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BOUNDARY VALUE PROBLEMS TO SECOND ORDER COMPLEX PARTIAL DIFFERENTIAL EQUATIONS IN A RING DOMAIN

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Abstract. The Green, Neumann, Poisson and Bergman kernel functions are obtained for a circular ring domain. The Dirichlet and Neumann problem for the Poisson equation, and the Dirichlet and Schwarz problem for the inhomogeneous Bitsadze equation are solved. All formulas are given in explicit form.

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1. Introduction

This paper is a continuation of [10], where basic boundary value problems for analytic functions and for the inhomogeneous complex Cauchy-Riemann equation are solved in a concentric circular ring domain.

In order to treat boundary value problems for second order complex partial differential equations, special kernel functions have to be constructed. The most important among them is the Green function. It gives the solution to the Dirichlet problem for harmonic functions and relates to the Schwarz operator (see, e.g. [2], [4], [7], [9]). The Poisson and Bergman kernels can be obtained from the known Green function.

It is well known that not any domain on the complex plane has the Green function (see, e.g. [4], [7]). But, in the case of ring domains, the Green function exists and can be expressed in explicit form. In [1], the Green function for a ring $\{z \in \mathbb{C} : 0 < r < |z| < 1\}$ is given via Weierstrass ζ-function. In [6], the procedure of constructing the Green function for a ring $\{z \in \mathbb{C} : 0 < \frac{1}{r} < |z| < 1\}$ is described. In the same way we construct the Green function for $\{z \in \mathbb{C} : 0 < r < |z| < 1\}$.

The second important kernel function is the Neumann function. The Neumann function for a circular ring $\{z \in \mathbb{C} : 0 < r < |z| < 1\}$ is also given here.

Both the Green and Neumann functions are certain fundamental solutions to the Laplace operator. They are used for solving boundary value problems via integral representation formulas for solutions (see, e.g. [2], [8]).

In this paper, we use the Green and Neumann functions to treat the Dirichlet and Neumann problem for the Poisson equation. Besides of these problems other boundary conditions can be prescribed for second order complex partial differential equations (Laplace, Poisson, Bitsadze). In the case of the unit disk, solutions of the problems are obtained in [2]. The main idea exploited there is to decompose a problem in a system of corresponding boundary value problems to first order partial differential equations and use known formulas to solve them.

The solution to the Dirichlet problem for the inhomogeneous Bitsadze equation in a circular ring domain are found here by the above said decomposition method of [2]. It should be emphasized that in principle this method can be used to solve other boundary value problems, but it is appeared not to be always effective. In the case of a circular ring domain, such procedure leads to highly complicated analysis and it is difficult to get the final result (for instance, it is happened when solving the Schwarz problem for the Poisson equation or the Neumann problem for the inhomogeneous Bitsadze equation). Thus the solution of certain problems needs to apply another approaches and is finished not yet.

One of the possible ways to handle such problems is to generalize the corresponding formulas for the unit disk. By using this technics the Schwarz problem for the inhomogeneous Bitsadze equation is solved here.

The paper is organized as follows. In Section 2, the general integral representation formulas of Cauchy-Pompeiu type are given. In Section 3, kernel functions for a circular ring domain are constructed. On their base the integral representation formulas via Green and Neumann functions for a certain class of functions are proved. Section 4 is devoted to the solution of some boundary value problems to the second order complex partial differential equations in a circular ring domain.

2. Integral representations

To treat boundary value problems, integral representation formulas are used. In the case of first order equations, the main tools for solving boundary value problems are Gauss theorem and Cauchy-Pompeiu representation formulas.

Integral representation formulas for solutions to first order complex partial differential equations can be used to get such formulas for higher order equations via iterations. The idea is based on the iterating procedure of the main theorem of calculus (see, e.g. [2], [5]).

Let us introduce the complex partial differential operators of first and second order. Let $z = x + iy$, $x, y \in \mathbb{R}$.

Two partial differential operators of first order, ∂_z and $\partial_{\overline{z}}$, are defined by

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{1}$$

There are two basic second order partial differential operators, the Laplace operator $\partial_z\partial_{\overline{z}}$ and the Bitsadze operator $\partial_{\overline{z}}^2$. The third one, ∂_z^2, is the complex conjugate of the Bitsadze operator.

Let D be a regular domain of the complex plane $\mathbb{C}$, i.e., bounded with smooth boundary ∂D.

Gauss theorem (complex form; [2]). *Let D be a regular domain, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, then*

$$\int_D w_{\overline{z}}(z)\mathrm{d}x\mathrm{d}y = \frac{1}{2i}\int_{\partial D} w(z)\mathrm{d}z \tag{2}$$

and

$$\int_D w_z(z)\mathrm{d}x\mathrm{d}y = -\frac{1}{2i}\int_{\partial D} w(z)\mathrm{d}\overline{z}. \tag{3}$$

From the Gauss theorem the following representation formulas can be derived.

Cauchy-Pompeiu representations ([2]). *Let D be a regular domain of $\mathbb{C}$, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, $\zeta = \xi + i\eta$. Then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta - z} \tag{4}$$

and

$$w(z) = -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_{D} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \tag{5}$$

hold for all $z \in D$.

Applying the iteration procedure to the integral representations (4), (5) leads to higher order integral representations of the Cauchy-Pompeiu type.

THEOREM 1 ([4]). *Let* $D \subset \mathbb{C}$ *be a regular domain,* $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$. *Then the representation formulas*

$$\begin{aligned} w(z) \;=\; & \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta \\ & + \frac{1}{\pi}\int\limits_{D} w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta \end{aligned} \tag{6}$$

and

$$\begin{aligned} w(z) \;=\; & \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\zeta}}(\zeta)\log|\zeta - z|^2 d\overline{\zeta} \\ & + \frac{1}{\pi}\int\limits_{D} w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2 d\xi d\eta \end{aligned} \tag{7}$$

hold for all $z \in D$.

REMARK. There are dual formulas to (6), (7). To deduce them, the complex conjugation of (6), (7) is taken and w is replaced by $\overline{w}$. Then the formulas

$$\begin{aligned} w(z) \;=\; & -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} + \frac{1}{2\pi i}\int\limits_{\partial D} w_{\zeta}(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}d\overline{\zeta} \\ & + \frac{1}{\pi}\int\limits_{D} w_{\zeta\zeta}(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}d\xi d\eta \end{aligned} \tag{8}$$

and

$$\begin{aligned} w(z) \;=\; & -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{2\pi i}\int\limits_{\partial D} w_{\zeta}(\zeta)\log|\zeta - z|^2 d\zeta \\ & + \frac{1}{\pi}\int\limits_{D} w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2 d\xi d\eta \end{aligned} \tag{9}$$

are valid.

3. Kernel functions for a circular ring domain

Definition 1 ([4]). *A real-valued function* $G(z,\zeta)=\frac{1}{2}G_1(z,\zeta)$ *in a regular domain* $D\subset\mathbb{C}$ *is called the Green function of* D, *or, more exactly, the Green function of* D *for the Laplace operator, if it possesses, for any fixed* $\zeta\in D$, *as a function of* z *the following properties:*

1°. $G(z,\zeta)$ *is harmonic in* $D\setminus\{\zeta\}$;

2°. $G(z,\zeta)+\log|\zeta-z|$ *is harmonic in* D;

3°. $\lim\limits_{z\to\partial D}G(z,\zeta)=0.$

The Green function has the additional properties [4]:

4°. $G(z,\zeta)>0$;

5°. $G(z,\zeta)=G(\zeta,z)$;

6°. *It is uniquely determined by* **1°** − **3°**.

Not any domain in the complex plane has a Green function. The existence of the Green function for a given domain $D\subset\mathbb{C}$ can be proved in the case when the Dirichlet problem for harmonic function is solvable for D (see, e.g. [4]).

The procedure of constructing the Green function for a circular ring $\{z\in\mathbb{C}:0<\frac{1}{r}<|z|<r\}$ is described in [6]. In our case, dealing with the circular ring $R=\{z\in\mathbb{C}:0<r<|z|<1\}$, it has essential distinctions given below.

Any harmonic function in domain can be represented as the real part of an analytic function. The desired Green function is sought in the form

$$-\frac{1}{2}\mathrm{Re}\,\{\log f(z)\}=-\frac{1}{2}\log|f(z)|^2$$

with the function $f(z)$ being analytic in R, having a simple zero in, say c, with modulus 1 on the boundary. Without loss of generality, c is assumed to be real and positive. In order to find enough function-theoretical properties to construct $f(z)$ explicitly, it is extended beyond the two circles $|z|=1$, $|z|=r$ by setting $f(z)=\overline{f(z^*)}$, where $z^*=\frac{1}{\bar z}$ for $|z|>1$, or $z^*=\frac{r^2}{\bar z}$ for $|z|<r$.

If z approaches a boundary point, the corresponding point z^* does approach it also. The function $f(z)$ may be considered as a real function taking complex conjugate values at complex conjugate points. This implies that $f(z)f\left(\frac{1}{z}\right)$ or $f(z)f\left(\frac{r^2}{z}\right)$ approaches the positive real value $|f(z_0)|^2$, whenever z tends to $z_0\in\partial R$. On the other hand, $f(z)$ has modulus 1 on the boundary. Thus, the functional equations

$$f(z)f\left(\frac{1}{z}\right)=1 \tag{10}$$

and

$$f(z)f\Big(\frac{r^2}{z}\Big)=1 \tag{11}$$

hold (identically) for all z.

Since c is a simple zero of $f(z)$, applying (10) and (11) successively shows that the function $f(z)$ has simple zeroes at the points

$$c, cr^2, \frac{c}{r^2}, cr^4, ..., cr^{2k}, \frac{c}{r^{2k}}, ...,$$

and simple poles at the points

$$\frac{1}{c}, \frac{r^2}{c}, \frac{1}{cr^2}, \frac{r^4}{c}, \frac{1}{cr^4}, ..., \frac{r^{2k}}{c}, \frac{1}{r^{2k}c}, \dots\ .$$

Thus, it coincides in its zeroes and poles with the function

$$F(z)=\frac{c-z}{1-cz}\prod_{k=1}^{\infty}\frac{(z-r^{2k}c)(c-r^{2k}z)}{(cz-r^{2k})(1-r^{2k}cz)}. \tag{12}$$

Moreover, the relation

$$F(z)F\Big(\frac{1}{z}\Big)=1$$

is valid.

From the other side,

$$\begin{aligned} F(z)F\Big(\frac{r^2}{z}\Big) &= \frac{1}{1-cz}\Bigg[\prod_{k=1}^{\infty}\frac{1-r^{2(k-1)}cz}{1-r^{2k}cz}\Bigg]\frac{1}{z-r^2c}\Bigg[\prod_{k=1}^{\infty}\frac{z-r^{2k}c}{z-r^{2(k+1)}c}\Bigg] \\ &\quad\times(cz-r^2)\Bigg[\prod_{k=1}^{\infty}\frac{cz-r^{2(k+1)}}{cz-r^{2k}}\Bigg](c-z)\Bigg[\prod_{k=1}^{\infty}\frac{c-r^{2k}z}{c-r^{2(k-1)}z}\Bigg] \\ &= \lim_{k\to\infty}\frac{(cz-r^{2(k+1)})(c-r^{2k}z)}{(1-r^{2k}cz)(z-r^{2(k+1)}c)}=c^2. \end{aligned}$$

So the function $F(z)$ does not satisfy (11).

Let us define

$$f(z)=az^bF(z),$$

$a, b\in\mathbb{R}$. Equations (10), (11) determine a system of equations for a and b. Its solutions are $a=\pm1$, $b=-\frac{\log c}{\log r}$. We choose $a=1$. Then

$$f(z)=z^{-\frac{\log c}{\log r}}F(z).$$

Now instead of the real $c \in R$ one can take an arbitrary $\zeta \in R$ (see [6]). Then the Green function for R is represented by $G(z,\zeta) = -\log \|f(z)\| = \frac{1}{2}\, G_1(z,\zeta)$ with

$$G_1(z,\zeta) = \frac{\log|z|^2 \log|\zeta|^2}{\log r^2} - \log\Big|\frac{\zeta - z}{1 - z\overline{\zeta}} \prod_{k=1}^{\infty} \frac{(z - r^{2k}\zeta)(\zeta - r^{2k}z)}{(z\overline{\zeta} - r^{2k})(1 - r^{2k}\overline{z}\zeta)}\Big|^2. \quad (13)$$

The Green function defined in (13) satisfies properties 1°, 2° and 5°. Its boundary behavior is

$$\begin{aligned}
&\lim_{\substack{|z|\to 1,\\ z\in R}} G_1(z,\zeta)\\
&\quad = -\lim_{\substack{|z|\to 1,\\ z\in R}} \log\left|\frac{\overline{z}\zeta - |z|^2}{\overline{z}^2(1-\overline{z}\zeta)} \prod_{k=1}^{\infty} \frac{(|z|^2 - r^{2k}\overline{z}\zeta)(\overline{z}\zeta - r^{2k}|z|^2)}{(z\overline{\zeta} - r^{2k})(1 - r^{2k}z\overline{\zeta})}\right|^2 = 0,\\
&\lim_{\substack{|z|\to r,\\ z\in R}} G_1(z,\zeta)\\
&\quad = \log|\zeta|^2 - \lim_{n\to\infty} \lim_{\substack{|z|\to r,\\ z\in R}} \log\left|(\zeta - z)\prod_{k=1}^{n} \frac{|z|^2 - r^{2k}\overline{z}\zeta}{1 - r^{2k}z\overline{\zeta}}\right.\\
&\quad \times \left.\frac{1}{1-\overline{z}\zeta}\prod_{k=1}^{n}\frac{\zeta - r^{2k}z}{|z|^2\overline{\zeta} - r^{2k}z}\right|^2\\
&\quad = \log|\zeta|^2 - \lim_{n\to\infty} \log\left|\frac{\zeta - r^{2n}z}{1 - r^{2n}\overline{z}\zeta}\right|^2 = 0.
\end{aligned}$$

The Green function is a fundamental solution to the Laplace equation. It is used for solving boundary value problems via integral representation formulas for solutions.

The next theorem gives the representation formula via Green function and will be used to solve the Dirichlet boundary value problem for the Poisson equation.

To formulate the theorem, the definition of the outward normal derivative at the boundary of a circular domain is needed. The direction of this derivative on a circle $|z-a| = r$ coincides with the direction of the radius vector. In the case of the circular ring R, the normal derivative is given by the formulas

$$\partial_{\nu_z} = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & |z| = 1,\\ -\frac{z}{r}\partial_z - \frac{\overline{z}}{r}\partial_{\overline{z}}, & |z| = r. \end{cases}$$

Theorem 2. *Any $w \in C^2(R;\mathbb{C}) \cap C^1(\overline{R};\mathbb{C})$ can be represented by*

$$w(z) = -\frac{1}{4\pi i}\int\limits_{\partial R} |\zeta|\partial_{\nu_\zeta} G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{R} w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta. \quad (14)$$

Proof. Let $z \in R$, and $\varepsilon > 0$ be so small that

$$\overline{B_\varepsilon(z)} \subset R, \quad B_\varepsilon(z) = \{\zeta \in \mathbb{C} : |\zeta - z| < \varepsilon\}.$$

Let us denote $R_\varepsilon = R \setminus \overline{B_\varepsilon(z)}$, and consider

$$\begin{aligned}
&\frac{1}{\pi}\int\limits_{R_\varepsilon} w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta \\
&= \frac{1}{2\pi}\int\limits_{R_\varepsilon}\Big[\partial_{\overline{\zeta}}\left[w_\zeta(\zeta)G_1(z,\zeta)\right] + \partial_\zeta\left[w_{\overline{\zeta}}(\zeta)G_1(z,\zeta)\right] \\
&\quad - w_\zeta(\zeta)G_{1\overline{\zeta}}(z,\zeta) - w_{\overline{\zeta}}(\zeta)G_{1\zeta}(z,\zeta)\Big]d\xi d\eta \\
&= \frac{1}{4\pi i}\int\limits_{\partial R_\varepsilon} G_1(z,\zeta)[w_\zeta(\zeta)d\zeta - w_{\overline{\zeta}}(\zeta)d\overline{\zeta}\,] \\
&\quad - \frac{1}{2\pi}\int\limits_{R_\varepsilon}\Big[\partial_\zeta[w(\zeta)G_{1\overline{\zeta}}(z,\zeta)] + \partial_{\overline{\zeta}}[w(\zeta)G_{1\zeta}(z,\zeta)] \\
&\quad - 2\partial_{\zeta\overline{\zeta}}G_1(z,\zeta)w(\zeta)\Big]d\xi d\eta \\
&= -\frac{1}{4\pi i}\int\limits_{|\zeta - z| = \varepsilon} G_1(z,\zeta)[w_\zeta(\zeta)d\zeta - w_{\overline{\zeta}}(\zeta)d\overline{\zeta}\,] \\
&\quad - \frac{1}{4\pi i}\int\limits_{\partial R_\varepsilon} w(\zeta)\Big[G_{1\zeta}(z,\zeta)d\zeta - G_{1\overline{\zeta}}(z,\zeta)d\overline{\zeta}\Big] \\
&= -\frac{1}{4\pi i}\int\limits_{|\zeta - z| = \varepsilon} G_1(z,\zeta)\Big[(\zeta - z)w_\zeta(\zeta) + (\overline{\zeta - z})w_{\overline{\zeta}}(\zeta)\Big]\frac{d\zeta}{\zeta - z} \\
&\quad - \frac{1}{4\pi i}\int\limits_{\partial R} w(\zeta)|\zeta|\Big(\frac{\zeta}{|\zeta|}\partial_\zeta + \frac{\overline{\zeta}}{|\zeta|}\partial_{\overline{\zeta}}\Big)G_1(z,\zeta)\frac{d\zeta}{\zeta} \\
&\quad + \frac{1}{4\pi i}\int\limits_{|\zeta - z| = \varepsilon}\Big[(\zeta - z)G_{1\zeta}(z,\zeta) + (\overline{\zeta - z})G_{1\overline{\zeta}}(z,\zeta)\Big]w(\zeta)\frac{d\zeta}{\zeta - z}.
\end{aligned}$$

Introducing polar coordinates $\zeta = z + \varepsilon e^{i\theta}$ leads to

$$\frac{1}{4\pi i}\int\limits_{|\zeta - z| = \varepsilon} G_1(z,\zeta)\Big[(\zeta - z)w_\zeta(\zeta) - (\overline{\zeta - z})w_{\overline{\zeta}}(\zeta)\Big]\frac{d\zeta}{\zeta - z}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \varepsilon [\mathrm{e}^{i\theta} w_\zeta(\zeta) + \mathrm{e}^{-i\theta} w_{\overline{\zeta}}(\zeta)] G_1(z,\zeta) \mathrm{d}\theta$$

which tends to zero as $\varepsilon \to 0$.

From property 2° of the Green function, the representation

$$G_1(z,\zeta) = -\log|\zeta - z|^2 + h_1(z,\zeta)$$

holds, with $h_1(z,\zeta)$ being harmonic in R as a function of z for any $\zeta \in R$. Using this formula and polar coordinates gives

$$\frac{1}{4\pi i} \int_{|\zeta - z| = \varepsilon} \Big[(\zeta - z) G_{1\zeta}(z,\zeta) + (\overline{\zeta - z}) G_{1\overline{\zeta}}(z,\zeta)\Big] w(\zeta) \frac{\mathrm{d}\zeta}{\zeta - z}$$
$$= \frac{1}{4\pi} \int_0^{2\pi} \Big[\varepsilon [\mathrm{e}^{i\theta}\partial_\zeta + \mathrm{e}^{-i\theta}\partial_{\overline{\zeta}}]\, h_1(z, z + \varepsilon \mathrm{e}^{i\theta}) - 2\Big] w(z + \varepsilon \mathrm{e}^{i\theta}) \mathrm{d}\theta,$$

and this tends to $-w(z)$ as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{R_\varepsilon} w_{\zeta\overline{\zeta}}(\zeta) G_1(z,\zeta) \mathrm{d}\xi \mathrm{d}\eta = -\frac{1}{4\pi i} \int_{\partial R} |\zeta|\, \partial_{\nu_\zeta} G_1(z,\zeta) w(\zeta) \frac{\mathrm{d}\zeta}{\zeta} - w(\zeta).$$

This proves (14).

Definition 2 ([4]). *A real-valued function $N(z,\zeta) = \frac{1}{2} N_1(z,\zeta)$ in a regular domain $D \subset \mathbb{C}$ is called the Neumann function of D (for the Laplace operator) if it has as a function of z the following properties:*

1°. *$N(z,\zeta)$ is harmonic in $D \setminus \{\zeta\}$;*

2°. *$N(z,\zeta) + \log|\zeta - z|$ is harmonic in D;*

3°. *$\partial_{\nu_z} N(z,\zeta)$ is constant on any boundary component of D for any $\zeta \in D$.*

Remark. The Neumann function is not uniquely defined by $1^\circ - 3^\circ$.

Lemma 1. *Let D be a regular domain in the complex plane $\mathbb{C}$, its boundary consist of $(n+1)$ components, $\partial D = \bigcup_{j=0}^{n} \Gamma_j$, and all $\Gamma_j, j = \overline{1,n}$ being inside of Γ_0, where $n \in \mathbb{N} \cup \{0\}$ is fixed. The Neumann function $N(z,\zeta)$ of D is uniquely defined by asking*

$$\frac{1}{2\pi} \int_{\Gamma_0} N(z,\zeta) \mathrm{d}s_z = 0, \quad \textit{for all} \quad \zeta \in D, \quad \mathrm{d}s_z = |\mathrm{d}z|. \tag{15}$$

Proof. Let $N_{(1)}(z,\zeta)$, $N_{(2)}(z,\zeta)$ be two Neumann functions of D. Consider

$$U(z) := N_{(1)}(z,\zeta) - N_{(2)}(z,\zeta)$$

for any fixed $\zeta \in D$. Using properties $1° - 3°$ of the Neumann function, it follows that $U(z)$ is harmonic in D and $\partial_{\nu_z} U(z) = 0$.

Applying the real Green formula

$$\begin{aligned} &\int_D (u(x,y) \triangle v(x,y) + <\operatorname{grad} u(x,y), \operatorname{grad} v(x,y)>) \mathrm{d}x\mathrm{d}y \\ &\quad = \int_{\partial D} u(x,y) \partial_\nu v(x,y) \mathrm{d}s \end{aligned}$$

for $u(x,y) \equiv v(x,y) \equiv U(x,y)$, where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, we obtain

$$\int_D \left| \operatorname{grad} U(x,y) \right|^2 \mathrm{d}x\mathrm{d}y = 0.$$

Hence

$$U(x,y) \equiv \text{const.}$$

Formula (15) immediately provides this constant to be zero.

Proposition 1. *The function $N(z,\zeta) = \frac{1}{2} N_1(z,\zeta)$ with*

$$\begin{aligned} &N_1(z,\zeta) \\ &\quad = -\log \Big| (\zeta - z)(1 - z\overline{\zeta}) \\ &\qquad \times \prod_{k=1}^{\infty} \frac{(z - r^{2k}\zeta)(z\overline{\zeta} - r^{2k})(\zeta - r^{2k}z)(1 - r^{2k}z\overline{\zeta})}{|z|^2|\zeta|^2} \Big|^2 \end{aligned} \tag{16}$$

represents the Neumann function for the ring domain R.

PROOF. Properties $1^\circ - 3^\circ$ of the Neumann function have to be checked.

The function $N(z,\zeta)$ is harmonic and satisfies 2°. The boundary behavior is observed from

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} zN_{1\,z}(z,\zeta) + \overline{z}N_{1\,\overline{z}}(z,\zeta), & |z|=1, \\ -\frac{z}{r}N_{1\,z}(z,\zeta) - \frac{\overline{z}}{r}N_{1\,\overline{z}}(z,\zeta), & |z|=r, \end{cases}$$

where

$$\begin{aligned} &zN_{1\,z}(z,\zeta) \\ &\quad = \frac{z}{\zeta - z} + \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} \\ &\quad - \sum_{k=1}^{\infty}\Big(\frac{z}{z - r^{2k}\zeta} + \frac{z\overline{\zeta}}{z\overline{\zeta} - r^{2k}} - \frac{r^{2k}z}{\zeta - r^{2k}z} - \frac{r^{2k}z\overline{\zeta}}{1 - r^{2k}z\overline{\zeta}} - 2\Big). \end{aligned} \tag{17}$$

Then, for all $\zeta \in R$,

$$\partial_{\nu_z} N_1(z,\zeta) = -2 \quad \text{on} \quad |z|=1, \tag{18}$$

$$\partial_{\nu_z} N_1(z,\zeta) = 0 \quad \text{on} \quad |z|=r. \tag{19}$$

The normalization condition (15) reads for R

$$\frac{1}{2\pi i}\int\limits_{|z|=1} N_1(z,\zeta)\frac{\mathrm{d}z}{z} = 0. \tag{20}$$

In order to prove it, consider

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{|z|=1} N_1(z,\zeta)\frac{\mathrm{d}z}{z} \\ &\quad = -\frac{1}{2\pi i}\int\limits_{|z|=1}\Big[\log|\zeta - z|^2 + \log|1 - z\overline{\zeta}|^2 \\ &\quad + \sum_{k=1}^{\infty}\log\Big|\frac{(z - r^{2k}\zeta)(z\overline{\zeta} - r^{2k})(\zeta - r^{2k}z)(1 - r^{2k}z\overline{\zeta})}{|z|^2|\zeta|^2}\Big|^2\Big]\frac{\mathrm{d}z}{z} \\ &\quad = -\frac{1}{2\pi i}\int\limits_{|z|=1}\Big[2\log|1 - z\overline{\zeta}|^2 \\ &\quad + \sum_{k=1}^{\infty}\Big(\log|z - r^{2k}\zeta|^2 + \log|z\overline{\zeta} - r^{2k}|^2 + \log|\zeta - r^{2k}z|^2 \\ &\quad + \log|1 - r^{2k}z\overline{\zeta}|^2 - 2\log|\zeta|^2\Big)\Big]\frac{\mathrm{d}z}{z} \end{aligned}$$

with

$$\frac{1}{2\pi i}\int\limits_{|z|=1}\log|1-z\overline{\zeta}|^2\frac{\mathrm{d}z}{z}=2\mathrm{Re}\left[\frac{1}{2\pi i}\int\limits_{|z|=1}\log(1-z\overline{\zeta})\frac{\mathrm{d}z}{z}\right]=0,$$
$$\frac{1}{2\pi i}\int\limits_{|z|=1}\log|z-r^{2k}\zeta|^2\frac{\mathrm{d}z}{z}=\frac{1}{2\pi i}\int\limits_{|z|=1}\log|1-r^{2k}z\overline{\zeta}|^2\frac{\mathrm{d}z}{z}=0,$$
$$\frac{1}{2\pi i}\int\limits_{|z|=1}\log|z\overline{\zeta}-r^{2k}|^2\frac{\mathrm{d}z}{z}=\frac{1}{2\pi i}\int\limits_{|z|=1}\log|\zeta-r^{2k}z|^2\frac{\mathrm{d}z}{z}=\log|\zeta|^2.$$

This completes the proof.

THEOREM 3. *Any $w\in C^2(R;\mathbb{C})\cap C^1(\overline{R};\mathbb{C})$ can be represented by the formula*

$$\begin{aligned} w(z) &= -\frac{1}{4\pi i}\int\limits_{\partial R}|\zeta|\partial_{\nu_\zeta}N_1(z,\zeta)w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\frac{1}{4\pi i}\int\limits_{\partial R}|\zeta|\,\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\ &\quad -\frac{1}{\pi}\int\limits_{R}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \end{aligned}\tag{21}$$

PROOF. Let $z\in R$, and $\varepsilon>0$ be so small that

$$\overline{B_\varepsilon(z)}\subset R,\quad B_\varepsilon(z)=\{\zeta\in\mathbb{C}:|\zeta-z|<\varepsilon\}.$$

We denote $R_\varepsilon=R\setminus\overline{B_\varepsilon(z)}$, and consider

$$\begin{aligned} &\frac{1}{\pi}\int\limits_{R_\varepsilon}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta\\ &=\frac{1}{2\pi}\int\limits_{R_\varepsilon}\Big[\partial_{\overline{\zeta}}\left[w_\zeta(\zeta)N_1(z,\zeta)\right]+\partial_\zeta\left[w_{\overline{\zeta}}(\zeta)N_1(z,\zeta)\right]\\ &\quad -w_\zeta(\zeta)N_{1\overline{\zeta}}(z,\zeta)-w_{\overline{\zeta}}(\zeta)N_{1\zeta}(z,\zeta)\Big]\mathrm{d}\xi\mathrm{d}\eta\\ &=\frac{1}{4\pi i}\int\limits_{\partial R_\varepsilon}N_1(z,\zeta)[w_\zeta\mathrm{d}\zeta-w_{\overline{\zeta}}\mathrm{d}\overline{\zeta}]\\ &\quad -\frac{1}{2\pi}\int\limits_{R_\varepsilon}\Big[\partial_\zeta[w(\zeta)N_{1\overline{\zeta}}(z,\zeta)]+\partial_{\overline{\zeta}}[w(\zeta)N_{1\zeta}(z,\zeta)]\\ &\quad -2\partial_{\zeta\overline{\zeta}}N_1(z,\zeta)w(\zeta)\Big]\mathrm{d}\xi\mathrm{d}\eta\\ &=\frac{1}{4\pi i}\int\limits_{\partial R}N_1(z,\zeta)[w_\zeta(\zeta)\mathrm{d}\zeta-w_{\overline{\zeta}}(\zeta)\mathrm{d}\overline{\zeta}\,] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\varepsilon} N_1(z,\zeta)[w_\zeta(\zeta)\mathrm{d}\zeta - w_{\overline{\zeta}}(\zeta)\mathrm{d}\overline{\zeta}\,] \\
&-\frac{1}{4\pi i}\int\limits_{\partial R_\varepsilon} w(\zeta)\Big[N_{1\zeta}(z,\zeta)\mathrm{d}\zeta - N_{1\overline{\zeta}}(z,\zeta)\mathrm{d}\overline{\zeta}\Big] \\
= &\frac{1}{4\pi i}\int\limits_{\partial R} N_1(z,\zeta)|\zeta|\Big(\frac{\zeta}{|\zeta|}\partial_\zeta + \frac{\overline{\zeta}}{|\zeta|}\partial_{\overline{\zeta}}\Big)w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&-\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\varepsilon} N_1(z,\zeta)\Big[(\zeta-z)w_\zeta(\zeta) + (\overline{\zeta-z})w_{\overline{\zeta}}(\zeta)\Big]\frac{\mathrm{d}\zeta}{\zeta-z} \\
&-\frac{1}{4\pi i}\int\limits_{\partial R} w(\zeta)|\zeta|\Big(\frac{\zeta}{|\zeta|}\partial_\zeta + \frac{\overline{\zeta}}{|\zeta|}\partial_{\overline{\zeta}}\Big)N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&+\frac{1}{4\pi i}\int\limits_{|\zeta-z|=\varepsilon} \Big[(\zeta-z)N_{1\zeta}(z,\zeta) + (\overline{\zeta-z})N_{1\overline{\zeta}}(z,\zeta)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z}.
\end{aligned}
$$

This gives formula (21), letting ε tend to zero, by the same arguments as have been used in the proof of Theorem 2.

Integral representations (14), (21) are used to solve related boundary value problems, but in the case of the Neumann problem solvability conditions will appear.

Definition 3. *Let D be a regular domain with a Green function $G(z,\zeta) = \frac{1}{2}G_1(z,\zeta)$. The Poisson kernel for D is defined by*

$$
g_1(z,\zeta) = -\frac{1}{2}\,\partial_{\nu_\zeta}\,G_1(z,\zeta), \quad z \in D, \quad \zeta \in \partial D. \tag{22}
$$

The Poisson kernel for R can be evaluated explicitly and has the form

$$
g_1(z,\zeta) = \begin{cases} -\mathrm{Re}\,\{\widehat{g}_1(z,\zeta)\}, & |\zeta| = 1, \\ \frac{1}{r}\mathrm{Re}\,\{\widehat{g}_1(z,\zeta)\}, & |\zeta| = r, \end{cases} \tag{23}
$$

with

$$
\begin{aligned}
\widehat{g}_1(z,\zeta) \quad := \quad & \frac{\log|z|^2}{\log r^2} - \frac{\zeta}{\zeta - z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta} \\
& + \sum_{k=1}^{\infty}\Big(\frac{r^{2k}\overline{z}\zeta}{r^{2k}\overline{z}\zeta - 1} + \frac{\overline{z}\zeta}{\overline{z}\zeta - r^{2k}} - \frac{\zeta}{\zeta - r^{2k}z} - \frac{r^{2k}\zeta}{r^{2k}\zeta - z}\Big).
\end{aligned}
$$

Lemma 2. *Let $G(z,\zeta)=\frac{1}{2}G_1(z,\zeta)$ be the Green function for R defined by* (13)*. Then the formulas*

$$G_{1\zeta}(z,\zeta)=\frac{\log|z|^2}{\log r^2}\frac{1}{\zeta}-K(z,\zeta)+K\big(\frac{1}{\overline{z}},\zeta\big) \tag{24}$$

and

$$G_{1\zeta}(z,\zeta)=\Big(\frac{\log|z|^2}{\log r^2}-1\Big)\frac{1}{\zeta}-K(z,\zeta)+K\big(\frac{r^2}{\overline{z}},\zeta\big) \tag{25}$$

hold, where

$$K(z,\zeta)=\frac{1}{\zeta-z}+\sum_{k=1}^{\infty}\left(\frac{r^{2k}}{r^{2k}\zeta-z}+\frac{r^{2k}z}{\zeta(\zeta-r^{2k}z)}\right)=\frac{1}{\zeta-z}+\widehat{K}(z,\zeta).$$

Proof. Formulas (24), (25) are proved by a direct calculation.

Using the Poisson kernel, the representation formula (14) is rewritten in the form

$$w(z)=\frac{1}{2\pi i}\int_{\partial R}|\zeta|g_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int_{R}w_{\zeta\overline{\zeta}}(\zeta)\,G_1(z,\zeta)d\xi d\eta. \tag{26}$$

The Poisson kernel for R possesses the following important property.

Theorem 4. *Let $g_1(z,\zeta)$ be the Poisson kernel for R defined by* (23)*. Then, for any $w\in C(\partial R;\mathbb{C})$,*

$$\lim_{z\to z_0,\ z\in R}\frac{1}{2\pi i}\int_{\partial R}g_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}=w(z_0),\quad z_0\in\partial R. \tag{27}$$

Proof. From the definition of the Poisson kernel and the normal derivative the equalities

$$\frac{1}{2\pi i}\int_{\partial R}|\zeta|g_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}=-\frac{1}{4\pi i}\int_{\partial R}(\zeta\partial_\zeta+\overline{\zeta}\partial_{\overline{\zeta}})G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}$$
$$=-\frac{1}{2}\Big[\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}+\overline{\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)\overline{w(\zeta)}\frac{d\zeta}{\zeta}}\Big]$$

follow.

Transformations of the integral

$$\frac{1}{2\pi i}\int_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)w(\zeta)\frac{d\zeta}{\zeta}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int\limits_{\partial R}\Big[\frac{\log|z|^2}{\log r^2} - \frac{\zeta}{\zeta - z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta} \\
&\quad -\zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}\big(\frac{1}{\overline{z}},\zeta\big)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Big[-\Big(\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big) + \frac{\log|z|^2}{\log r^2} \\
&\quad -\zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}\big(\frac{1}{\overline{z}},\zeta\big)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&\quad -\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Big[\frac{\log|z|^2}{\log r^2}\frac{1}{\zeta} - \frac{1}{\zeta - z} + \frac{|z|^2}{|z|^2\zeta - z} \\
&\quad -\widehat{K}(z,\overline{\zeta}) + \widehat{K}\big(\frac{1}{\overline{z}},\zeta\big)\Big]w(\zeta)\mathrm{d}\zeta,
\end{aligned}
\tag{28}
$$

or

$$
\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{\partial R}\zeta\partial_\zeta G_1(z,\zeta)w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\int\limits_{\partial R}\Big[\Big(\frac{\log|z|^2}{\log r^2} - 1\Big) - \frac{\zeta}{\zeta - z} - \frac{\overline{z}\zeta}{1-\overline{z}\zeta} \\
&\quad -\zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}\big(\frac{r^2}{\overline{z}},\zeta\big)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Big[\Big(\frac{\log|z|^2}{\log r^2} - 1\Big) - \frac{\overline{z}\zeta}{\overline{z}\zeta - |z|^2} + \frac{\overline{z}\zeta}{\overline{z}\zeta - r^2} \\
&\quad -\zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}\big(\frac{r^2}{\overline{z}},\zeta\big)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\
&\quad -\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\Big[\Big(\frac{\log|z|^2}{\log r^2} - 1\Big) - \Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big) \\
&\quad -\zeta\widehat{K}(z,\overline{\zeta}) + \zeta\widehat{K}\big(\frac{r^2}{\overline{z}},\zeta\big)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}
\end{aligned}
\tag{29}
$$

are a result of applying formulas (24) and (25), respectively.

The function $\widehat{K}(z,\zeta)$ is uniformly continuous on z, $|z| > r$, for any fixed $\zeta \in \partial R$, what is observed from the estimates

$$
\begin{aligned}
&\Big|\widehat{K}(z_1,\zeta) - \widehat{K}(z_2,\zeta)\Big| \\
&\quad = \Big|\sum_{k=1}^{\infty} r^{2k}(z_1 - z_2)
\end{aligned}
$$

$$\times\Big(\frac{1}{(r^{2k}\zeta - z_1)(r^{2k}\zeta - z_2)} + \frac{1}{(\zeta - r^{2k}z_1)(\zeta - r^{2k}z_1)}\Big)\Big|$$

$$\leqslant \begin{cases} |z_1 - z_2| \\ \quad \times \sum\limits_{k=1}^{\infty} r^{2k}\Big(\frac{1}{(|z_1|-r^{2k})(|z_2|-r^{2k})} + \frac{1}{(1-r^{2k}|z_1|)(1-r^{2k}|z_2|)}\Big), \\ \hfill |\zeta| = 1, \\ |z_1 - z_2| \\ \quad \times \sum\limits_{k=1}^{\infty} r^{2k}\Big(\frac{1}{(|z_1|-r^{2k+1})(|z_2|-r^{2k+1})} + \frac{1}{r^2(1-r^{2k-1}|z_1|)(1-r^{2k-1}|z_2|)}\Big), \\ \hfill |\zeta| = r, \end{cases}$$

$$\leqslant \Big(\frac{1}{(1-r)^2} + \frac{1}{(1-r^2)^2}\Big)\frac{|z_1 - z_2|}{1-r^2}, \quad \forall\, z_1, z_2 \in R.$$

Hence

$$\lim_{\substack{z \to z_0, \\ z \in R}} \widehat{K}(z,\zeta) - \widehat{K}\big(\frac{1}{\overline{z}},\zeta\big) = 0, \quad |z_0| = 1, \tag{30}$$

$$\lim_{\substack{z \to z_0, \\ z \in R}} \widehat{K}(z,\zeta) - \widehat{K}\big(\frac{r^2}{\overline{z}},\zeta\big) = 0, \quad |z_0| = r, \tag{31}$$

as $\big|z - \frac{1}{\overline{z}}\big|$ tends to zero, when $z \to z_0$, $|z_0| = 1$, and $\big|z - \frac{r^2}{\overline{z}}\big|$ approaches zero, when $z \to z_0$, $|z_0| = r$.

The statement of the theorem follows from (30), (31) and the properties of the Poisson kernel

$$\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1$$

for circles $|\zeta| = 1$, $|\zeta| = r$ (see, e.g. [4]), taking the limit of (28), when z tends to z_0, $|z_0| = 1$, and the limit of (29), when z approaches z_0, $|z_0| = r$.

Definition 4 ([4]). *For a domain $D \subset \mathbb{C}$ with Green function $G(z,\zeta)$, the function*

$$k(z,\overline{\zeta}) = -\frac{2}{\pi}\, G_{z\overline{\zeta}}(z,\zeta)$$

is called the Bergman kernel function of D.

The Bergman kernel $k(z,\overline{\zeta})$ is analytic in z and $\overline{\zeta}$ for z, $\zeta \in D$, and has a singularity at $z = \zeta \in D$.

From the symmetry of the Green function, the property of the Bergman kernel

$$k(z,\overline{\zeta}) = \overline{k(\zeta,\overline{z})}$$

follows immediately. The Bergman kernel possesses the so-called reproducing property in the space of bounded analytic functions.

THEOREM 5 ([4]). *For any bounded analytic function w in D, the representation formula*

$$w(z) = \int_D w(\zeta)k(z,\overline{\zeta})\mathrm{d}\xi\mathrm{d}\eta, \quad z \in D,$$

holds.

The Bergman kernel for the ring R is expressed explicitly by

$$k(z,\zeta) = -\frac{1}{\pi}\left[\frac{1}{z\overline{\zeta}\log r^2} - \frac{1}{(1-z\overline{\zeta})^2} - \sum_{k=1}^{\infty}\left(\frac{1}{(1-r^{2k}z\overline{\zeta})^2} + \frac{1}{(r^{2k}-z\overline{\zeta})^2}\right)\right].$$

4. Boundary value problems

The second order differential operators, i.e., the Laplace operator $\partial_z\partial_{\overline{z}}$ and the Bitsadze operator $\partial^2_{\overline{z}}$, determine second order partial differential equations, namely, the Laplace, the Poisson (i.e., inhomogeneous Laplace), the Bitsadze and inhomogeneous Bitsadze equations. Different boundary value problems can be formulated for these equations.

In this section, the Dirichlet and the Neumann problem for the Poisson equation, the Dirichlet and the Schwarz problem for inhomogeneous Bitsadze equations will be solved. For the particular ring domain $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$, the solutions to these problems and the solvability conditions, if any, are given in explicit form.

THEOREM 6. *The Dirichlet problem for the Poisson equation in R*

$$w_{z\overline{z}} = f \quad \text{in} \quad R, \quad w = \gamma \quad \text{on} \quad \partial R, \tag{32}$$

for given $f \in L_1(R;\mathbb{C})$ and $\gamma \in C(\partial R;\mathbb{C})$, is uniquely solvable by

$$w(z) = -\frac{1}{4\pi i}\int_{\partial R} |\zeta|\partial_{\nu_\zeta} G_1(z,\zeta)\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{\pi}\int_R f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{33}$$

PROOF. Theorem 2, properties of the Poisson kernel (Theorem 4) and the Green function immediately provide the solution to the Dirichlet problem (32), expressed by (33).

THEOREM 7. *The Neumann problem for the Poisson equation in the ring R*

$$w_{z\bar z} = f \quad in \quad R, \quad |z|\partial_{\nu_z} w(z) = \gamma \quad on \quad \partial R, \quad \frac{1}{2\pi i}\int\limits_{|z|=1} w(z)\frac{\mathrm{d}z}{z} = c \quad (34)$$

for given $f \in L_1(R;\mathbb{C})$, $\gamma \in C(\partial R;\mathbb{C})$, $c \in \mathbb{C}$, *is solvable if and only if*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = \frac{2}{\pi}\int\limits_{R} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta \quad (35)$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = 0. \quad (36)$$

In this case, the unique solution is given by

$$w(z) = c + \frac{1}{4\pi i}\int\limits_{\partial R} \gamma(\zeta)N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{R} f(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \quad (37)$$

PROOF. If w is a solution to (34), then formula (37) is obtained from the representation (21), where (18) and (19) are used.

From properties of the Neumann function, it follows that (37) is the solution to the Poisson equation.

The boundary and normalization conditions in (34) have to be checked. For the beginning, the normal derivative of the Neumann function is considered for $\zeta \in \partial R$.

Observing

$$\begin{aligned} &|z|\partial_{\nu_z} N_1(z,\zeta) \\ &\quad = 2\mathrm{Re}\Big[\frac{z}{\zeta - z} + \frac{z\bar\zeta}{1 - z\bar\zeta} \\ &\quad -\sum_{k=1}^{\infty}\Big(\frac{z}{z - r^{2k}\zeta} + \frac{z\bar\zeta}{z\bar\zeta - r^{2k}} - \frac{r^{2k}z}{\zeta - r^{2k}z} - \frac{r^{2k}z\bar\zeta}{1 - r^{2k}z\bar\zeta} - 2\Big)\Big], \end{aligned}$$

it follows that

$$|z|\partial_{\nu_z} N_1(z,\zeta) = 2\Big[\frac{\zeta}{\zeta - z} + \frac{\bar\zeta}{\overline{\zeta - z}} - 2\Big], \quad \text{for } |z| = 1, \quad |\zeta| = 1, \quad (38)$$

$$|z|\partial_{\nu_z} N_1(z,\zeta) = 0, \quad \text{for } |z| = r, \quad |\zeta| = 1, \quad (39)$$

$$|z|\partial_{\nu_z} N_1(z,\zeta) = -4, \quad \text{for } |z| = 1, \quad |\zeta| = r, \quad (40)$$

$$|z|\partial_{\nu_z} N_1(z,\zeta) = 2\Big[\frac{\zeta}{\zeta - z} + \frac{\bar\zeta}{\overline{\zeta - z}} - 2\Big], \quad \text{for } |z| = r, \quad |\zeta| = r. \quad (41)$$

Taking the normal derivative of both sides of (37), we obtain

$$|z|\partial_{\nu_z} w(z) = \frac{1}{4\pi i}\int\limits_{\partial R} \gamma(\zeta)|z|\partial_{\nu_z} N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{\pi}\int\limits_R f(\zeta)|z|\partial_{\nu_z} N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

Then, on $|z| = 1$,

$$\begin{aligned}
&|z|\partial_{\nu_z} w(z)\\
&\quad= \frac{1}{4\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)|z|\partial_{\nu_z} N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\
&\qquad -\frac{1}{4\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)|z|\partial_{\nu_z} N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta} + \frac{2}{\pi}\int\limits_R f(\zeta)\mathrm{d}\xi\mathrm{d}\eta\\
&\quad= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta}\\
&\qquad -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta} + \frac{1}{\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta} + \frac{2}{\pi}\int\limits_R f(\zeta)\mathrm{d}\xi\mathrm{d}\eta,
\end{aligned}$$

and, on $|z| = r$,

$$\begin{aligned}
&|z|\partial_{\nu_z} w(z)\\
&\quad= \frac{1}{4\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)|z|\partial_{\nu_z} N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{4\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)|z|\partial_{\nu_z} N_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\
&\quad= \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta}.
\end{aligned}$$

Hence the boundary condition is valid if and only if the solvability conditions (35) and (36) hold.

Evaluating

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|z|=1} w(z)\frac{\mathrm{d}z}{z} &= c + \frac{1}{4\pi i}\int\limits_{\partial R} \gamma(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=1} N_1(z,\zeta)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}\zeta}{\zeta}\\
&\quad -\frac{1}{\pi}\int\limits_R f(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=1} N_1(z,\zeta)\frac{\mathrm{d}z}{z}\mathrm{d}\xi\mathrm{d}\eta,
\end{aligned}$$

the normalization condition follows due to (20).

Now the Dirichlet problem for the Bitsadze equation will be solved. This Dirichlet problem is different from that for the Poisson equation. It is taken in such a formulation that it can be treated by reducing it to a system of first order partial differential equations.

THEOREM 8. *The Dirichlet problem for the inhomogeneous Bitsadze equation in R*

$$w_{\overline{z}\,\overline{z}} = f \quad in \quad R, \quad w = \gamma_0, \quad w_{\overline{z}} = \gamma_1 \quad on \quad \partial R, \tag{42}$$

is solvable if and only if, for all $z \in R$,

$$\begin{aligned}&\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\mathrm{d}\zeta}{1-\overline{z}\zeta}\\&\quad=\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{1-\overline{z}\zeta}\mathrm{d}\zeta-\frac{r^2\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\&\quad\quad-\frac{\overline{z}}{\pi}\int_{R}f(\zeta)\frac{\overline{\zeta - z}}{1-\overline{z}\zeta}\mathrm{d}\xi\mathrm{d}\eta+\frac{r^2\overline{z}}{\pi}\int_{R}f(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta},\end{aligned} \tag{43}$$

and

$$\begin{aligned}&\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\mathrm{d}\zeta}{r^2-\overline{z}\zeta}\\&\quad=\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{r^2-\overline{z}\zeta}\mathrm{d}\zeta-\frac{\overline{z}}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\&\quad\quad-\frac{\overline{z}}{\pi}\int_{R}f(\zeta)\frac{\overline{\zeta - z}}{r^2-\overline{z}\zeta}\mathrm{d}\xi\mathrm{d}\eta+\frac{\overline{z}}{\pi}\int_{R}f(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta}.\end{aligned} \tag{44}$$

The unique solution, for given $f \in L_1(R;\mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial R;\mathbb{C})$, is expressed by

$$\begin{aligned}w(z) \;=\; &\frac{1}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z}-\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}\mathrm{d}\zeta-\frac{r^2}{2\pi i z}\int_{\partial R}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\&+\frac{1}{\pi}\int_{R}f(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta+\frac{r^2}{\pi z}\int_{R}f(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta}.\end{aligned} \tag{45}$$

PROOF. The problem is reduced to the system

$$w_{\overline{z}} = \varphi \quad \text{in} \quad R, \quad w = \gamma_0 \quad \text{on} \quad \partial R, \tag{46}$$

$$\varphi_{\overline{z}} = f \quad \text{in} \quad R, \quad \varphi = \gamma_1 \quad \text{on} \quad \partial R \tag{47}$$

of Dirichlet problems for the inhomogeneous Cauchy-Riemann equation. Using the result of Theorem 7 from [10], under the solvability conditions

$$\frac{1}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\overline{z}\mathrm{d}\zeta}{1-\overline{z}\zeta}=\frac{1}{\pi}\int_R\varphi(\zeta)\frac{\overline{z}\mathrm{d}\xi\mathrm{d}\eta}{1-\overline{z}\zeta}, \tag{48}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\overline{z}\mathrm{d}\zeta}{r^2-\overline{z}\zeta}=\frac{1}{\pi}\int_R\varphi(\zeta)\frac{\overline{z}\mathrm{d}\xi\mathrm{d}\eta}{r^2-\overline{z}\zeta}, \tag{49}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{z}\mathrm{d}\zeta}{1-\overline{z}\zeta}=\frac{1}{\pi}\int_R f(\zeta)\frac{\overline{z}\mathrm{d}\xi\mathrm{d}\eta}{1-\overline{z}\zeta}, \tag{50}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\overline{z}\mathrm{d}\zeta}{r^2-\overline{z}\zeta}=\frac{1}{\pi}\int_R f(\zeta)\frac{\overline{z}\mathrm{d}\xi\mathrm{d}\eta}{r^2-\overline{z}\zeta}, \tag{51}$$

we obtain that

$$w(z)=\frac{1}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z}-\frac{1}{\pi}\int_R\varphi(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z}, \tag{52}$$

$$\varphi(z)=\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z}-\frac{1}{\pi}\int_R f(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z}. \tag{53}$$

Substituting (53) into (52) gives

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial R}\gamma_0(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z}+\frac{1}{2\pi i}\int_{\partial R}\gamma_1(\widetilde{\zeta})\frac{1}{\pi}\int_R\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(\zeta-z)}\mathrm{d}\widetilde{\zeta} \\ &\quad -\frac{1}{\pi}\int_R f(\widetilde{\zeta})\frac{1}{\pi}\int_R\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(\zeta-z)}\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta} \end{aligned} \tag{54}$$

with

$$\frac{1}{\pi}\int_R\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(\zeta-z)}=\frac{1}{\widetilde{\zeta}-z}\left[\frac{1}{\pi}\int_R\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-\widetilde{\zeta}}-\frac{1}{\pi}\int_R\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z}\right]. \tag{55}$$

The Cauchy-Pompeiu integral representation (4) is used to evaluate the integral

$$\frac{1}{\pi}\int_R\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta-z}=-\overline{z}+\frac{1}{2\pi i}\int_{\partial R}\frac{\overline{\zeta}\mathrm{d}\zeta}{\zeta-z}=-\overline{z}+\frac{r^2}{z}.$$

Then the integral (55) can be easily found. Substituting it to (54), the formula (45) for the solution is obtained.

The function $\varphi(z)$ is plugged into (48) and (49) to get the solvability conditions. This leads to

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}\mathrm{d}\zeta}{1-\overline{z}\zeta} &= -\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\widetilde{\zeta})\frac{1}{\pi}\int\limits_{R}\frac{\overline{z}\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)}\mathrm{d}\widetilde{\zeta}\\ &+\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{\pi}\int\limits_{R}\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)}\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}\mathrm{d}\zeta}{r^2-\overline{z}\zeta} &= -\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\widetilde{\zeta})\frac{1}{\pi}\int\limits_{R}\frac{\overline{z}\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)}\mathrm{d}\widetilde{\zeta}\\ &+\frac{1}{\pi}\int\limits_{R}f(\widetilde{\zeta})\frac{1}{\pi}\int\limits_{R}\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)}\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}\end{aligned}$$

with

$$\begin{aligned}\frac{1}{\pi}\int\limits_{R}\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)} &= -\frac{\overline{\widetilde{\zeta}-z}}{1-\overline{z}\widetilde{\zeta}}+\frac{1}{2\pi i}\int\limits_{\partial R}\frac{(\overline{\zeta-z})\mathrm{d}\zeta}{(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)}\\ &= -\frac{\overline{\widetilde{\zeta}-z}}{1-\overline{z}\widetilde{\zeta}}+\frac{1}{2\pi i}\int\limits_{\partial R}\frac{(|\zeta|^2-\overline{z}\zeta)\mathrm{d}\zeta}{\zeta(\zeta-\widetilde{\zeta})(1-\overline{z}\zeta)}\\ &= \frac{r^2}{\widetilde{\zeta}}-\frac{\overline{\widetilde{\zeta}-z}}{1-\overline{z}\widetilde{\zeta}},\\ \frac{1}{\pi}\int\limits_{R}\frac{\mathrm{d}\xi\mathrm{d}\eta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)} &= -\frac{\overline{\widetilde{\zeta}-z}}{r^2-\overline{z}\widetilde{\zeta}}+\frac{1}{2\pi i}\int\limits_{\partial R}\frac{(\overline{\zeta-z})\mathrm{d}\zeta}{(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)}\\ &= -\frac{\overline{\widetilde{\zeta}-z}}{r^2-\overline{z}\widetilde{\zeta}}+\frac{1}{2\pi i}\int\limits_{\partial R}\frac{(|\zeta|^2-\overline{z}\zeta)\mathrm{d}\zeta}{\zeta(\zeta-\widetilde{\zeta})(r^2-\overline{z}\zeta)}\\ &= \frac{1}{\widetilde{\zeta}}-\frac{\overline{\widetilde{\zeta}-z}}{r^2-\overline{z}\widetilde{\zeta}}.\end{aligned}$$

From this solvability conditions (43) and (44) follow.

THEOREM 9. *The Schwarz problem for the inhomogeneous Bitsadze equation in R*

$$w_{\overline{z}\,\overline{z}}=f \quad in \quad R,\quad \mathrm{Re}\,w=\gamma_0,\quad \mathrm{Re}\,w_{\overline{z}}=\gamma_1 \quad on \quad \partial R, \tag{56}$$

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\mathrm{Im}\,w(z)\frac{\mathrm{d}z}{z}=c_0,\quad \frac{1}{2\pi i}\int\limits_{|z|=r}\mathrm{Im}\,w_{\overline{z}}(z)\frac{\mathrm{d}z}{z}=c_1 \tag{57}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)\mathrm{d}\xi\mathrm{d}\eta=0, \tag{58}$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{\mathrm{d}\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{\mathrm{d}\zeta}{\zeta} -\frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta+\overline{\zeta})\mathrm{d}\xi\mathrm{d}\eta=0. \tag{59}$$

For given $f\in L_1(R;\mathbb{C})$, $\gamma_0,\gamma_1\in C(\partial R;\mathbb{C})$, c_0, $c_1\in\mathbb{R}$, *the unique solution is expressed by*

$$\begin{aligned} w(z) \;=\; & ic_0+i(z+\overline{z})c_1+\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)K_1(z,\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\ & -\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)K_1(z,\zeta)(\zeta-z+\overline{\zeta-z})\frac{\mathrm{d}\zeta}{\zeta} \\ & +\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}(K_1(z,\zeta)-1)(\zeta-z+\overline{\zeta-z})\mathrm{d}\xi\mathrm{d}\eta \\ & +\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(K_2(z,\zeta)-1)(\zeta-z+\overline{\zeta-z})\mathrm{d}\xi\mathrm{d}\eta \\ & -\frac{2r^2}{4\pi i(1-r^2)}\int\limits_{\partial R}\gamma_1(\zeta)\Big(\frac{1-\zeta^2}{\zeta}-\frac{1-\overline{\zeta}^{\,2}}{\overline{\zeta}}\Big)\frac{\mathrm{d}\zeta}{\zeta} \\ & +\frac{2r^2}{2\pi(1-r^2)}\int\limits_{R}\Big(f(\zeta)\,\frac{1-\zeta^2}{\zeta^2}-\overline{f(\zeta)}\,\frac{1-\overline{\zeta}^{\,2}}{\overline{\zeta}^{\,2}}\Big)\mathrm{d}\xi\mathrm{d}\eta, \end{aligned} \tag{60}$$

where

$$K_1(z,\zeta)=\frac{\zeta+z}{\zeta-z}+2\sum_{n=1}^{\infty}\bigg(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z}\bigg),$$

$$K_2(z,\zeta)=\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}+2\sum_{n=1}^{\infty}\bigg(\frac{r^{2n}}{r^{2n}-z\overline{\zeta}}+\frac{r^{2n}z\overline{\zeta}}{1-r^{2n}z\overline{\zeta}}\bigg).$$

REMARK. Observing the formula for the solution to the Schwarz problem for the Bitsadze equation (see [2], Theorem 14), we note that extra term in the form of series appears when we pass from the problem in the unit disk to a circular ring domain. Formula (60) represents a generalization of the mentioned result in the case of R. Therefore, it is sufficient to show that, under conditions (58) and (59), formula (60) gives the solution to (56) and (57), and to see the uniqueness of such a solution.

PROOF. From

$$\begin{aligned} w_{\bar z}(z) &= \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)K_1(z,\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}(K_1(z,\zeta)-1)d\xi d\eta \\ &\quad -\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}(K_2(z,\zeta)-1)d\xi d\eta + ic_1, \end{aligned} \tag{61}$$

by using the properties of the Pompeiu operator, it is seen that (60) is the solution to the Bitsadze equation.

Let us verify the first boundary condition. Using the following expression

$$\begin{aligned} \operatorname{Re} w(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})] \\ &\quad\times\Bigg[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1 \\ &\quad+\sum_{n=1}^{\infty}\Bigg(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}+\frac{r^{2n}z}{\zeta-r^{2n}z} \\ &\quad-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\Bigg)\Bigg]\frac{d\zeta}{\zeta} \\ &\quad-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})] \\ &\quad\times\Bigg[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1 \\ &\quad+\sum_{n=1}^{\infty}\Bigg(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2(n-1)}|z|^2\zeta}{r^{2(n-1)}|z|^2\zeta-z} \\ &\quad+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2(n+1)}z}{|z|^2\zeta-r^{2(n+1)}z}\Bigg)\Bigg]\frac{d\zeta}{\zeta} \\ &\quad+\frac{1}{2\pi}\int\limits_{R}\operatorname{Re}\Bigg(\frac{f(\zeta)}{\zeta}\Bigg[\frac{\zeta+z}{\zeta-z}-\frac{|z|^2\zeta+z}{|z|^2\zeta-z} \end{aligned}$$

$$
-2+2\sum_{n=1}^{\infty}\left(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}\right.
$$
$$
\left.+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\right)\Big]\Big]\Big)(\zeta-z+\overline{\zeta-z})\mathrm{d}\xi\mathrm{d}\eta,
$$

we can find that

$$
\begin{aligned}
\lim_{\substack{|z|\to 1,\\ z\in R}}\operatorname{Re}w(z) &= \gamma_0(z)+\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})]\frac{\mathrm{d}\zeta}{\zeta}\\
&\quad -\frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})\mathrm{d}\xi\mathrm{d}\eta
\end{aligned}
$$

and

$$
\lim_{\substack{|z|\to r,\\ z\in R}}\operatorname{Re}w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})]\frac{\mathrm{d}\zeta}{\zeta}+\gamma_0(z).
$$

Hence, the first boundary condition is satisfied if and only if

$$
\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{|\zeta|=r}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})]\frac{\mathrm{d}\zeta}{\zeta}\\
&\qquad -\frac{1}{2\pi}\int\limits_{R}\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})\mathrm{d}\xi\mathrm{d}\eta\\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta-z+\overline{\zeta-z})]\frac{\mathrm{d}\zeta}{\zeta}=0
\end{aligned}
\tag{62}
$$

holds. In the same way, we obtain that

$$
\begin{aligned}
\operatorname{Re}w_{\overline{z}}(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}\\
&\quad -1+\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}\\
&\quad +\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\Big)\Big]\frac{\mathrm{d}\zeta}{\zeta}\\
&\quad -\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma_1(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}
\end{aligned}
$$

$$-1+\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2(n-1)}|z|^2\zeta}{r^{2(n-1)}|z|^2\zeta-z}$$
$$+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2(n+1)}z}{|z|^2\zeta-r^{2(n+1)}z}\Big)\Big]\frac{\mathrm{d}\zeta}{\zeta}$$
$$-\frac{1}{2\pi}\int_R \mathrm{Re}\,\Big(\frac{f(\zeta)}{\zeta}\Big[\frac{\zeta+z}{\zeta-z}-\frac{|z|^2\zeta+z}{|z|^2\zeta-z}$$
$$-2+2\sum_{n=1}^{\infty}\Big(\frac{r^{2n}\zeta}{r^{2n}\zeta-z}-\frac{r^{2n}|z|^2\zeta}{r^{2n}|z|^2\zeta-z}$$
$$+\frac{r^{2n}z}{\zeta-r^{2n}z}-\frac{r^{2n}z}{|z|^2\zeta-r^{2n}z}\Big)\Big]\Big)\mathrm{d}\xi\mathrm{d}\eta.$$

Hence, the equalities

$$\lim_{\substack{|z|\to 1,\\ z\in R}} \mathrm{Re}\, w_{\overline{z}}(z)=\gamma_1(z)+\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\frac{1}{2\pi}\int_R\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)\mathrm{d}\xi\mathrm{d}\eta$$

and

$$\lim_{\substack{|z|\to r,\\ z\in R}} \mathrm{Re}\, w_{\overline{z}}(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\gamma_1(z)$$

are valid. This shows that the second boundary condition is satisfied if and only if (58) holds.

Condition (62) can be rewritten in the form

$$\frac{1}{2\pi i}\int_{|\zeta|=r}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{\mathrm{d}\zeta}{\zeta}-\frac{1}{2\pi}\int_R\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)(\zeta+\overline{\zeta})\mathrm{d}\xi\mathrm{d}\eta$$
$$+(z+\overline{z})\Big[\frac{1}{2\pi i}\int_{|\zeta|=r}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\frac{1}{2\pi}\int_R\Big(\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big)\mathrm{d}\xi\mathrm{d}\eta\Big]$$
$$=\frac{1}{2\pi i}\int_{|\zeta|=1}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{\mathrm{d}\zeta}{\zeta}+(z+\overline{z})\frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta}=0,$$

what is equivalent to (59), due to (58).

Now the normalization conditions (57) will be checked. Let us find the integral

$$\frac{1}{2\pi i}\int_{|z|=r} w_{\overline{z}}(z)\frac{\mathrm{d}z}{z}.$$

Using (61), we obtain

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|z|=r} w_{\overline{z}}(z)\frac{\mathrm{d}z}{z} &= \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r} K_1(z,\zeta)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}\zeta}{\zeta} \\
&\quad -\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int\limits_{|z|=r}(K_1(z,\zeta)-1)\frac{\mathrm{d}z}{z}\mathrm{d}\xi\mathrm{d}\eta \\
&\quad -\frac{1}{2\pi}\int\limits_{R}\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1}{2\pi i}\int\limits_{|z|=r}(K_2(z,\zeta)-1)\frac{\mathrm{d}z}{z}\mathrm{d}\xi\mathrm{d}\eta + ic_1
\end{aligned}$$

with

$$\frac{1}{2\pi i}\int\limits_{|z|=r} K_1(z,\zeta)\frac{\mathrm{d}z}{z} = 1, \quad \frac{1}{2\pi i}\int\limits_{|z|=r}(K_2(z,\zeta)-1)\frac{\mathrm{d}z}{z} = 0. \tag{63}$$

Then

$$\frac{1}{2\pi i}\int\limits_{|z|=r} w_{\overline{z}}(z)\frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta} + ic_1,$$

which gives

$$\frac{1}{2\pi i}\int\limits_{|z|=r}\operatorname{Im} w_{\overline{z}}(z)\frac{\mathrm{d}z}{z} = c_1.$$

The other normalization condition is obtained, considering

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|z|=r} w(z)\frac{\mathrm{d}z}{z} &= ic_0 + \frac{ic_1}{2\pi i}\int\limits_{|z|=r}(z+\overline{z})\,\frac{\mathrm{d}z}{z} \\
&\quad +\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{1}{2\pi i}\int\limits_{|z|=r} K_1(z,\zeta)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}\zeta}{\zeta} \\
&\quad -\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_1(\zeta) \\
&\quad \times\frac{1}{2\pi i}\int\limits_{|z|=r} K_1(z,\zeta)(\zeta - z + \overline{\zeta - z})\frac{\mathrm{d}z}{z}\frac{\mathrm{d}\zeta}{\zeta} \\
&\quad +\frac{1}{2\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta} \\
&\quad \times\frac{1}{2\pi i}\int\limits_{|z|=r}(K_1(z,\zeta)-1)(\zeta - z + \overline{\zeta - z})\frac{\mathrm{d}z}{z}\mathrm{d}\xi\mathrm{d}\eta
\end{aligned}$$

$$+\frac{1}{2\pi}\int_R \frac{\overline{f(\zeta)}}{\overline{\zeta}}$$

$$\times\frac{1}{2\pi i}\int_{|z|=r}(K_2(z,\zeta)-1)(\zeta-z+\overline{\zeta-z})\frac{dz}{z}d\xi d\eta$$

$$-\frac{r^2}{2\pi i(1-r^2)}\int_{\partial R}\gamma_1(\zeta)\Big(\frac{1-\zeta^2}{\zeta^2}d\zeta-\frac{1-\overline{\zeta}^2}{\overline{\zeta}^2}d\overline{\zeta}\Big)$$

$$+\frac{r^2}{\pi(1-r^2)}\int_R\Big(f(\zeta)\,\frac{1-\zeta^2}{\zeta^2}-\overline{f(\zeta)}\,\frac{1-\overline{\zeta}^{\,2}}{\overline{\zeta}^{\,2}}\Big)d\xi d\eta$$

with

$$\frac{1}{2\pi i}\int_{|z|=r}K_1(z,\zeta)(\zeta-z+\overline{\zeta-z})\frac{dz}{z}$$

$$=(\zeta+\overline{\zeta})\frac{1}{2\pi i}\int_{|z|=r}K_1(z,\zeta)\frac{dz}{z}-\frac{1}{2\pi i}\int_{|z|=r}K_1(z,\zeta)dz$$

$$-\frac{1}{2\pi i}\int_{|z|=r}K_1(z,\zeta)\frac{dz}{z^2}$$

$$=\zeta+\overline{\zeta}-\frac{2r^2(1-\zeta^2)}{(1-r^2)\zeta},$$

$$\frac{1}{2\pi i}\int_{|z|=r}(K_2(z,\zeta)-1)(\zeta-z+\overline{\zeta-z})\frac{dz}{z}$$

$$=(\zeta+\overline{\zeta})\frac{1}{2\pi i}\int_{|z|=r}(K_2(z,\zeta)-1)\frac{dz}{z}-\frac{1}{2\pi i}\int_{|z|=r}K_2(z,\zeta)dz$$

$$-\frac{r^2}{2\pi i}\int_{|z|=r}K_2(z,\zeta)\frac{dz}{z^2}$$

$$=\frac{2r^2(1-\overline{\zeta}^{\,2})}{(1-r^2)\overline{\zeta}},$$

where (63) have been used. Then

$$\frac{1}{2\pi i}\int_{|z|=r}w(z)\frac{dz}{z} = ic_0+\frac{1}{2\pi i}\int_{|\zeta|=1}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi i}\int_{|\zeta|=r}[\gamma_0(\zeta)-\gamma_1(\zeta)(\zeta+\overline{\zeta})]\frac{d\zeta}{\zeta}$$

$$
\begin{aligned}
&+\frac{r^2}{(1-r^2)\pi i}\int\limits_{\partial R}\gamma_1(\zeta)\frac{1-\zeta^2}{\zeta^2}\mathrm{d}\zeta\\
&-\frac{r^2}{(1-r^2)2\pi i}\\
&\times\int\limits_{\partial R}\gamma_1(\zeta)\Big[\frac{1-\zeta^2}{\zeta^2}\mathrm{d}\zeta+\frac{1-\overline{\zeta}^2}{\overline{\zeta}^2}\mathrm{d}\overline{\zeta}\Big], \qquad (64)
\end{aligned}
$$

and from this the first normalization condition follows if one uses (59) and takes the imaginary part of (64).

To show the uniqueness of the solution (60), the Schwarz problem for the homogeneous Bitsadze equation with homogeneous boundary and normalization conditions is considered, i.e.,

$$
\begin{aligned}
&w_{\overline{z}\,\overline{z}}=0 \quad \text{in} \quad R, \quad \operatorname{Re} w=0, \quad \operatorname{Re} w_{\overline{z}}=0 \quad \text{on} \quad \partial R,\\
&\frac{1}{2\pi i}\int\limits_{|z|=r}\operatorname{Im} w(z)\frac{\mathrm{d}z}{z}=0, \quad \frac{1}{2\pi i}\int\limits_{|z|=r}\operatorname{Im} w_{\overline{z}}(z)\frac{\mathrm{d}z}{z}=0.
\end{aligned}
$$

It is decomposed to a system of Schwarz problems for inhomogeneous and homogeneous Cauchy-Riemann equations

$$
\begin{aligned}
&w_{\overline{z}}=\varphi \quad \text{in} \quad R, \quad \operatorname{Re} w=0 \quad \text{on} \quad \partial R, \quad \frac{1}{2\pi i}\int\limits_{|z|=r}\operatorname{Im} w(z)\frac{\mathrm{d}z}{z}=0,\\
&\varphi_{\overline{z}}=0 \quad \text{in} \quad R, \quad \operatorname{Re}\varphi=0 \quad \text{on} \quad \partial R, \quad \frac{1}{2\pi i}\int\limits_{|z|=r}\operatorname{Im}\varphi(z)\frac{\mathrm{d}z}{z}=0,
\end{aligned}
$$

which has only the trivial solutions if one takes into account the results of Theorem 2 and Theorem 6 in [10]. The proof is completed.

Remark. This result differs from the one for simply connected domains (see, [2]) as solvability conditions appear. They exclude functions which are not determined in a unique way by their respective boundary data. Let us consider, for instance, the function $\log z$. It is the solution to the Bitsadze equation, having vanishing Schwarz data on $|z|=1$ and the data $\log r$ on $|z|=r$, while its derivative with respect to $\overline{z}$ is identically zero. The second condition in (59) is not valid as the integral equals to $\log r$. Formula (60) is not applied for this function as the right hand side is $\log r+\pi i$. The function log z is a multi-valued one with single-valued real part. One can not expect this function to be determined by its real part on the boundary.

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References

[1] N. I. Akhiezer, *Elements of the Theory of Elliptic Functions*, AMS, Providence, Rhode Insland, 1990.

[2] H. Begehr, Boundary value problems in complex analysis. I, *F. Bol. Asoc. Mat. Venezolana* **XII**, No. 1 (2005), 65–85.

[3] H. Begehr, Boundary value problems in complex analysis. II, *F. Bol. Asoc. Mat. Venezolana* **XII**, No. 2 (2005), 217–250.

[4] H. Begehr, *Complex Analytic Methods for Partial Differential Equations. An Introductionary Text*, World Scientific, Singapore, 1994.

[5] H. Begehr, The main theorem of calculus in complex analysis, *Ann. EAS* (2005), 184–210.

[6] R. Courant, D. Hilbert, *Methods of Mathematical Physics. I*, Interscience Publishers. INC, New York, 1953.

[7] F. D. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford, 1966.

[8] A. Krausz, *Integraldarstellungen mit Greenschen Funktionen hoeherer Ordnung in Gebieten und Polygebieten*, Ph. D. Thesis, FU Berlin, 2005; http://www.diss.fu-berlin.de/2005/128/.

[9] V. V. Mityushev, S. V. Rogosin, *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions: Theory and Applications*, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[10] T. S. Vaitekhovich, Boundary value problems to first order complex partial differential equations in a ring domain, *Integral Transforms Spec. Funct.* (submitted).

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