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THE DISCRETE UNIVERSALITY OF THE DERIVATIVES OF L-FUNCTIONS OF ELLIPTIC CURVES

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Abstract. In the paper, a discrete universality theorem of the Voronin type for the derivatives of L-functions of elliptic curves over the field of rational numbers is obtained.

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1. Introduction

Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = ax^3 + bx + c, \quad a, b, c \in \mathbb{Z}.$$

We suppose that the discriminant $\Delta = -16(4a^3 + 27b^2)$ of E is non-zero. Then the elliptic curve E is non-singular.

For every prime p, denote by $\nu(p)$ the number of solutions of the congruence

$$y^2 \equiv ax^3 + bx + c (\bmod p),$$

and let $\lambda(p) = p - \nu(p)$. Then by a Hasse result

$$|\lambda(p)| < 2\sqrt{p}. \tag{1}$$

The L-function $L_E(s)$, $s = \sigma + it$, of the elliptic curve E is defined by

$$L_E(s) = \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod_{p \mid \Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1}.$$

Inequality (1) shows that the function $L_E(s)$ is analytic in the half-plane $\sigma > \frac{3}{2}$. Moreover, the theory of new forms, see, for example, [9], and the proof of the Shimura-Taniyama conjecture [2] imply analytic continuation of $L_E(s)$ to an entire function.

In [4]–[8], the universality in the Voronin sense of the function $L_E(s)$ has been obtained. We recall the mentioned results. Let $D = \{s \in \mathbb{C} : 1 < \sigma < \frac{3}{2}\}$, and meas$\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

THEOREM A ([?]). *Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L_E(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Theorem A means a continuous universality of the function $L_E(s)$. Also, a discrete universality for $L_E(s)$ is true. Let $h > 0$ be a fixed number.

THEOREM B ([?], [?]). *Let K and $f(s)$ be the same as in* Theorem A. *Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant m \leqslant N : \sup_{s\in K} |L_E(s+imh) - f(s)| < \varepsilon\right\} > 0.$$

In [8], the continuous universality of the derivative $L'_E(s)$ has been considered.

THEOREM C ([?]). *Let K be the same as in* Theorem A, *and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |L'_E(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

This note is devoted to a discrete universality of $L'_E(s)$. We limit ourselves only by $h>0$ such that $\exp\{\frac{2\pi k}{h}\}$ is irrational for all $k\in\mathbb{Z}\setminus\{0\}$.

THEOREM 1. *Let* K *and* $f(s)$ *be the same as in* Theorem C. *Then, for every* $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant m\leqslant N:\sup_{s\in K}|L'_E(s+imh)-f(s)|<\varepsilon\right\}>0.$$

2. A discrete limit theorem for $L'_E(s)$

The proof of Theorem 1 is based on a discrete limit theorem in the sense of weak convergence of probability measures in the space $H(D)$ of analytic on D functions equipped with the topology of uniform convergence on compacta as well as on the properties of the limit measure in this theorem.

Denote by $\mathcal{B}(S)$ the class of Borel sets of a space S. Define

$$\Omega=\prod_p\gamma_p,$$

where $\gamma_p=\{s\in\mathbb{C}:|s|=1\}\stackrel{def}{=}\gamma$ for all primes p. By the Tikhonov theorem, with the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega,\mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and this leads to a probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by $\omega(p)$ the projection of $\omega\in\Omega$ to the coordinate space γ_p. Then $\{\omega(p):p \text{ is prime}\}$ is a sequence of independent complex-valued random variables defined of the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$.

On $(\Omega,\mathcal{B}(\Omega),m_H)$, define an $H(D)$-valued random element $L'_E(s,\omega)$ by the formula

$$\begin{aligned}L'_E(s,\omega) &= \prod_{p\nmid\Delta}\left(1-\frac{\lambda(p)\omega(p)}{p^s}+\frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}\prod_{p\mid\Delta}\left(1-\frac{\lambda(p)\omega(p)}{p^s}\right)^{-1}\\ &\quad\times\Bigg(-\sum_{p\nmid\Delta}\left(\frac{\lambda(p)\omega(p)\log p}{p^s}-\frac{2\omega^2(p)\log p}{p^{2s-1}}\right)\\ &\quad\times\left(1-\frac{\lambda(p)\omega(p)}{p^s}+\frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}\\ &\quad-\sum_{p\mid\Delta}\frac{\lambda(p)\omega(p)\log p}{p^s}\left(1-\frac{\lambda(p)\omega(p)}{p^s}\right)^{-1}\Bigg).\end{aligned}$$

Moreover, let

$$P_N(A) \stackrel{def}{=} \frac{1}{N+1} \# \left\{ 0 \leqslant m \leqslant N : L'_E(s+imh) \in A \right\}, \quad A \in \mathcal{B}(H(D)).$$

Theorem 2. *Suppose that* $\exp\{\frac{2\pi k}{h}\}$ *is an irrational number for all* $k \in \mathbb{Z} \setminus \{0\}$. *Then the probability measure* P_N *converges weakly to the distribution of the random element* $L'_E(s,\omega)$ *as* $N \to \infty$.

Proof. Define

$$Q_N(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant m \leqslant N : L_E(s+imh) \in A \right\}, \quad A \in \mathcal{B}(H(D)).$$

Then in [6] it was proved that, under hypotheses of Theorem 2, the probability measure Q_N converges weakly to the distribution Q_{L_E} of the $H(D)$-valued random element

$$L_E(s,\omega) = \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1} \prod_{p \mid \Delta} \left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1}$$

as $N \to \infty$. Clearly, the function $u : H(D) \to H(D)$ defined by the formula $u(g(s)) = g'(s)$, $g(s) \in H(D)$, in continuous. Therefore, the weak convergence of the measure Q_N and Theorem 5.1 from [1] show that the probability measure P_N converges weakly to $Q_{L_E} u^{-1}$ as $N \to \infty$. Since $Q_{L_E} u^{-1}$ is the distribution of the random element $L'_E(s,\omega)$, the theorem is proved.

3. The support of $L'_E(s,\omega)$

The space $H(D)$ is separable. Therefore, the support of the distribution $P_{L'_E}$ of $L'(s,\omega)$ is a minimal closed set $S_{P_{L'_E}} \subset H(D)$ such that $P_{L'_E}\left(S_{P_{L'_E}}\right) = 1$. The support of $P_{L'_E}$ is called the support of $L'_E(s,\omega)$.

For the proof of Theorem 1, we have to know the support of the measure $P_{L'_E}$. For this, we will apply an information on the support of the random element $L_E(s,\omega)$. Let

$$S = \left\{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \right\}.$$

Lemma 3. *The support of the random element* $L_E(s,\omega)$ *is the set* S.

The lemma is proved in [6], Lemma 5.

Now we recall the definition of a metric on $H(D)$ which induces the topology of uniform convergence on compacta.

It is well known, see, for example, [3] that there exists a sequence $\{K_m : m \in \mathbb{N}\}$ of compact subsets of the strip D such that

$$D = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$, $m \in \mathbb{N}$, and, for any compact subset K of D, there exists an m for which $K \subseteq K_m$ holds. Then

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup\limits_{s \in K_m} |(g_1(s) - g_2(s))|}{1 + \sup\limits_{s \in K_m} |(g_1(s) - g_2(s))|}, \quad g_1, g_2 \in H(D),$$

is a metric on $H(D)$ which induces its topology.

Also, we recall the Mergelyan theorem on the approximation of analytic functions by polynomials, see [10].

LEMMA 4. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then the function $g(s)$ can be approximated uniformly on K by polynomials in s.*

THEOREM 5. *The support of the random element $L'_E(s, \omega)$ is the whole of $H(D)$.*

PROOF. Let a function $u : S \to H(D)$ be given by the formula

$$u(g(s)) = g'(s), \quad g(s) \in S.$$

Then it follows easily from the integral Cauchy formula that the function u is continuous. Therefore, for any open set $G \subset H(D)$, the set $u^{-1}G$ is an open subset of S. It remains to prove that the set $u^{-1}G$ is non empty.

Let $g \in u^{-1}G$. Then, by the definition of u, we have that $u(g) \in G$.

Let K be a compact subset of the strip D with connected complement. Then, in view of Lemma 4, there exists a polynomial $p(s)$ which approximates $u(g(s))$ with a given accuracy uniformly on K. Hence, it follows that $p(s) \in G$. Thus, we obtain that there exists a polynomial $q(s) \in u^{-1}(p(s))$ and $q(s) \neq 0$ on D. This shows that the set $u^{-1}G$ is non empty.

4. Proof of Theorem 1

By Theorem 2, the probability measure P_N converges weakly to the distribution $P_{L'_E}$ of the random element $L'(s,\omega)$ as $N \to \infty$. Hence, by Theorem 2.1 from [1], we have that, for every open set $A \subset H(D)$,

$$\liminf_{N\to\infty} P_N(A) \geqslant P_{L'_E}(A). \tag{2}$$

By Lemma 4, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3}$$

Let G be a set in $H(D)$ defined by

$$G = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G is an open set. Moreover, since the support of the measure $P_{L'_E}$ consists of all $g \in H(D)$ such that, for every neighbourhood G of g, the inequality $P_{L'_E}(\mathcal{G}) > 0$ is satisfied, and, by Theorem 5, $p(s) \in S_{P_{L'_E}}$, we have that $P_{L'_E}(G) > 0$. Therefore, inequality (2) shows that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant m \leqslant N : \sup_{s\in K} |L'(s+imh) - f(s)| < \frac{\varepsilon}{2} \right\} > 0. \tag{4}$$

However, in virtue of (3),

$$\left\{ 0 \leqslant m \leqslant N : \sup_{s\in K} |L'_E(s+imh) - f(s)| < \varepsilon \right\}$$
$$\supseteq \left\{ 0 \leqslant m \leqslant N : \sup_{s\in K} \left| L'_E(s+imh) - p(s) \right| < \frac{\varepsilon}{2} \right\}.$$

Hence and from (4) we have that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant m \leqslant N : \sup_{s\in K} |L'(s+imh) - f(s)| < \varepsilon \right\} > 0.$$

The theorem is proved.

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