that a subset of natural numbers $\mathcal{A}_x$ may depend on $x \geqslant 2$ and other parameters. Then

$$\nu_x(\mathcal{A}_x) = \frac{1}{[x]} \sum_{\substack{m \leqslant x \\ m \in \mathcal{A}_x}} 1$$

denotes a frequency, and

$$\mu_x\left(\mathcal{A}_x\right) = \left\{ \sum_{m \leqslant x} \frac{1}{m} \right\}^{-1} \sum_{\substack{m \leqslant x \\ m \in \mathcal{A}_x}} \frac{1}{m}$$

denotes a logarithmic frequency of the set $\mathcal{A}_x$.

Let, further, f_x be a set of integer–valued strongly additive functions (s.a.f.). It is evident that $\nu_x(f_x(m) < u)$ and $\mu_x(f_x(m) < u)$ are distribution functions. Because s.a.f. f_x are integer valued, possible limit laws for $\nu_x(f_x(m) < u)$ and $\mu_x(f_x(m) < u)$ must be also integer valued. The best–known such a law is the Poisson law with distribution function

$$\Pi_\lambda(u) = \mathrm{e}^{-\lambda} \sum_{k=0}^{[u]} \frac{\lambda^k}{k!} \,,$$

where λ is some positive parameter. In [2], see also [3], the following assertion was obtained on the weak convergence of $\nu_x(f_x(m) < u)$ to the Poisson law.

THEOREM 1. *Let f_x be a set of s.a.f., and $f_x(p) \in \{0,1\}$ for every prime number p. The weak convergence*

$$\nu_x(f_x(m) < u) \underset{x\to\infty}{\Longrightarrow} \Pi_\lambda(u)$$

holds if and only if the next three conditions are satisfied:

$$\lim_{x\to\infty} \max_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p} = 0, \tag{1}$$

$$\lim_{x\to\infty} \sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p} = \lambda, \tag{2}$$

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{\log p}{p} = 0, \tag{3}$$

In [4] a similar criteria was obtained for the weak convergence of distribution functions $\mu_x(f_x(m) < u)$ to the Poisson law.

THEOREM 2. *Let f_x be a set of s.a.f., and $f_x(p) \in \{0,1\}$ for every prime number p. For the weak convergence*

$$\mu_x(f_x(m) < u) \underset{x\to\infty}{\Longrightarrow} \Pi_\lambda(u),$$

the same three conditions (1)–(3) *are necessary and sufficient.*

It is evident that, for every fixed real number $\alpha \in [0,1]$, the equality

$$\nu_x^\alpha(\mathcal{A}_x) = \Big\{ \sum_{m\leqslant x} \frac{1}{m^\alpha} \Big\}^{-1} \sum_{\substack{m\leqslant x\\ m\in\mathcal{A}_x}} \frac{1}{m^\alpha}$$

defines a new frequency on the set of positive integers numbers. Hence, for every set of additive functions f_x,

$$\nu_x^{\alpha}(f_x(m)<u)=\left\{\sum_{m\leqslant x}\frac{1}{m^{\alpha}}\right\}^{-1}\sum_{\substack{m\leqslant x\\ f_x(m)<u}}\frac{1}{m^{\alpha}}$$

is a new set of distribution functions. In the present paper, we consider the weak convergence of the above distribution functions to the Poisson limit law. The next assertion is the main in the present work.

THEOREM 3. *Let $\alpha\in[0,1]$ be a fixed real number, f_x be a set of s.a.f., and $f_x(p)\in\{0,1\}$ for every prime number p. The distribution functions $\nu_x^{\alpha}(f_x(m)<u)$ converge weakly to the Poisson law with parameter λ if and only if the above three conditions* (1)–(3) *are satisfied.*

2. Proof of Theorem 3. Necessity.

If $\alpha=0$ or $\alpha=1$, the statement of our theorem immediately follows from Theorems 1 and 2. Therefore, we can assert that $\alpha\in(0,1)$. In this section, we suppose that

$$\nu_x^{\alpha}(f_x(m)<u)\underset{x\to\infty}{\Longrightarrow}\Pi_{\lambda}(u), \tag{4}$$

and we derive step by step conditions (1)–(3).

I. At first, we prove that

$$\sum_{\substack{p\leqslant x\\ f_x(p)=1}}\frac{1}{p}\leqslant c_{1,\lambda} \tag{5}$$

for all $x\geqslant 2$ with some quantity $c_{1,\lambda}$ dependent on the parameter λ. The next lemma (see [1]) is favorable for our purpose.

LEMMA 1. *Let $h(m)$ be an arbitrary real–valued additive function which may be dependent on x. Then there exists some absolute constant c_2 such that*

$$\sup_{y}\sum_{\substack{m\leqslant x\\ h(m)=y}}1\leqslant c_2x\Bigg(\sum_{\substack{p\leqslant x\\ h(p)\neq 0}}\frac{1}{p}\Bigg)^{-\frac{1}{2}}$$

for all $x\geqslant 2$.

From this lemma, we have that

$$\begin{aligned}\sum_{\substack{m\leqslant x\\ f_x(m)=0}} \frac{1}{m^\alpha} &= \frac{1}{x^\alpha}\sum_{\substack{m\leqslant x\\ f_x(m)=0}} 1+\alpha\int_1^x\Big(\sum_{\substack{m\leqslant u\\ f_x(m)=0}} 1\Big)\frac{\mathrm{d}u}{u^{\alpha+1}}\\ &\leqslant c_2x^{1-\alpha}\Big(\sum_{\substack{p\leqslant x\\ f_x(p)=1}}\frac{1}{p}\Big)^{-\frac{1}{2}}\\ &\quad +\alpha\int_1^{\sqrt{x}} u^{-\alpha}\mathrm{d}u+\alpha c_2\int_{\sqrt{x}}^x u^{-\alpha}\Big(\sum_{\substack{p\leqslant u\\ f_x(p)=1}}\frac{1}{p}\Big)^{-\frac{1}{2}}\mathrm{d}u\\ &\leqslant \frac{c_2x^{1-\alpha}}{1-\alpha}\Big(\sum_{\substack{p\leqslant \sqrt{x}\\ f_x(p)=1}}\frac{1}{p}\Big)^{-\frac{1}{2}}+\mathrm{O}_\alpha\big(x^{\frac{1-\alpha}{2}}\big).\end{aligned}$$

Hence,

$$\nu_x^\alpha(f_x(m)=0)\leqslant c_{3,\alpha}\Big(\sum_{\substack{p\leqslant \sqrt{x}\\ f_x(p)=1}}\frac{1}{p}\Big)^{-\frac{1}{2}}+\mathrm{o}_\alpha(1),$$

and

$$\sum_{\substack{p\leqslant \sqrt{x}\\ f_x(p)=1}}\frac{1}{p}\leqslant c_{3,\alpha}^2\left(\nu_x^\alpha(f_(m)=0)+\mathrm{o}_\alpha(1)\right)^{-2}.$$

According to the condition (4),

$$\lim_{x\to\infty}\nu_x^\alpha(f_x(m)=0)=\mathrm{e}^{-\lambda}.$$

The desired estimate (5) now follows from two last relations.

II. For positive integer number l, denote

$$\varphi_{x,l}=\frac{1-\alpha}{x^{1-\alpha}}\sum_{m\leqslant x}\frac{1}{m^\alpha}f_x(m)(f_x(m)-1)...(f_x(m)-l+1).$$

In this subsection, we prove that

$$\lim_{x\to\infty}\varphi_{x,l}=\lambda^l \tag{6}$$

for every fixed l.

Throughout of this paper, the star $*$ over the sign of sum means that the summation is extended over primes for which $f_x(p)=1$. Using the obtained estimate (5), for fixed positive integer l, we have

$$\begin{aligned}
\varphi_{x,l} &= \frac{1-\alpha}{x^{1-\alpha}} \sum_{m\leqslant x} \frac{1}{m^\alpha} \sideset{}{^*}\sum_{p_1|m} \sideset{}{^*}\sum_{\substack{p_2|m\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_l|m\\ p_l\neq p_1,p_2,\dots,p_{l-1}}} 1\\
&= \sideset{}{^*}\sum_{p_1\leqslant x} \sideset{}{^*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}}} \frac{1-\alpha}{x^{1-\alpha}} \frac{1}{p_1^\alpha p_2^\alpha \dots p_l^\alpha} \sum_{n\leqslant \frac{x}{p_1p_2\dots p_l}} \frac{1}{n^\alpha}\\
&\leqslant \left(\sideset{}{^*}\sum_{p\leqslant x} \frac{1}{p} \right)^l \leqslant c_{1,\lambda}^l.
\end{aligned}$$

Since, for every fixed $\alpha \in (0,1)$,

$$\sum_{m\leqslant x} \frac{1}{m^\alpha} = \frac{x^{1-\alpha}}{1-\alpha} + \mathrm{O}_\alpha(1), \tag{7}$$

choosing $K > l+2$, we get from (4) that

$$\begin{aligned}
\varphi_{x,l} &= \sum_{k=l}^{K} k(k-1)\dots(k-l+1) \frac{1-\alpha}{x^{1-\alpha}} \sum_{\substack{m\leqslant x\\ f_x(m)=k}} \frac{1}{m^\alpha}\\
&\quad + \frac{1-\alpha}{x^{1-\alpha}} \sum_{\substack{m\leqslant x\\ f_x(m)>K}} \frac{1}{m^\alpha} f_x(m)(f_x(m)-1)\dots(f_x(m)-l+1) \frac{f_x(m)-l}{f_x(m)-l}\\
&= \sum_{k=l}^{K} k(k-1)\dots(k-l+1) \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} + \mathrm{o}_{K,\alpha}(1) + \mathrm{O}\left(\frac{\varphi_{l+1,x}}{K-l} \right)\\
&= \lambda^l + \mathrm{o}_{K,\alpha}(1) + \mathrm{O}\left(\frac{\lambda^{K+1}}{(K+1-l)!} \right) + \mathrm{O}\left(\frac{c_{1,\lambda}^{l+1}}{K-l} \right).
\end{aligned}$$

We can obtain the desired relation (6) from the last equality taking at first the limit when x tends to infinity, and then when K tends to infinity.

III. Using (7) once again, we get

$$\begin{aligned}
\varphi_{x,l} &= \sideset{}{^*}\sum_{p_1\leqslant x} \sideset{}{^*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}}} \frac{1-\alpha}{x^{1-\alpha}} \frac{1}{p_1^\alpha p_2^\alpha \dots p_l^\alpha} \sum_{n\leqslant \frac{x}{p_1p_2\dots p_l}} \frac{1}{n^\alpha}\\
&= \sideset{}{^*}\sum_{p_1\leqslant x} \sideset{}{^*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}} \cdots \sideset{}{^*}\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\dots p_l\leqslant x}} \frac{1}{p_1p_2\dots p_l}
\end{aligned}$$

$$+\mathrm{O}_\alpha\Bigg(\frac{1}{x^{1-\alpha}}\sum_{p_1\leqslant x}^{*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*}\cdots\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\dots p_l\leqslant x}}^{*}\frac{1}{p_1^\alpha p_2^\alpha\dots p_l^\alpha}\Bigg).$$

The following Landau's inequality (see, for example, [5]) is useful to estimate the remainder term of $\varphi_{x,l}$.

Lemma 2. *Let $\omega(m)$ denote the number of distinct prime divisors of positive integer m. Then, for $x\geqslant 3$ and every fixed positive integer l,*

$$\nu_x(\omega(m)=l)\ll_l\frac{(\log\log x)^{l-1}}{\log x}.$$

According to this lemma, we have for sufficiently large x

$$\begin{aligned}
&\frac{1}{x^{1-\alpha}}\sum_{p_1\leqslant x}^{*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*}\cdots\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\dots p_l\leqslant x}}^{*}\frac{1}{p_1^\alpha p_2^\alpha\dots p_l^\alpha}\\
&=\frac{1}{l!x^{1-\alpha}}\sum_{\substack{n\leqslant x,n\ \text{is square free}\\ p|n\Rightarrow f_x(p)=1\\ \omega(n)=l}}\frac{1}{n^\alpha}\leqslant\frac{1}{l!x^{1-\alpha}}\sum_{\substack{n\leqslant x\\ \omega(n)=l}}\frac{1}{n^\alpha}\\
&\ll_l\frac{1}{x^{1-\alpha}}\Bigg(\sum_{n\leqslant\sqrt{x}}\frac{1}{n^\alpha}+\sum_{\substack{\sqrt{x}\leqslant n\leqslant x\\ \omega(n)=l}}\frac{1}{n^\alpha}\Bigg)\\
&\leqslant\Bigg(\frac{1}{x^{1-\alpha}}\frac{x^{\frac{1-\alpha}{2}}}{1-\alpha}+\frac{1}{x}\sum_{\substack{\sqrt{x}\leqslant n\leqslant x\\ \omega(n)=l}}1+\frac{\alpha}{x^{1-\alpha}}\int_{\sqrt{x}}^{x}\sum_{\substack{\sqrt{x}\leqslant n\leqslant u\\ \omega(n)=l}}1\,\frac{\mathrm{d}u}{u^{\alpha+1}}\Bigg)\\
&\ll_l\Bigg(\frac{1}{(1-\alpha)x^{\frac{1-\alpha}{2}}}+\frac{(\log\log x)^{l-1}}{\log x}+\frac{2\alpha}{x^{1-\alpha}}\frac{(\log\log x)^{l-1}}{\log x}\int_{\sqrt{x}}^{x}\frac{\mathrm{d}u}{u^\alpha}\Bigg)\\
&\ll_{\alpha,l}\frac{(\log\log x)^{l-1}}{\log x}.
\end{aligned}$$

Consequently,

$$\varphi_{x,l}=\sum_{p_1\leqslant x}^{*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*}\cdots\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\dots p_l\leqslant x}}^{*}\frac{1}{p_1p_2\dots p_l}+\mathrm{O}_{\alpha,l}\Bigg(\frac{(\log\log x)^{l-1}}{\log x}\Bigg).\qquad(8)$$

The last equality and relation (6) imply

$$\lim_{x\to\infty}\sideset{}{^*}\sum_{p_1\leqslant x}\sideset{}{^*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}\dots\sideset{}{^*}\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\dots p_l\leqslant x}}\frac{1}{p_1p_2\dots p_l}=\lambda^l \tag{9}$$

for every fixed positive integer l.

IV. In this subsection, we derive from the relation (9) conditions (1)–(3) of our theorem. Taking in (9) $l=1$ and $l=2$, we get

$$\lim_{x\to\infty}\sideset{}{^*}\sum_{p\leqslant x}\frac{1}{p}=\lambda,$$

$$\lim_{x\to\infty}\sideset{}{^*}\sum_{p_1\leqslant x}\sideset{}{^*}\sum_{\substack{p_2\leqslant \frac{x}{p_1}\\ p_2\neq p_1}}\frac{1}{p_1p_2}=\lambda^2.$$

Condition (2) follows immediately from the first obtained equality. Further, the obtained relations imply

$$\lim_{x\to\infty}\left\{\left(\sideset{}{^*}\sum_{p\leqslant x}\frac{1}{p}\right)^2-\sideset{}{^*}\sum_{p_1\leqslant x}\sideset{}{^*}\sum_{\substack{p_2\leqslant \frac{x}{p_1}\\ p_2\neq p_1}}\frac{1}{p_1p_2}\right\}=0.$$

Since

$$\left(\sideset{}{^*}\sum_{p\leqslant x}\frac{1}{p}\right)^2-\sideset{}{^*}\sum_{p_1\leqslant x}\sideset{}{^*}\sum_{\substack{p_2\leqslant \frac{x}{p_1}\\ p_2\neq p_1}}\frac{1}{p_1p_2}\geqslant\sideset{}{^*}\sum_{p\leqslant x}\frac{1}{p^2},$$

the last relations show that

$$\lim_{x\to\infty}f_x(p)=0$$

for every fixed prime number p, and condition (1) follows.

For every pair of indices $1\leqslant i<j\leqslant l$,

$$\begin{aligned}\sideset{}{^*}\sum_{p_1\leqslant x}\dots\sideset{}{^*}\sum_{p_i\leqslant x}\dots\sideset{}{^*}\sum_{\substack{p_j\leqslant x\\ p_j=p_i}}\dots\sideset{}{^*}\sum_{\substack{p_l\leqslant x\\ p_1p_2\dots p_l\leqslant x}}\frac{1}{p_1\dots p_i\dots p_j\dots p_l}\\ \leqslant\max_{\substack{p\leqslant x\\ f_x(p)=1}}\frac{1}{p}\left(\sideset{}{^*}\sum_{p\leqslant x}\frac{1}{p}\right)^{l-1}.\end{aligned} \tag{10}$$

Hence, the proved equalities (1), (2) and (9) imply that, for every fixed positive integer l,

$$\lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \sum_{p_2\leqslant x}^{*} \dots \sum_{\substack{p_l\leqslant x \\ p_1p_2\dots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\dots p_l} = \lambda^l. \tag{11}$$

Hence, for such l,

$$\lim_{x\to\infty} \left(\left(\sum_{p_1\leqslant x}^{*} \frac{1}{p} \right)^l - \sum_{p_1\leqslant x}^{*} \sum_{p_2\leqslant x}^{*} \dots \sum_{\substack{p_l\leqslant x \\ p_1p_2\dots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\dots p_l} \right) = 0.$$

It is evident that

$$\begin{aligned} &\left(\sum_{p\leqslant x}^{*} \frac{1}{p} \right)^l - \sum_{p_1\leqslant x}^{*} \sum_{p_2\leqslant x}^{*} \dots \sum_{\substack{p_l\leqslant x \\ p_1p_2\dots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\dots p_l} \\ &= \sum_{p_1\leqslant x}^{*} \sum_{p_2\leqslant x}^{*} \dots \sum_{\substack{p_l\leqslant x \\ p_1p_2\dots p_l> x}}^{*} \frac{1}{p_1p_2\dots p_l} \\ &\geqslant \left(\sum_{x^{\frac{1}{l}}<p\leqslant x}^{*} \frac{1}{p} \right)^l. \end{aligned}$$

Therefore, for each fixed positive integer l,

$$\lim_{x\to\infty} \sum_{x^{\frac{1}{l}}<p\leqslant x}^{*} \frac{1}{p} = 0, \tag{12}$$

and condition (3) follows.

3. Proof of Theorem 3. Sufficiency.

In this section, we derive the weak convergence (4) from conditions (1)–(3). Condition (3) is equivalent to the relation (12). Hence, the following estimate

$$0 \leqslant \left(\sum_{p\leqslant x}^{*} \frac{1}{p} \right)^l - \sum_{p_1\leqslant x}^{*} \sum_{p_2\leqslant x}^{*} \dots \sum_{\substack{p_l\leqslant x \\ p_1p_2\dots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\dots p_l}$$

$$= \sum_{p_1\leqslant x}^{*}\sum_{p_2\leqslant x}^{*}\dots\sum_{\substack{p_l\leqslant x\\ p_1p_2\dots p_l>x}}^{*}\frac{1}{p_1p_2\dots p_l}$$
$$\ll_l \sum_{x^{\frac{1}{l}}<p\leqslant x}^{*}\frac{1}{p}\bigg(\sum_{p\leqslant x}^{*}\frac{1}{p}\bigg)^{l-1}$$

and condition (2) imply the relation (11). According to (10) and conditions (1), (2), we have that, for every pair of indices $1\leqslant i<j\leqslant l$,

$$\lim_{x\to\infty}\sum_{p_1\leqslant x}^{*}\dots\sum_{p_i\leqslant x}^{*}\dots\sum_{\substack{p_j\leqslant x\\ p_j=p_i}}^{*}\dots\sum_{\substack{p_l\leqslant x\\ p_1p_2\dots p_l\leqslant x}}^{*}\frac{1}{p_1\dots p_i\dots p_j\dots p_l}=0.$$

Therefore, the relation (11) implies the equality (9), and, finally, the equality (8) imply (6), i.e.,

$$\lim_{x\to\infty}\varphi_{x,l}=\lambda^l$$

for every fixed positive integer l.

Let, for $x\geqslant 2$ and $t\in\mathbb{R}$,

$$\psi_x(t)=\frac{1-\alpha}{x^{1-\alpha}}\sum_{m\leqslant x}\frac{\mathrm{e}^{itf_x(m)}}{m^\alpha}.$$

If r and n are positive integers, then

$$\bigg|\mathrm{e}^{itr}-1-\sum_{j=1}^{n-1}\binom{r}{j}(\mathrm{e}^{it}-1)^j\bigg|\leqslant\binom{r}{n}\big|\mathrm{e}^{it}-1\big|^n.$$

Thus, for positive integer L,

$$\bigg|\psi_x(t)-1-\sum_{l=1}^{L}\frac{(\mathrm{e}^{it}-1)^l}{l!}\varphi_{x,l}\bigg|\leqslant\frac{|\mathrm{e}^{it}-1|^{L+1}\varphi_{L+1,x}}{(L+1)!}.$$

According to (6), we obtain

$$\bigg|\psi_x(t)-1-\sum_{l=1}^{L}\frac{(\mathrm{e}^{it}-1)^l}{l!}\lambda^l\bigg|\leqslant \mathrm{o}_L(1)+\mathrm{O}\bigg(\frac{c_{4,\lambda}^{L+1}}{(L+1)!}\bigg).$$

From the last equality it follows that

$$\lim_{x\to\infty}\bigg(\sum_{m\leqslant x}\frac{1}{m^\alpha}\bigg)^{-1}\sum_{m\leqslant x}\frac{\mathrm{e}^{itf_x(m)}}{m^\alpha}=\mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}$$

for each real number t. Hence, the distribution functions $\nu_x^\alpha(f_x(m)<u)$ converge weakly to the Poisson law with parameter λ. The theorem is proved.

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