THE DISCRETE UNIVERSALITY
OF THE DERIVATIVES OF L-FUNCTIONS
OF ELLIPTIC CURVES

Igoris BELOVAS$^1$, Virginija GARBALIAUSKIENĖ$^2$, 
RŪTA IVANAUSKAITE$^3$

$^1$Institute of Mathematics and Informatics, Akademijos 4, 
LT-08663 Vilnius, Lithuania; e-mail: igor_belov@talas.lt

$^2$Šiauliai University, P. Višninkio 19, LT-77156 Šiauliai, Lithuania; 
e-mail: virginija@fn.su.lt

$^3$Insurance Supervisory Commission of the Republic of Lithuania, 
Ukmergės 222, LT-07157 Vilnius, Lithuania; e-mail: iv.ruta@gmail.com

Abstract. In the paper, a discrete universality theorem of the Voronin type for the derivatives of $L$-functions of elliptic curves over the field of rational numbers is obtained.

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1. Introduction

Let $E$ be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = ax^3 + bx + c, \quad a, b, c \in \mathbb{Z}.$$

We suppose that the discriminant $\Delta = -16(4a^3 + 27b^2)$ of $E$ is non-zero. Then the elliptic curve $E$ is non-singular.

For every prime $p$, denote by $\nu(p)$ the number of solutions of the congruence

$$y^2 \equiv ax^3 + bx + c \pmod{p},$$

$$y^2 = ax^3 + bx + c, \quad a, b, c \in \mathbb{Z}.$$
and let $\lambda(p) = p - \nu(p)$. Then by a Hasse result

$$|\lambda(p)| < 2\sqrt{p}. \tag{1}$$

The $L$-function $L_E(s), s = \sigma + it$, of the elliptic curve $E$ is defined by

$$L_E(s) = \prod_{p \mid \Delta} \left( 1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}} \right)^{-1} \prod_{p \mid \Delta} \left( 1 - \frac{\lambda(p)}{p^s} \right)^{-1}.$$  

Inequality (1) shows that the function $L_E(s)$ is analytic in the half-plane $\sigma > \frac{3}{2}$. Moreover, the theory of new forms, see, for example, [9], and the proof of the Shimura-Taniyama conjecture [2] imply analytic continuation of $L_E(s)$ to an entire function.

In [4]–[8], the universality in the Voronin sense of the function $L_E(s)$ has been obtained. We recall the mentioned results. Let $D = \{s \in \mathbb{C} : 1 < \sigma < \frac{3}{2} \}$, and $\text{meas}\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

**Theorem A ([7]).** Let $K$ be a compact subset of the strip $D$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on $K$ which is analytic in the interior of $K$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |L_E(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Theorem A means a continuous universality of the function $L_E(s)$. Also, a discrete universality for $L_E(s)$ is true. Let $h > 0$ be a fixed number.

**Theorem B ([7], [9]).** Let $K$ and $f(s)$ be the same as in Theorem A. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq m \leq N : \sup_{s \in K} |L_E(s + imh) - f(s)| < \varepsilon \right\} > 0.$$  

In [8], the continuous universality of the derivative $L'_E(s)$ has been considered.

**Theorem C ([9]).** Let $K$ be the same as in Theorem A, and let $f(s)$ be a continuous function on $K$ which is analytic in the interior of $K$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |L'_E(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$
This note is devoted to a discrete universality of $L'_E(s)$. We limit ourselves only by $h > 0$ such that $\exp\left\{\frac{2\pi k}{h}\right\}$ is irrational for all $k \in \mathbb{Z} \setminus \{0\}$. 

**Theorem 1.** Let $K$ and $f(s)$ be the same as in Theorem C. Then, for every $\varepsilon > 0$, 

\[
\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq m \leq N : \sup_{s \in K} |L'_E(s + imh) - f(s)| < \varepsilon \right\} > 0.
\]

2. A discrete limit theorem for $L'_E(s)$

The proof of Theorem 1 is based on a discrete limit theorem in the sense of weak convergence of probability measures in the space $H(D)$ of analytic on $D$ functions equipped with the topology of uniform convergence on compacta as well as on the properties of the limit measure in this theorem.

Denote by $\mathcal{B}(S)$ the class of Borel sets of a space $S$. Define

$$
\Omega = \prod_p \gamma_p,
$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\} \overset{\text{def}}{=} \gamma$ for all primes $p$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the infinite-dimensional torus $\Omega$ is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure $m_H$ exists, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space $\gamma_p$. Then $\{\omega(p) : p \text{ is prime}\}$ is a sequence of independent complex-valued random variables defined of the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

On $(\Omega, \mathcal{B}(\Omega), m_H)$, define an $H(D)$-valued random element $L'_E(s, \omega)$ by the formula

\[
L'_E(s, \omega) = \prod_{p|\Delta} \left( 1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1} \prod_{p|\Delta} \left( 1 - \frac{\lambda(p)\omega(p)}{p^s} \right)^{-1} \times \left( - \sum_{p|\Delta} \frac{(\lambda(p)\omega(p) \log p)}{p^s} + \frac{2\omega^2(p) \log p}{p^{2s-1}} \right) \times (1 - \frac{\lambda(p)\omega(p)}{p^s})^{-1} \times \left( 1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1} \times \left( - \sum_{p|\Delta} \frac{\lambda(p)\omega(p) \log p}{p^s} \left( 1 - \frac{\lambda(p)\omega(p)}{p^s} \right)^{-1} \right).
\]
Moreover, let

\[ P_N(A) \overset{\text{def}}{=} \frac{1}{N+1} \# \{ 0 \leq m \leq N : L_E'(s + imh) \in A \}, \quad A \in \mathcal{B}(H(D)). \]

**Theorem 2.** Suppose that \( \exp \{ \frac{2\pi k}{h} \} \) is an irrational number for all \( k \in \mathbb{Z} \setminus \{0\} \). Then the probability measure \( P_N \) converges weakly to the distribution of the random element \( L_E'(s, \omega) \) as \( N \to \infty \).

**Proof.** Define

\[ Q_N(A) = \frac{1}{N+1} \# \{ 0 \leq m \leq N : L_E(s + imh) \in A \}, \quad A \in \mathcal{B}(H(D)). \]

Then in [6] it was proved that, under hypotheses of Theorem 2, the probability measure \( Q_N \) converges weakly to the distribution \( Q_{L_E} \) of the \( H(D) \)-valued random element

\[ L_E(s, \omega) = \prod_{p \mid \Delta} \left( 1 - \frac{\lambda(p) \omega(p)}{p} + \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1} \prod_{p \mid \Delta} \left( 1 - \frac{\lambda(p) \omega(p)}{p^s} \right)^{-1} \]

as \( N \to \infty \). Clearly, the function \( u : H(D) \to H(D) \) defined by the formula \( u(g(s)) = g'(s) \), \( g(s) \in H(D) \), is continuous. Therefore, the weak convergence of the measure \( Q_N \) and Theorem 5.1 from [1] show that the probability measure \( P_N \) converges weakly to \( Q_{L_E} u^{-1} \) as \( N \to \infty \). Since \( Q_{L_E} u^{-1} \) is the distribution of the random element \( L_E'(s, \omega) \), the theorem is proved.

### 3. The support of \( L_E'(s, \omega) \)

The space \( H(D) \) is separable. Therefore, the support of the distribution \( P_{L_E'} \) of \( L'(s, \omega) \) is a minimal closed set \( S_{P_{L_E'}} \subset H(D) \) such that

\[ P_{L_E'} \left( S_{P_{L_E'}} \right) = 1. \]

The support of \( P_{L_E'} \) is called the support of \( L_E'(s, \omega) \).

For the proof of Theorem 1, we have to know the support of the measure \( P_{L_E'} \). For this, we will apply an information on the support of the random element \( L_E(s, \omega) \). Let

\[ S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}. \]

**Lemma 3.** The support of the random element \( L_E(s, \omega) \) is the set \( S \).
The discrete universality of the derivatives...

The lemma is proved in [6], Lemma 5.

Now we recall the definition of a metric on $H(D)$ which induces the
topology of uniform convergence on compacta.

It is well known, see, for example, [3] that there exists a sequence \{${K_m : m \in \mathbb{N}}$\} of compact subsets of the strip $D$ such that

$$D = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subseteq K_{m+1}, m \in \mathbb{N},$ and, for any compact subset $K$ of $D$, there exists an $m$ for which $K \subseteq K_m$ holds. Then

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s \in K_m} \|g_1(s) - g_2(s)\|}{1 + \sup_{s \in K_m} \|g_1(s) - g_2(s)\|}, \quad g_1, g_2 \in H(D),$$

is a metric on $H(D)$ which induces its topology.

Also, we recall the Mergelyan theorem on the approximation of analytic functions by polynomials, see [10].

**Lemma 4.** Let $K \subset \mathbb{C}$ be a compact set with connected complement, and let $g(s)$ be a continuous function on $K$ which is analytic in the interior of $K$. Then the function $g(s)$ can be approximated uniformly on $K$ by polynomials

**Theorem 5.** The support of the random element $L_E(s, \omega)$ is the whole of $H(D)$.

**Proof.** Let a function $u : S \rightarrow H(D)$ be given by the formula

$$u(g(s)) = g'(s), \quad g(s) \in S.$$

Then it follows easily from the integral Cauchy formula that the function $u$ is continuous. Therefore, for any open set $G \subset H(D)$, the set $u^{-1}G$ is an open subset of $S$. It remains to prove that the set $u^{-1}G$ is non empty.

Let $g \in u^{-1}G$. Then, by the definition of $u$, we have that $u(g) \in G$.

Let $K$ be a compact subset of the strip $D$ with connected complement. Then, in view of Lemma 4, there exists a polynomial $p(s)$ which approximates $u(g(s))$ with a given accuracy uniformly on $K$. Hence, it follows that $p(s) \in G$. Thus, we obtain that there exists a polynomial $q(s) \in u^{-1}(p(s))$ and $q(s) \neq 0$ on $D$. This shows that the set $u^{-1}G$ is non empty.
4. Proof of Theorem 1

By Theorem 2, the probability measure $P_N$ converges weakly to the distribution $P_{L_E'}$ of the random element $L'(s, \omega)$ as $N \to \infty$. Hence, by Theorem 2.1 from [1], we have that, for every open set $A \subset H(D)$,

\[
\liminf_{N \to \infty} P_N(A) \geq P_{L_E'}(A).
\] (2)

By Lemma 4, there exists a polynomial $p(s)$ such that

\[
\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}.
\] (3)

Let $G$ be a set in $H(D)$ defined by

\[
G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.
\]

Then $G$ is an open set. Moreover, since the support of the measure $P_{L_E'}$ consists of all $g \in H(D)$ such that, for every neighbourhood $G$ of $g$, the inequality $P_{L_E'}(G) > 0$ is satisfied, and, by Theorem 5, $p(s) \in S_{P_{L_E'}}$, we have that $P_{L_E'}(G) > 0$. Therefore, inequality (2) shows that

\[
\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq m \leq N : \sup_{s \in K} |L'(s + imh) - f(s)| < \frac{\varepsilon}{2} \right\} > 0.
\] (4)

However, in virtue of (3),

\[
\left\{ 0 \leq m \leq N : \sup_{s \in K} |L'_E(s + imh) - f(s)| < \varepsilon \right\} \\
\supseteq \left\{ 0 \leq m \leq N : \sup_{s \in K} |L'_E(s + imh) - p(s)| < \varepsilon \right\}.
\]

Hence and from (4) we have that

\[
\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq m \leq N : \sup_{s \in K} |L'(s + imh) - f(s)| < \varepsilon \right\} > 0.
\]

The theorem is proved.

References


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