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CONTRIBUTIONS TO THE THEORY OF THE HURWITZ-LERCH ZETA-FUNCTION

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Abstract. In this paper, we shall prove useful analogues of known results on the Hurwitz zeta-function as presented by [14]. Naturally there can be two natural analogues, one with an additive character (the exponential function) and the other with a multiplicative character (Dirichlet character) both to the fixed modulus q. It turns out that in the case of the perturbed Dirichlet L-function, there appear the generalized Bernoulli polynomials introduced by Leopoldt and the results obtained are in conformity with what has been known, while the remaining case of the Lerch zeta-function, what appears is a generalization of the cotangent function in the context of [12] and presents new aspects comparable to those of [15]. The main results are of the same form as before and manifest the principle that an essential part already describes the whole, i.e., from the integral representation for the partial sum, we may immediately deduce that of the whole zeta-function and its approximation by the partial sum, together with the Ewald expansion which in turn implies the functional equation. The main tool is an analogue of the Euler-Maclaurin summation formula for periodic sequences similar to that developed by Berndt [4] and Berndt and Schoenfeld [5], but here we use a unified form due to Rane ([23], [24]), in simplified and handier form, whose proof is to appear in the author thesis.

Key words and phrases: Dirichlet L-function, integral representation, Hurwitz-Lerch transcendent, periodic summation formulas.

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1. Introduction and summation formulas

We recall from [14, p. 4, (19)] the Fourier expansion for the periodic Bernoulli polynomial $\bar{B}_k(x) = B_k(x-[x])$ of degree k, where $[x]$ denotes the integral part of x and $B_k(x) = \sum\limits_{r=1}^{k} \binom{k}{r} B_k x^{k-r}$ is the kth Bernoulli polynomial, $B_k = B_k(0)$ being the kth Bernoulli number:

$$\begin{aligned} \bar{B}_k(x) &= -\frac{k!}{(2\pi i)^k} \sum_{n=-\infty}^{\infty}{}' \frac{\mathrm{e}^{2\pi i n x}}{n^k} \\ &= \begin{cases} (-1)^{\frac{k+1}{2}} \frac{2(k!)}{(2\pi)^k} \sum\limits_{n=1}^{\infty} \frac{\sin 2\pi n x}{n^k} & \text{if} \quad k \quad \text{odd}, \\ (-1)^{\frac{k}{2}-1} \frac{2(k!)}{(2\pi)^k} \sum\limits_{n=1}^{\infty} \frac{\cos 2\pi n x}{n^k} & \text{if} \quad k \quad \text{even}, \end{cases} \end{aligned} \tag{1.1}$$

where the prime on the summation sign means that the term with $n = 0$ is to be excluded.

In particular, the first periodic Bernoulli polynomial is often used:

$$\bar{B}_1(t) = t - [t] - \frac{1}{2} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n}, \tag{1.2}$$

where the second equality holds true for $t \notin \mathbb{Z}$; for $t \in \mathbb{Z}$, the Fourier series converges to 0, while $\bar{B}_1(t) = -\frac{1}{2} = B_1$ for $t \in \mathbb{Z}$.

In this section, we shall state some useful periodic summation formulas due to Rane ([23], [24]), which are more or less presented earlier by Berndt and Berndt-Schoenfeld (cf. also [4], [5]). We shall omit the proofs of the results, which are given in much simplified and clarified form in my thesis [9]. For rudiments of the theory of periodic sequences, we refer to [11].

Noting the relation

$$\frac{\mathrm{d}}{\mathrm{d}x} \bar{B}_k(x) = k \bar{B}_{k-1}(x),$$

we may prove the classical Euler-Maclaurin sum formula: for $f \in C^l$,

$$\begin{aligned} \sum_{a<n\leqslant b} f(n) &= \int_a^b f(t)\mathrm{d}t + \sum_{r=1}^{l} \frac{(-1)^r}{r} \left[\bar{B}_r(t) f^{(r-1)}(t)\right]_a^b \\ &\quad + \frac{(-1)^{l+1}}{l!} \int_a^b \bar{B}_l(t) f^{(l)}(t)\mathrm{d}t. \end{aligned} \tag{1.3}$$

We are now in a position to state a periodic analogue of (1.3).

THEOREM 1 (Euler-Maclaurin analogue). *For $f \in C^l$ and periodic χ with period q, $q > 1$, we have, for $X > 0$,*

$$\begin{aligned} S(X) &= \sum_{m \leqslant X} \sum_{\frac{a}{m} < n \leqslant \frac{b}{m}} \chi(mn) f(mn) \\ &= \sum_{r_1=1}^{q} \sum_{r_2=1}^{q} \chi(r_1 r_2) \left\{ \sum_{\substack{m \leqslant X \\ m \equiv r_1 (\bmod q)}} \frac{1}{qm} \int_a^b f(u) \mathrm{d}u \right. \\ &\quad + \sum_{r=1}^{l} \frac{(-1)^r}{r!} q^{r-1} \\ &\quad \times \sum_{\substack{m \leqslant X \\ m \equiv r_1 (\bmod q)}} m^{r-1} \left[\bar{B}_r \left(\frac{\frac{u}{m} - r_2}{q} \right) f^{(r-1)}(u) \right]_a^b \\ &\quad + \frac{(-1)^{l+1}}{l!} q^{l-1} \\ &\quad \left. \times \sum_{\substack{m \leqslant X \\ m \equiv r_1 (\bmod q)}} m^{l-1} \int_a^b \bar{B}_l \left(\frac{\frac{u}{m} - r_2}{q} \right) f^{(l)}(u) \mathrm{d}u \right\}. \end{aligned}$$

The proof rests on the following statement.

LEMMA 1. *For $f \in C^1$, we have*

$$\begin{aligned} &\sum_{\substack{\frac{a}{m} < n \leqslant \frac{b}{m} \\ m \equiv r (\bmod q)}} f(mn) \\ &= \frac{1}{qm} \int_a^b f(u) \mathrm{d}u - \left[\bar{B}_1 \left(\frac{\frac{u}{m} - r}{q} \right) f(u) \right]_a^b + \int_a^b \bar{B}_1 \left(\frac{\frac{u}{m} - r}{q} \right) \mathrm{d}f(u). \end{aligned}$$

REMARK 1. Note that the special case of Lemma 1 with $m = q = 1$ and $f \in C^1$ gives rise to Euler's summation formula

$$\sum_{a < n \leqslant b} f(n) = \int_a^b f(u) \mathrm{d}u - \left[\bar{B}_1(u) f(u) \right]_a^b + \int_a^b \bar{B}_1(u) f'(u) \mathrm{d}u. \tag{1.4}$$

If $f \in C^l$, then we apply integration by parts repeatedly to the last integral on the right of (1.4), thereby using (1.2), to derive (1.3).

The proof of Theorem 1 is obtained in the same way: first derive a counterpart of (1.4) and then apply integration by parts.

We now state a corollary to Theorem 1, the special case of $X = 1$, which will play the main role in the subsequent discussion.

COROLLARY 1. *For $f \in C^l$ and periodic χ with period q, we have*

$$\begin{aligned}\sum_{a<n\leqslant b} \chi(n)f(n) &= \sum_{j=1}^{q} \chi(j) \Bigg\{ \frac{1}{q} \int_a^b f(u)\mathrm{d}u \\ &\quad + \sum_{r=1}^{l} \frac{(-1)^r}{r!} q^{r-1} \left[\bar{B}_r \left(\frac{u-j}{q} \right) f^{(r-1)}(u) \right]_a^b \\ &\quad + \frac{(-1)^{l+1}}{l!} q^{l-1} \int_a^b \bar{B}_l \left(\frac{u-j}{q} \right) f^{(l)}(u)\mathrm{d}u \Bigg\}.\end{aligned}$$

2. The Hurwitz-Lerch transcendent

Let $\phi(s,a,\mu)$ denote the Hurwitz-Lerch transcendent defined by

$$\phi(s,a,\mu) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2\pi i n\mu}}{(n+a)^s}, \qquad \sigma = \operatorname{Re} s > 1, \tag{2.1}$$

where $a > 0$ and $\mu \in \mathbb{R}$.

The absolute convergence of the series in (2.1) follows by partial summation or by Euler's summation formula (1.4).

In this paper, we shall consider the special case where

$$\mu = \frac{\nu}{q}, \quad 0 \leqslant \nu \leqslant q, \quad q \in \mathbb{N}, \quad \nu \in \mathbb{Z},$$

and obtain a counterpart of [14, Theorem 3.1].

Let $\Gamma(s)$ be the gamma-function defined in the first instance by

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-x} x^{s-1}\mathrm{d}x, \quad \sigma > 0.$$

Theorem 2 ([14]).

i) *For the partial sum with* $u \neq -1$

$$L_u(x,a,\mu) = \sum_{0 \leqslant n \leqslant x} \mathrm{e}^{2\pi i \mu n}(n+a)^u$$

of $\varphi(-u,a,\mu)$, *we have the integral representation*

$$\begin{aligned} L_u(x,a,\mu) \quad = \quad & \begin{cases} \frac{1}{u+1}(x+a)^{u+1} & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad 0 < \nu < q, \end{cases} \\ & + \varphi(-u,a,\mu) \\ & + \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} q^r \\ & \times \frac{1}{q} \sum_{j=1}^{q} \mathrm{e}^{2\pi i \mu j} \bar{B}_r \left(\frac{x-j}{q} \right) (x+a)^{u-r+1} \\ & + \frac{(-1)^l}{l!} q^l \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \\ & \times \int_x^{\infty} \frac{1}{q} \sum_{j=1}^{q} \mathrm{e}^{2\pi i \mu j} \bar{B}_l \left(\frac{t-j}{q} \right) (t+a)^{u-l} \mathrm{d}t, \end{aligned} \tag{2.2}$$

and the asymptotic formula

$$\begin{aligned} L_u(x,a,\mu) \quad = \quad & \begin{cases} \frac{1}{u+1}(x+a)^{u+1} & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad 0 < \nu < q, \end{cases} \\ & + \varphi(-u,a,\mu) \\ & + \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} q^{r-1} \\ & \times \sum_{j=1}^{q} \mathrm{e}^{2\pi i \mu j} \bar{B}_r \left(\frac{x-j}{q} \right) (x+a)^{u-r+1} \\ & + \mathrm{O}\left(q^{l-1} (x+a)^{\operatorname{Re} u - l} \right), \end{aligned} \tag{2.3}$$

where $l > \operatorname{Re} u + 1$ *and* $l \in \mathbb{N}$.

ii) *We have the integral representation for the Hurwitz-Lerch transcendent*

$$\varphi(-u,a,\mu) \quad = \quad \begin{cases} -\frac{a^{u+1}}{u+1} & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad 0 < \nu < q, \end{cases}$$

$$+a^u - \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} q^{r-1}$$
$$\times \sum_{j=1}^{q} \mathrm{e}^{2\pi i \mu j} \bar{B}_r \left(-\frac{j}{q}\right) a^{u-r+1}$$
$$+\frac{(-1)^{l+1}}{l!} q^{l-1} \frac{\Gamma(u+1)}{\Gamma(u+1-l)}$$
$$\times \int_0^\infty \sum_{j=1}^{q} \mathrm{e}^{2\pi i \mu j} \bar{B}_l \left(\frac{t-j}{q}\right) (t+a)^{u-l} \mathrm{d}t, \quad (2.4)$$

where the last integral is absolutely convergent for $\mathrm{Re}\,(-u) > 1-l$ *and boundedly convergent for* $\mathrm{Re}\,(-u) > -l$.

iii) *The special case* $u=-1$, $\mu=0$ *corresponds to* [14, p. 55, (3.6)]

$$L_{-1}(x,a,0) = \log(x+a) - \psi(a) - \sum_{r=1}^{l} \frac{1}{r} B_r (x+a)^{-r} + \int_x^\infty \bar{B}_l(t)(t+a)^{-l-1} \mathrm{d}t, \quad (2.5)$$

where the digamma-function $\psi(a)$ *has two representations*

$$-\psi(a) = \lim_{x\to\infty} \left(\sum_{0\leqslant n\leqslant x} \frac{1}{n+a} - \log(x+a) \right)$$
$$= -\frac{1}{2}a^{-1} + \sum_{r=2}^{l} \frac{1}{r} B_r a^{-r} - \log a - \int_0^\infty \bar{B}_l(t)(t+a)^{-l-1} \mathrm{d}t. \quad (2.6)$$

Proof. The proof is in the spirit of that of [14, Theorem 3.1]. Applying Corollary 1 with $\chi(n) = \mathrm{e}^{2\pi i \mu n}$ and $f(t) = (t+a)^u$, we obtain

$$L_u(x,a,\mu) = a^u + \frac{1}{q} \sum_{j=1}^{q} \mathrm{e}^{2\pi i \mu j} \begin{cases} \frac{(x+a)^{u+1}}{u+1} - \frac{a^{u+1}}{u+1} & \text{if} \quad u \neq -1, \\ \log(x+a) - \log a & \text{if} \quad u = -1, \end{cases}$$
$$+ \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} q^{r-1}$$

$$
\begin{aligned}
&\times \sum_{j=1}^{q} \mathrm{e}^{2\pi i\mu j} \bar{B}_r \left(\frac{x-j}{q}\right) (x+a)^{u-r+1} \\
&- \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} q^{r-1} \\
&\times \sum_{j=1}^{q} \mathrm{e}^{2\pi i\mu j} \bar{B}_r \left(-\frac{j}{q}\right) a^{u-r+1} \\
&+ \frac{(-1)^{l+1}}{l!} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \\
&\times \int_0^x q^{l-1} \bar{B}_l \left(\frac{t-j}{q}\right) (t+a)^{u-l} \mathrm{d}t. \qquad (2.7)
\end{aligned}
$$

Now suppose $\operatorname{Re} u < -1$, and let $x \to \infty$. In this case, the last integral on the right of (2.7) may be written as $\int_0^\infty - \int_x^\infty$, and the left-hand side converges to $\varphi(-u, \mu, a)$, whence we deduce (2.4) or (2.6) as the case may be.

Then substituting (2.4) or (2.6) back into (2.7), we conclude (2.2) or (2.5) as the case may be, thereby noting that in the case $\mu = 0$, (2.7) reads

$$
\begin{aligned}
L_{-1}(x, a, 0) \quad = \quad & \log(x+a) - \psi(a) \\
& - \sum_{r=1}^{l} \frac{1}{r} q^{r-1} \sum_{j=1}^{q} \bar{B}_r \left(\frac{x-j}{q}\right) (x+a)^{-r} \\
& + \int_x^\infty q^{l-1} \sum_{j=1}^{q} \bar{B}_r \left(\frac{t-j}{q}\right) (t+a)^{-l-1} \mathrm{d}t,
\end{aligned}
$$

which reduces to (2.5) in view of the Kubert relation [14, p. 4, (1.8)]

$$
q^{r-1} \sum_{j=1}^{q} \bar{B}_r \left(\frac{x+j}{q}\right) = \bar{B}_r(x).
$$

Formula (2.3) follows on estimating the integral in (2.2). This complete the proof of Theorem 2.

Remark 2. For possible simplification of the coefficients

$$
q^{r-1} \sum_{j=1}^{q} \mathrm{e}^{2\pi i\mu j} \bar{B}_r \left(\frac{x-j}{q}\right),
$$

cf. Remark 3.

Let $\Gamma(s,z)$ denote the incomplete gamma-function of the second kind

$$\Gamma(s,z)=\int_z^\infty e^{-x}x^{s-1}dx, \quad |\arg z|<\pi,$$

for which we refer to [8, Chapter III, p. 133–], [25, p. 277], [7, p. 37–].

We may now prove an Ewald expansion for $\varphi(s,a,\mu)$.

Theorem 3. *For the Hurwitz-Lerch transcendent $\varphi(s,a,\mu)$, $a>0$, with $q>1$, we have*

$$\varphi(s,a,\mu)=a^{-s}+\begin{cases}\frac{a^{1-s}}{s-1}, & \nu=0,\\ 0, & 0<\nu<q,\end{cases}$$
$$+q^{1-s}\sum_{\substack{n=-\infty\\ n\equiv\nu(\bmod q)}}^{\infty}\frac{e^{-\frac{2\pi ina}{q}}}{(-2\pi in)^{1-s}}\Gamma\left(1-s,-\frac{2\pi ina}{q}\right), \tag{2.8}$$

where in the case $q=1$ we must have one more term $-\frac{1}{2}a^{-s}$ on the right of (2.8).

Proof. The case $q=1$ being treated in [14], we may assume $q>1$. Formula (2.4) with $l=1$ and s for $-u$ reads

$$\varphi(s,a,\mu) = \begin{cases}\frac{a^{1-s}}{s-1} & \text{if } \nu=0,\\ 0 & \text{if } 0<\nu<q,\end{cases}$$
$$+a^{-s}\sum_{j=1}^{q}e^{2\pi i\mu j}\bar{B}_1\left(-\frac{j}{q}\right)+\sum_{j=1}^{q}e^{2\pi i\mu j}I_j(s), \tag{2.9}$$

say, where

$$I_j(s)=-s\int_0^\infty \bar{B}_1\left(\frac{t-j}{q}\right)(t+a)^{-s-1}dt.$$

Recall the Eisenstein formula (cf. e.g., [19])

$$\bar{B}_1\left(\frac{a}{q}\right)=-\frac{1}{2q}\sum_{\nu=1}^{q-1}\sin\frac{2ak}{q}\pi\cot\frac{k}{q}\pi$$

whose Fourier inversion is

$$\sum_{j=1}^{q}e^{2\pi i\mu j}\bar{B}_1\left(-\frac{j}{q}\right)=\begin{cases}-\frac{1}{2} & \text{if } \nu=0,\\ \frac{i}{2}\cot\frac{\nu}{q}\pi & \text{if } 0<\nu<q,\end{cases} \tag{2.10}$$

([14, (8.41), p. 178]), which evaluates the second term on the right of (2.8).

We apply the Fourier expansion (1.1) to transform $I_j(s)$ into

$$I_j(s) = -\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\frac{e^{2\pi i\frac{a+j}{q}n}}{n}\left(\frac{1}{z_n}\right)^{-s}(-s)\int_{az_n}^{\infty} e^{-x}x^{-s-1}dx, \qquad (2.11)$$

after a suitable change of symbols, where

$$z_n = -2\pi i\frac{n}{q}.$$

Applying integration by parts to the last integral in (2.11), we find that

$$\begin{aligned} I_j(s) &= -\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\frac{e^{2\pi i\frac{a+j}{q}n}}{n}\left(\frac{1}{z_n}\right)^{-s}\left(-e^{-az_n}(az_n)^{-s}+\Gamma(1-s,az_n)\right) \\ &= -a^{-s}\bar{B}_1\left(-\frac{j}{q}\right)+\frac{1}{q}\sum_{n=-\infty}^{\infty}\frac{e^{2\pi i\frac{a+j}{q}n}}{n}\Gamma(1-s,az_n). \qquad (2.12)\end{aligned}$$

Now we substitute (2.12) into (2.9) to find that

$$\begin{aligned} \varphi(s,a,\mu) &= a^{-s}+\begin{cases}\frac{a^{1-s}}{s-1} & \text{if} \quad \nu=0, \\ 0 & \text{if} \quad 0<\nu<q,\end{cases} \\ &\quad +\frac{1}{q}\sum_{n=-\infty}^{\infty}\frac{e^{-2\pi i\frac{a}{q}n}}{n}\Gamma(1-s,az_n)\sum_{j=1}^{q}e^{2\pi i\frac{\nu-n}{q}j}, \qquad (2.13)\end{aligned}$$

since the terms involving $\sum_{j=1}^{q} e^{2\pi i\mu j}\bar{B}_1\left(-\frac{j}{q}\right)$, which is evaluated in (2.10), cancel each other.

The last term on the right of (2.13) becomes

$$q^{1-s}\sum_{\substack{n=-\infty \\ n\equiv\nu \ (\bmod q)}}^{\infty}\frac{e^{-2\pi i\frac{a}{q}n}}{(-2\pi in)^{1-s}}\Gamma\left(1-s,-2\pi i\frac{n}{q}a\right)$$

in conformity with the corresponding incomplete gamma series in [14, (3.65)], and this gives the last term on the right of (2.8).

Remark 3. In connection with (2.10), it is interesting to consider a more general sum

$$P = P(x) = \sum_{j=1}^{q} e^{2\pi i\frac{\nu}{q}j}\bar{B}_1\left(-\frac{x+j}{q}\right). \qquad (2.14)$$

If we appeal to a (general) partial fraction expansion formula (cf. [3] etc.)

$$\frac{e^{2\pi i w}}{e^{2\pi i w}-1}+\frac{\delta(a)}{2}=\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\frac{e^{-2\pi i a m}}{m+w}, \tag{2.15}$$

valid for all $w \notin \mathbb{Z}$, where $\delta(a)$ is 1 or 0 according as a is 1 or not, then we may evaluate (2.14) as follows. We have

$$\sum_{j=1}^{q} e^{2\pi i \frac{\nu}{q} j}\bar{B}_1\left(-\frac{x+j}{q}\right)=-\frac{1}{2i}\cot\frac{\nu}{q}\pi+\frac{1}{2}-e^{2\pi i \frac{x}{q}\nu}\frac{\delta(x)}{2}. \tag{2.16}$$

Indeed, by the Fourier series for $\bar{B}_1(x)$, we have that

$$P=-\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\frac{1}{n}\sum_{j=1}^{n} e^{2\pi i\frac{\nu-n}{q}j}e^{-2\pi i\frac{x}{q}n}=-\frac{q}{2\pi i}\sum_{\substack{n=-\infty\\ n\equiv\nu\ (\bmod q)}}^{\infty}\frac{1}{n}e^{-2\pi i\frac{x}{q}n}.$$

Writing $n=lq+\nu$, $l\in\mathbb{Z}$, we transform P into

$$P=-\frac{1}{2\pi i}e^{-2\pi i\frac{\nu}{q}j}\sum_{\substack{m=-\infty\\ mq+\nu\neq 0}}^{\infty}\frac{e^{-2\pi i x m}}{m+\frac{\nu}{q}},$$

which becomes by (2.15),

$$P=-\frac{1}{e^{2\pi i\frac{\nu}{q}}-1}-e^{-2\pi i\frac{\nu}{q}x}\frac{\delta(x)}{2}. \tag{2.17}$$

By a simple calculation, we may deduce that

$$-\frac{1}{e^{2\pi i\mu}-1}-\frac{1}{2}=-\frac{1}{2i}\cot\mu\pi. \tag{2.18}$$

Combining (2.17) and (2.18) proves (2.16).

Now we are in a position to state and prove the functional equation for $\phi(s,a,\mu)$ (cf. [8, p. 26 and p. 29]).

THEOREM 4. *For $\sigma<0$, $0<a<1$,*

$$\begin{aligned}\phi(s,a,\mu) &= \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{\frac{\pi}{2}i(1-s)}\sum_{l=0}^{\infty}\frac{e^{-2\pi i a(l+\mu)}}{(l+\mu)^{1-s}}\right.\\ &\left.\quad+e^{-\frac{\pi}{2}i(1-s)}{\sum_{l=0}^{\infty}}'\frac{e^{2\pi i a(l+1-\mu)}}{(l+1-\mu)^{1-s}}\right),\end{aligned} \tag{2.19}$$

where in the latter summation, the prime indicates that the term corresponding to $l=0$ is to be omitted if $\mu=1$.

Proof is similar to the proof of the functional equation in [14, (3.67)] for the Hurwitz zeta-function ($q = 1$ case) using the incomplete gamma-function of the first kind

$$\gamma(s,z) = \int_{\sigma}^{z} \mathrm{e}^{-\mu} u^{s-1} \mathrm{d}u = z^s \int_0^1 \mathrm{e}^{-z\mu} u^{s-1} \mathrm{d}u,$$

and we may assume $q > 1$. Then in (2.8), we replace $\Gamma(1-s, az_n)$ by $\Gamma(1-s) - \gamma(1-s, az_n)$ to deduce that

$$\begin{aligned} \varphi(s,a,\mu) &= a^{-s} + \begin{cases} \frac{a^{1-s}}{s-1} & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad 0 < \nu < q, \end{cases} \\ &\quad -q^{1-s} \sum_{\substack{n=-\infty \\ n \equiv \nu (\bmod q)}}^{\infty} \frac{\mathrm{e}^{-2\pi i \frac{a}{q} n}}{(-2\pi i n)^{1-s}} \gamma\left(1-s, -\frac{2\pi i n a}{q}\right) \\ &\quad +q^{1-s}\Gamma(1-s) \sum_{\substack{n=-\infty \\ n \equiv \nu (\bmod q)}}^{\infty} \frac{\mathrm{e}^{-2\pi i \frac{a}{q} n}}{(-2\pi i n)^{1-s}}. \end{aligned}$$

Then, by the argument similar to that in [14, p. 75], we deduce that

$$\varphi(s,a,\mu) = q^{1-s}\Gamma(1-s) \sum_{\substack{n=-\infty \\ n \equiv \nu (\bmod q)}}^{\infty} \frac{\mathrm{e}^{-\frac{2\pi i n a}{q}}}{(-2\pi i n)^{1-s}}. \tag{2.20}$$

Writing $n = lq + \nu$, $l = 0, 1, \ldots$, for $n > 0$, and $n = (1+l)q - \nu$, $l = 0, 1, \ldots$, for $n < 0$, we see that the right-hand side of (2.20) is just another expansion for that of (2.19). This completes the proof.

Now we introduce two important zeta-functions as special cases of the Hurwitz-Lerch transcendent $\phi(s,a,\mu)$:

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

and

$$l_s(a) = \phi(s,a,1) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2\pi i a n}}{n^s}$$

for $\sigma > 1$, where $\zeta(s,a)$ is called the Hurwitz zeta-function and $l_s(a)$ the polylogarithm function (or the Lerch zeta-function). Their common special case $\zeta(s,1) = l_s(1) = \zeta(s)$ is the Riemann zeta-function

$$\zeta(s) = \sum_{n=}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$.

In this context, Theorem 4 reduces to the functional equation (or the Hurwitz formula).

Corollary 2. *For $0 < a < 1$, we have*

$$\zeta(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left(\mathrm{e}^{\frac{1-s}{2}\pi i} \ell_{1-s}(1-a) + \mathrm{e}^{-\frac{1-s}{2}\pi i} \ell_{1-s}(a) \right). \tag{2.21}$$

For this, cf. e.g., [14, p. 75]. Another natural way to understand (2.21) is to view it as the relation between two bases of the vector space of Kubert functions ([20]).

Corollary 2 further reduces to the asymptotic form of the functional equation for the Riemann zeta-function, which is indeed equivalent to it modulo a Mellin transform (cf. [14, Theorem 5.5, p. 102]).

Corollary 3. *We have*

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Remark 4. For more details on the Lerch zeta-function, cf. [1], [2], [16] and [12]. For zeta-functions, cf. [14] and references therein.

3. The perturbed Dirichlet L-function

Let $q > 2$ be a fixed integer, and let $\mathbb{Z}/q\mathbb{Z}$ denote the quotient ring, consisting of residue classes $a+q\mathbb{Z}, a \in \mathbb{Z}$. Let $(\mathbb{Z}/q\mathbb{Z})^\times$ denote the unit group of $\mathbb{Z}/q\mathbb{Z}$, consisting of $\phi(q)$ reduced residue classes mod q: $a+q\mathbb{Z}, (a,q) = 1$ (the greatest common divisor (a,q) being 1), where $\phi(q)$ is the Euler function counting the number of integers a, $1 \leqslant a \leqslant q$, relatively prime to q. Let χ be a homomorphism $(\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ (non-zero complex numbers). Then the values $\chi(a)$ are the qth root of 1, and so $|\chi(a)| = 1$. We extend the domain of definition of χ by defining $\chi(a) = \chi(a+q\mathbb{Z})$, which is to be 0 if $(a,q) > 1$. The multiplicative function χ thus defined is called a Dirichlet character mod q.

Let χ_0 denote the principal character induced by the trivial character χ_0, $\chi_0(a+q\mathbb{Z})=1$ for all a, $(a,q)=1$.

Let $L(s,\chi)$ denote the Dirichlet L-function associated to χ

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},\quad \sigma>1. \tag{3.1}$$

First we confirm the absolute convergence of the series in (3.1). By (1.4) for $\sigma>1$, $b>1$,

$$\sum_{n\leqslant b}\left|\frac{\chi(n)}{n^s}\right|\leqslant\sum_{n\leqslant b}n^{-\sigma}=1+\int_1^b u^{-\sigma}\mathrm{d}[u],$$

where the second term becomes

$$\left[u^{-\sigma}[u]\right]_1^b+\sigma\int_1^b u^{-\sigma-1}[u]\mathrm{d}u\ll 1+b^{1-\sigma}\to 1$$

as $b\to\infty$. We note that the above proof simultaneously assures the absolute convergence of the Dirichlet series $\sum_{n=0}^{\infty}\frac{\mathrm{e}^{2\pi i\mu n}}{(n+a)^s}$, $a>0$, in (2.1), and *a fortiori*, that of $l_s(x)$ and of $\zeta(s)$ for $\sigma>1$.

To state a theorem on the perturbed Dirichlet L-function

$$L(s,\chi,a)=\sum_{n=1}^{\infty}\frac{\chi(n)}{(n+a)^s},\quad \sigma>0,\quad a>0,$$

we introduce the generalized Bernoulli polynomials first introduced by Leopoldt [18] and developed by Carlitz [6] et al: for a primitive character mod q,

$$B_{n,\chi}(x)=q^{n-1}\sum_{j=1}^{q}\chi(j)B_n\left(\frac{x+j}{q}\right). \tag{3.2}$$

The nth generalized Bernoulli number $B_{n,\chi}$ is defined to be $B_{n,\chi}(0)$ and can be introduced in various ways as ordinary Bernoulli numbers (cf. [14, Chapter I]).

These are as important as ordinary ones in view of the fact that

$$-nL(1-n,\chi)=B_{n,\chi},$$

for $n \in \mathbb{N}$ (for this, cf. e.g. [10, Theorem 1, p. 11]); the corresponding formula is $-n\zeta(1-n,\chi) = \bar{B}_n(x)$, $n \in \mathbb{N}$, (cf. [14, (4.1), p. 77]).

Now note that by the change of variable $q-j \leftrightarrow j$ in (3.2), we see that

$$\chi(-1)\, q^{r-1} \sum_{j=1}^{q} \chi(j) \bar{B}_n\left(\frac{x+a-j}{q}\right) = \chi(-1)\, q^{r-1} \sum_{j=1}^{q} \chi(j) \bar{B}_n\left(\frac{x+a+j}{q}\right),$$

which, by [4, Theorem 3.1, p. 421 and Definition 2, p. 422], is equal to $\chi(-1)\, \bar{B}_{n,\bar{\chi}}(x+a)$, whence

$$q^{r-1} \sum_{j=1}^{q} \chi(j) \bar{B}_n\left(\frac{x+a-j}{q}\right) = \chi(-1)\, \bar{B}_{n,\bar{\chi}}(x+a).$$

We may state a counterpart of Theorem 2.

THEOREM 5. *Let χ be a primitive Dirichlet character* mod$q$. *Then, for the partial sum $L_u(x,a,\chi) = \sum_{n \leqslant x} \chi(n)(n+a)^u$ of the perturbed Dirichlet L-function,*

$$L(s,a,\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{(n+a)^s}, \quad a>0, \quad \sigma>0,$$

the following are valid:

i) *We have the integral representation*

$$\begin{aligned} L_u(x,a,\chi) &= L(-u,a,\chi) \\ &\quad + \chi(-1) \sum_{r=1}^{l} \frac{(-1)^r}{r!} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \\ &\quad \times \bar{B}_{r,\bar{\chi}}(x+a)\,(x+a)^{u-r+1} \\ &\quad + \chi(-1) \frac{(-1)^l}{l!} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \\ &\quad \times \int_x^{\infty} \bar{B}_{l,\chi}(t+a)\,(t+a)^{u-l}\mathrm{d}t, \end{aligned} \tag{3.3}$$

and the asymptotic formula

$$\begin{aligned} L_u(x,a,\chi) &= L(-u,a,\chi) \\ &\quad + \chi(-1) \sum_{r=1}^{l} \frac{(-1)^r}{r!} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \end{aligned}$$

$$\times\bar{B}_{r,\bar{\chi}}\,(x+a)\,(x+a)^{u-r+1}$$
$$+\mathrm{O}\left(q^{l-1}(x+a)^{\mathrm{Re}\,u-l}\right), \tag{3.4}$$

where $l>\mathrm{Re}\,u+1$ *and* $l\in\mathbb{N}$.

ii) *We have the integral representation for the perturbed Dirichlet* L*-function*

$$\begin{aligned} L(-u,a,\chi) &= -\chi\,(-1)\sum_{r=1}^{l}\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+2-r)} \\ &\quad\times\bar{B}_{r,\bar{\chi}}\,(x+a)\,(x+a)^{u-r+1} \\ &\quad+\chi\,(-1)\,\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)} \\ &\quad\times\int_x^\infty \bar{B}_{l,\chi}\,(t+a)\,(t+a)^{u-l}\mathrm{d}t, \end{aligned} \tag{3.5}$$

where the last integral is absolutely convergent for $\mathrm{Re}\,(-u)>1-l$ *and boundedly convergent for* $\mathrm{Re}\,(-u)>-l$.

Proof. Proof follows on the same lines as those of the proof of Theorem 2. Choosing $a=1$, $b=x\geqslant 1$ and $f\,(t)=(t+a)^u$ in Corollary 1, we obtain that

$$\begin{aligned} L_u(x,a,\chi) &= \chi\,(-1)\sum_{r=1}^{l}\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\left[\bar{B}_{r,\bar{\chi}}\,(t+a)\right]_0^x \\ &\quad+\chi\,(-1)\,\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)} \\ &\quad\times\int_x^\infty \bar{B}_{l,\bar{\chi}}\,(t+a)\,(t+a)^{u-l}\mathrm{d}t, \end{aligned}$$

after simplification.

By [4, (5.28)], $\bar{B}_{l,\bar{\chi}}\,(x)$ is uniformly bounded, and so, on letting $x\to\infty$ in the region $\mathrm{Re}\,(u)<-1$, we arrive at (3.6) as in the proof of Theorem 2.

Other results can be read off from (3.6), completing the proof.

Theorem 6. *Let* χ *be a primitive Dirichlet character* mod q. *Then for the partial sum* $L_u(x,\chi)=\sum_{n\leqslant x}\chi\,(n)\,n^u$ *of the Dirichlet* L*-function* (3.1), *the following are valid:*

i) *We have the integral representation*

$$\begin{aligned} L_u(x,\chi) &= L(-u,\chi) \\ &+\chi(-1)\sum_{r=1}^{l}\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\bar{B}_{r,\bar{\chi}}(x+a)(x)^{u-r+1} \\ &+\chi(-1)\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\int_x^\infty \bar{B}_{l,\chi}(t)\,t^{u-l}\mathrm{d}t, \end{aligned}$$

and the asymptotic formula

$$\begin{aligned} L_u(x,\chi) &= L(-u,\chi) \\ &+\chi(-1)\sum_{r=1}^{l}\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\bar{B}_{r,\bar{\chi}}(x)\,x^{u-r+1} \\ &+\mathrm{O}\left(q^{l-1}x^{\mathrm{Re}\,u-l}\right), \end{aligned} \tag{3.6}$$

where $l > \mathrm{Re}\,u + 1$ *and* $l \in \mathbb{N}$.

ii) *We have the integral representation for the Dirichlet* L*-function*

$$\begin{aligned} L(-u,\chi) &= -\chi(-1)\sum_{r=1}^{l}\frac{(-1)^r}{r!}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\bar{B}_{r,\bar{\chi}} \\ &\quad+\chi(-1)\frac{(-1)^l}{l!}\frac{\Gamma(u+1)}{\Gamma(u+1-l)} \\ &\quad\times\int_x^\infty \bar{B}_{l,\chi}(t)\,t^{u-l}\mathrm{d}t, \end{aligned} \tag{3.7}$$

the last integral being absolutely convergent for $\mathrm{Re}(-u) > 1-l$ *and boundedly convergent for* $\mathrm{Re}(-u) > -l$.

Remark 5. From the integral representations (3.5) and (3.7), we may deduce the Ewald expansion for $L(s,\chi,a)$ and $L(s,\chi)$, respectively. These are similar to, what Lavrik describes, Kuz'min's equation (or the Hardy-Littlewood integral). The approximate functional equations (3.4) and (3.6) are of the Hardy-Littlewood integral type. For more details, cf. [17, pp. 113–114].

For a more general perturbed Lerch L-function

$$L(s,a,\chi,\mu) = \sum_{n=0}^{\infty}\frac{\chi(n)\,\mathrm{e}^{2\pi i\mu z}}{(n+a)^s}, \quad \sigma > 1, \quad \mu = \frac{\nu}{q},$$

cf. [13, (1.5), etc.].

For the hitherto most general Hurwitz-Lerch L-function $L\,(s,\chi,b,a)$ defined by

$$L\,(s,a,\chi,\mu)=\sum_{n=0}^{\infty}\frac{\chi\,(n)\,b^n}{(n+a)^s},\quad \sigma>1,\quad |b|\leqslant 1,$$

cf. [14] or [22].

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