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VALUE DISTRIBUTION OF SOME NON-CLASSICAL ZETA-FUNCTIONS

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Abstract. This paper is a survey of author's thesis [7], where the value-distribution of some non-classical zeta-functions is considered. The main attention is devoted to the universality property of the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$. Also, a bicomplex Hurwitz zeta-function is defined for the first time and convergence of its series and analytical continuation are investigated. Moreover, in the thesis we provide a link between the Euclidean action and the geometric zeta-function. Also, a distribution of poles of the geometric zeta-function is investigated.

Key words and phrases: bicomplex numbers, bicomplex zeta-function, Hurwitz zeta-function, non-classical zeta-functions, periodic zeta-function, universality.

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Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, non-negative integers, real and complex numbers, respectively. Denote by $\mathfrak{a} = \{a_m : a_m \in \mathbb{C}, m \in \mathbb{N}_0\}$ a periodic with the minimal period $k \in \mathbb{N}$ sequence. Let $s = \sigma + it$ be a complex variable. The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$, for $\sigma > 1$, is defined by

$$\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

where $0 < \alpha \leqslant 1$ is a fixed real number. If $k = 1$ and $a_0 = 1$, we obtain the classical Hurwitz zeta-function

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1.$$

If $\alpha = 1$, then the function $\zeta(s, \alpha; \mathfrak{a})$ reduces to the periodic zeta-function

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_{m-1}}{m^s}, \quad \sigma > 1.$$

It is shown in the thesis that the function $\zeta(s, \alpha; \mathfrak{a})$ can be analytically continued to the whole complex plane, except, maybe, for a simple pole $s = 1$ with residue

$$a = \frac{1}{k} \sum_{r=0}^{k-1} a_r.$$

If $a = 0$, then $\zeta(s, \alpha; \mathfrak{a})$ is an entire function.

The universality is one of the most important and interesting properties in the theory of zeta and L-functions. S. M. Voronin was the first who discovered [29] the universality of the Riemann zeta-function $\zeta(s)$. Let $0 < r < \frac{1}{4}$, and suppose that $f(s)$ is a continuous non-vanishing on the disc $\{s \in \mathbb{C} : |s| \leqslant r\}$ function which is analytic in the interior of this disc. Then he proved that, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leqslant r} \left| \zeta \left(s + \frac{3}{4} + i\tau \right) - f(s) \right| < \varepsilon.$$

Later, the Voronin theorem was improved and generalized for other zeta and L-functions as well as for some classes of Dirichlet series by B. Bagchi, H. Bauer, V. Garbaliauskienė, R. Garunkštis, J. Genys, A. Good, S. M. Gonek, J. Ignatavičiūtė, R. Kačinskaitė, A. Laurinčikas, K. Matsumoto, R. Macaitienė, H. Mishou, H. Nagoshi, A. Reich, W. Schwarz, J. Steuding, R. Šleževičienė and others. There exists the Linnik-Ibragimov conjecture that all functions in some half-planes given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some natural growth conditions are universal in the Voronin sense. Now we know sufficiently many results which support the Linnik-Ibragimov conjecture.

The last version of the Voronin theorem is the following. Let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \ldots\},$$

where $\text{meas}\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and in place of dots a condition satisfied by τ is to be written. Then we have the following statement, see [14].

THEOREM A. *Suppose that K is a compact subset of the strip $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau) - f(s))| < \varepsilon \right) > 0.$$

So, roughly speaking, by the Voronin theorem, any analytic function is approximated uniformly on compact subsets by shifts of the Riemann zeta-function.

Later, the universality of the Hurwitz zeta-function $\zeta(s,\alpha)$ with transcendental or rational parameter α was obtained in [4] and [1], see, also [20]. Let us state the theorem from [20] when α is transcendental.

THEOREM A_1. *Suppose α is transcendental. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s))| < \varepsilon \right) > 0.$$

Note that the Hurwitz zeta-function with transcendental parameter α has no the Euler product over primes, therefore, in Theorem A_1 the function $f(s)$ is not necessarily non-vanishing.

The first attempt to obtain the universality of the periodic zeta-function $\zeta(s;\mathfrak{a})$ was made in [27]. Let us formulate the main result from the thesis [27].

THEOREM A_2. *Suppose that k is an odd prime, a_m is not a multiple of a character* mod k*, and $a_k = 0$. Let K and $f(s)$ be the same as in* Theorem A_1*. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s))| < \varepsilon \right) > 0.$$

The investigations of the universality for $\zeta(s;\mathfrak{a})$ were continued in [23], and the following results were obtained. We recall that the sequence $\mathfrak{a}$ is completely multiplicative if $a_1 = 1$ and, for all $m, n \in \mathbb{N}$,

$$a_{mn} = a_m a_n,$$

and it is multiplicative if the latter equality holds for all coprimes $m, n \in \mathbb{N}$. Denote by p a prime number.

THEOREM A_3. *Suppose that the sequence* $\mathfrak{a}$ *is completely multiplicative. Let* K *and* $f(s)$ *be the same as in* Theorem A. *Then the assertion of* Theorem A *is true for the function* $\zeta(s;\mathfrak{a})$.

A similar result is also true in the case of multiplicative sequence $\mathfrak{a}$.

THEOREM A_4. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative, and*

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\frac{\alpha}{2}}} \leqslant c_2 < 1.$$

Let K *and* $f(s)$ *be the same as in* Theorem A. *Then the assertion of* Theorem A *is true for the function* $\zeta(s;\mathfrak{a})$.

In the case of a non-multiplicative sequence $\mathfrak{a}$, a conditional universality of the function $\zeta(s;\mathfrak{a})$ was obtained in [23].

In the thesis, first we obtain the universality for one class of periodic Hurwitz zeta-functions.

THEOREM 1. *Suppose that* α *is a transcendental number and* $\min\limits_{0\leqslant m\leqslant k-1} |a_m| > 0$. *Let* K *be a compact subset of the strip* $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ *with connected complement, and let* $f(s)$ *be a continuous function on* K *which is analytic in the interior of* K. *Then, for any* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \right) > 0.$$

The proof of Theorem 1 is sufficiently simple [8]. To prove Theorem 1 we need a limit theorem. For this purpose, we introduce some notation.

Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S.

Define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where γ_m is the unit circle $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ for every $m \in \mathbb{N}_0$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m. Since the Haar measure m_H is the product of the Haar measures on the coordinate spaces γ_m, $\{\omega(m) : m \in \mathbb{N}_0\}$ is a sequence of independent complex-valued random variables uniformly distributed on the circle γ. For $\sigma > \frac{1}{2}$, define

$$\zeta(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

It is easy to prove that $\zeta(s, \alpha, \omega; \mathfrak{a})$ is an $H(D)$-valued random element. We denote by P_ζ the distribution of $\zeta(s, \alpha, \omega; \mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H\left(\omega \in \Omega \,:\, \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Now we can formulate a limit theorem.

THEOREM 2. *Let α be transcendental. Then the probability measure*

$$P_T(A) = \nu_T\left(\zeta(s + i\tau, \alpha; \mathfrak{a}) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ as $T \to \infty$.

After we had obtained Theorem 1, we asked ourselves: Is it possible to remove a condition $\min\limits_{1 \leqslant m \leqslant k} |a_m| > 0$ from Theorem 1? And the answer was positive [9].

THEOREM 3. *Suppose that α is a transcendental number. Let K and $f(s)$ be the same as in* Theorem 1. *Then for the function $\zeta(s, \alpha; \mathfrak{a})$ the assertion of* Theorem 1 *holds.*

A more complicated problem is the joint approximation of a collection of analytic functions by shifts of zeta or L-functions. S. M. Voronin was also the first who observed [30] the joint universality of Dirichlet L-functions $L(s, \chi)$.

Let us recall some definitions from the Dirichlet character theory. Let χ be a character modulo k, and $k|l$. Then the character χ can be extended to a character χ_1 modulo l defined by

$$\chi_1(m) = \begin{cases} 0 & \text{if} \quad (m, l) > 0, \\ \chi(m) & \text{if} \quad (m, l) = 1. \end{cases}$$

In this case, we say that the character χ_1 is generated by the character χ modulo k. A character χ_1 modulo l is called primitive if it is not generated by any character modulo k, $k < l$. Two characters are said to be equivalent if they are generated by the same primitive character. Otherwise, they are called non-equivalent characters.

We recall a modern version of a theorem from [30].

THEOREM B. *Let $\chi_1, \dots, \chi_r$ be pairwise non-equivalent Dirichlet characters, $K_1, \dots, K_r$ be compact subsets of the strip $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements, and let, for each $j = 1, \dots, r$, $f_j(s)$ be a continuous non-vanishing function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon \right) > 0.$$

The joint universality of Dirichlet L-functions independently also was obtained by S. M. Gonek [4] and B. Bagchi [1], [2]. In the proof of Theorem B, a certain independence of L-functions based on the orthogonality of Dirichlet characters is used essentially. Therefore, this way can not be applied to obtain the joint universality for other functions.

The joint universality for Lerch zeta-functions with some parameters was obtained in [20]. Some results in the field for Matsumoto zeta-functions are given in [15]. A conditional analogue of Theorem B for zeta-functions of certain cusp forms is proved in [21]. In [16], [17] and [3] the joint universality of general Dirichlet series has been investigated. The best results on generalization of Theorem B are related to twists with Dirichlet characters of some Dirichlet series. This case for Dirichlet series with multiplicative coefficients is considered in [28], while [22] is devoted to automorphic L-functions. We state the main result of [22].

THEOREM C. *Let $F(z)$ be a holomorphic normalized Hecke-eigen new form of weight κ with respect to*

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 (\bmod M) \right\},$$

where M is a positive integer, and let the Fourier expansion of $F(z)$ be given by

$$F(z) = \sum_{n=1}^{\infty} c(n) \mathrm{e}^{2\pi i n z}, \quad c(1) = 1.$$

Let q_j be positive integers prime to M, $1 \leqslant j \leqslant m$, χ_j be pairwise non-equivalent Dirichlet characters mod q_j, $1 \leqslant j \leqslant m$, *and define the twisted L-functions*

$$L_j(s,F)=\sum_{n=1}^{\infty} c(n)\chi_j(n)n^{-s},$$

which can be continued to the whole complex plane. For each j, let K_j be compact subset of the strip $\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$ with connected complement. Let, for each $j = 1, \ldots, r$, $f_j(s)$ be a continuous non-vanishing function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{1\leqslant j\leqslant m}\sup_{s\in K_j} |L_j(s+i\tau,F)-f_j(s)|<\varepsilon\right)>0.$$

The first results on the joint universality for periodic Hurwitz zeta-functions were obtained in [18] and [19]. Let $\mathfrak{a}_j = \{a_{mj} : m \in \mathbb{N}_0\}$ be a periodic with the least period $k_j \in \mathbb{N}$ sequence of complex numbers, α_j, $0 < \alpha_j \leqslant 1$, be a fixed number, and, for $\sigma > 1$,

$$\zeta(s,\alpha_j;\mathfrak{a}_j)=\sum_{m=0}^{\infty}\frac{a_{mj}}{(m+\alpha_j)^s}, \quad j=1,\ldots,r.$$

In [18], the case of $k_1 = \ldots = k_r = k$ and $\alpha_1 = \ldots = \alpha_r = \alpha$ has been considered. Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1r}\\ a_{21} & a_{22} & \ldots & a_{2r}\\ \ldots & \ldots & \ldots & \ldots\\ a_{k1} & a_{k2} & \ldots & a_{kr}\end{pmatrix}.$$

Denote, for brevity, $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$.

Theorem D. *Suppose that α is transcendental and* rank$\,(A) = r$. *Let, for each $j = 1, \ldots, r$, K_j be a compact subset of the strip D with connected complement, and let $f_j(s)$ be a continuous on K_j function which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j} |\zeta(s+i\tau,\alpha;\mathfrak{a}_j)-f_j(s)|<\varepsilon\right)>0.$$

In [19], a similar result has been obtained for periodic Hurwitz zeta-functions with different periods k_j. In the thesis, we obtain the joint universality for $\zeta(s,\alpha_1;\mathfrak{a}_1),\ldots,\zeta(s,\alpha_r;\mathfrak{a}_r)$ with different both k_j and α_j, $j =$

$1, \ldots, r$. We recall that the numbers $a_1, \ldots, a_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$ if the coefficients of every polynomial p with rational coefficients such that $p(a_1, \ldots, a_r) = 0$ are zeros. For example, the numbers $(\frac{1}{2})^{\sqrt[3]{2}}$ and $(\frac{1}{2})^{\sqrt[3]{4}}$ are algebraically independent over $\mathbb{Q}$. Denote by k the least common multiple of $k_1, \ldots, k_r$.

THEOREM 4. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(A) = r$. *Let $K_1, \ldots, K_r$ and $f_1(s), \ldots, f_r(s)$ be the same as in* Theorem D. *Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon \right) > 0.$$

To prove Theorem 4, we need a joint limit theorem in the sense of weak convergence of probability measures in the space of analytic functions for the functions $\zeta(s, \alpha_1; \mathfrak{a}_1), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_r)$. Let $H^r(D) = \underbrace{H(D) \times \ldots \times H(D)}_{r}$. We define $\Omega^r = \Omega_1 \times \ldots \times \Omega_r$, where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Then Ω^r is a compact topological Abelian group. Denote by m_{Hr} the probability Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$, and on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_{Hr})$ define an $H^r(D)$-valued random element $\zeta(s, \boldsymbol{\omega})$ by

$$\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{a}}) = (\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_1), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_r)),$$

where

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_j(m)}{(m+\alpha_j)^s},$$

$\omega_j \in \Omega$, $j = 1, \ldots, r$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\boldsymbol{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$. Let $P_{\boldsymbol{\zeta}}$ denote the distribution of the random element $\boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{a}})$, i.e.,

$$P_{\boldsymbol{\zeta}}(A) = m_{Hr}(\boldsymbol{\omega} \in \Omega^r : \boldsymbol{\zeta}(s, \boldsymbol{\alpha}, \boldsymbol{\omega}; \boldsymbol{\mathfrak{a}}) \in A), \quad A \in \mathcal{B}(H^r(D)),$$

and define the probability measure

$$P_T(A) = \nu_T ((\zeta(s+i\tau, \alpha_1; \mathfrak{a}_1), \ldots, \zeta(s+i\tau, \alpha_r; \mathfrak{a}_r)) \in A), \quad A \in \mathcal{B}(H^r(D)).$$

Now we can formulate a joint limit theorem.

THEOREM 5. *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measure P_T converges weakly to $P_{\boldsymbol{\zeta}}$ as $T \to \infty$.*

We proceed the paper with a generalization of Hurwitz zeta-function over bicomplex numbers. In [25], D. Rochon defined the first bicomplex zeta-function, i.e., the bicomplex Riemann zeta-function.

The history of bicomplex numbers began in 1882, when Corrado Segre had published a paper [26]. There he treated an infinite family of algebras whose elements are called bicomplex numbers, tricomplex numbers,... , n-complex numbers. Let us define bicomplex numbers as follows:

$$\begin{aligned}\boldsymbol{T} \ := \ &\{a+b\mathbf{i_1}+c\mathbf{i_2}+d\mathbf{j} : \mathbf{i_1}^2=\mathbf{i_2}^2=-1,\, \mathbf{j}^2=1,\\ &\mathbf{i_2}\mathbf{j}=\mathbf{j}\mathbf{i_2}=-\mathbf{i_1},\, \mathbf{i_1}\mathbf{j}=\mathbf{j}\mathbf{i_1}=-\mathbf{i_2},\, \mathbf{i_2}\mathbf{i_1}=\mathbf{i_2}\mathbf{j}=\mathbf{j}\},\end{aligned}$$

where $a, b, c, d \in \mathbb{R}$.

Let $z_1, z_2 \in \mathbb{C}_1 := \{x + y\mathbf{i_1} : \mathbf{i_1}^2 = -1\}$ and $\omega = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ with $\mathrm{Re}(z_1) > 1$ and $|\mathrm{Im}(z_2)| <\mathrm{Re}(z_1) - 1$. We can write a bicomplex number $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}$ as $(a + b\mathbf{i_1}) + (c + d\mathbf{i_1})\mathbf{i_2} = z_1 + z_2\mathbf{i_2}$, where $z_1, z_2 \in \mathbb{C}_1 := \{x + y\mathbf{i_1} : \mathbf{i_1}^2 = -1\}$.

It is easy to see [24] that $\boldsymbol{T}$ is a commutative unitary ring with the following characterization for the non-invertible elements. Let $\omega = a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} \in \boldsymbol{T}$. Then ω is non-invertible if and only if $a = -d$ and $b = c$ or $a = d$ and $b = -c$ [24].

Let $\omega = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ with $\mathrm{Re}\,(z_1) > 1$ and $|\mathrm{Im}\,(z_2)| < \mathrm{Re}\,(z_1) - 1$, then the bicomplex Riemann zeta-function $\zeta(\omega)$ is defined by the following convergent series

$$\zeta(\omega) = \sum_{n=1}^{\infty} \frac{1}{n^{\omega}}.$$

In the thesis, we define the bicomplex Hurwitz zeta-function and investigate its basic properties.

Firstly, following the standard procedure let us define the derivative of the bicomplex function f at a point ω_0 as the limit

$$f'(\omega_0) = \lim_{\omega \to \omega_0} \frac{f(\omega) - f(\omega_0)}{\omega - \omega_0}.$$

A problem arises, because the fraction $\frac{f(\omega)-f(\omega_0)}{\omega-\omega_0}$ is not defined for each non-invertible $\omega - \omega_0$ in the neighbourhood where the limit is taken. An obvious remedy is to confine limit taking only to invertible $\omega - \omega_0$. So, we have that

$$f'(\omega_0) = \lim_{\substack{\omega \to \omega_0 \\ \omega-\omega_0 \text{ is invertible}}} \frac{f(\omega) - f(\omega_0)}{\omega - \omega_0}.$$

Let $\omega = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ with $\mathrm{Re}\,(z_1) > 1$ and $|\mathrm{Im}\,(z_2)| < \mathrm{Re}\,(z_1) - 1$. We define a bicomplex Hurwitz zeta-function $\zeta(\omega, \alpha)$ by the following convergent series

$$\zeta(\omega, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^{\omega}}.$$

This definition is justified by the following theorem.

THEOREM 6. *Let* $\omega = z_1 + z_2\mathbf{i_2} \in \boldsymbol{T}$ *with* $\mathrm{Re}\,(z_1 - z_2\mathbf{i_1}) > 1$ *and* $\mathrm{Re}\,(z_1 + z_2\mathbf{i_1}) > 1$. *Then the series*

$$\sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^{\omega}}$$

converges, and

$$\sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{\omega}} = \left(\sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{z_1 - z_2\mathbf{i_1}}}\right)\mathbf{e_1} + \left(\sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{z_1 + z_2\mathbf{i_1}}}\right)\mathbf{e_2}.$$

Let us define a region $1 + \mathcal{O}_2$ in such a way that $\omega \in 1 + \mathcal{O}_2$ if and only if $z_1 - z_2\mathbf{i_1} = 1$ or $z_1 + z_2\mathbf{i_1} = 1$ [25]. Then we prove the following theorem.

THEOREM 7. *The analytic continuation of*

$$\zeta(\omega, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^{\omega}}$$

on $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$ *is unique.*

We end the investigation of the bicomplex Hurwitz zeta-function with a theorem which confirms that the domain $\boldsymbol{T}\backslash\{1 + \mathcal{O}_2\}$ is the best.

THEOREM 8. *Let* $\omega_0 \in 1 + \mathcal{O}_2$, *then*

$$\lim_{\substack{\omega \to \omega_0 \\ \omega \notin 1 + \mathcal{O}_2}} |\zeta(\omega, \alpha)| = \infty.$$

The proofs and details of the Theorems 6, 7 and 8 can be found in [6].

Our next step is an investigation of other non-classical zeta-function, i.e., the geometric zeta-function. Investigation of the geometric zeta function was begun by M. L. Lapidus and M. van Frankenhuysen in their papers [10], [11] and [12]. In their book [13], the authors developed a theory of distribution of complex dimensions (they can be viewed as poles of a certain zeta-function) of a fractal string. A fractal string L is a bounded open subset of $\mathbb{R}$, which consists of countably many open intervals, the lengths of which are denoted by $l_1 \geqslant l_2 \geqslant ... > 0$, and $l_j^{-1} \in \mathbb{N}$, $j = 1, 2, ...$. Then the geometric zeta-function $\zeta_L(s)$ of the fractal string L is given by

$$\zeta_L(s) = \sum_{j=1}^{\infty} l_j^s.$$

In the thesis, we find a way how the Euclidean action which is widely used in physics could be investigated through the distribution of complex dimensions of its geometric zeta function (see [5]).

To investigate the distribution of poles of the geometric zeta-function, we must generalize a concept of a fractal string and the geometric zeta-function. We begin with several definitions. The screen S is the contour

$$S(t) = r(t) + it, \quad t \in \mathbb{R},$$

with some continuous function $r : \mathbb{R} \to [-\infty, D_L]$ where the quantity D_L is called the dimension of the fractal string L and is defined by

$$D_L = \inf \left\{ \sigma > 0 : \sum_{j=1}^{\infty} l_j^{\sigma} < \infty \right\}.$$

The set

$$W = \{ s \in \mathbb{C} : \sigma \geqslant r(t) \}$$

is called the window.

Given a complex measure η, there exists a positive measure denoted by $|\eta|$ which measures the total variation of η

$$|\eta|(J) = \sup \sum_i |\eta(J_i)|,$$

where the supremum is taken over all partitions $\cup J_i$ of J into measurable subsets J_i. The measure $|\eta|$ is called the total variation measure associated with η. Recall that $|\eta| = \eta$ if η is positive.

A local positive measure is just a standard positive Borel measure on $(0, \infty)$ which satisfies the following local boundedness condition: for all bounded subintervals J of $(0, \infty)$,

$$\eta(J) < \infty.$$

More generally, we will say that a set function η on $(0, \infty)$ is a local complex measure on $(0, \infty)$ if following conditions are satisfied: (i) $\eta(A)$ is well defined for any Borel subset A of $[a,b]$, and (ii) the restriction of η to the Borel subsets of $[a,b]$ is a complex measure on $[a,b]$ in the traditional sense.

Now we can define a generalized fractal string. It is either a local complex, or a local positive measure η on $(0,\infty)$ such that

$$|\eta|(0,x_0)=0$$

for some positive number x_0. The dimension of η, denoted by $D=D_\eta$, is

$$D=D_\eta=\inf\left\{\sigma\in\mathbb{R}:\int_0^\infty x^{-\sigma}|\eta|(\mathrm{d}x)<\infty\right\}.$$

Then the geometric zeta-function $\zeta_\eta(s)$ of η is given, for $\sigma>D_\eta$, by

$$\zeta_\eta(s)=\int_0^\infty x^{-s}\eta(\mathrm{d}x).$$

For the investigation of dimensions of the geometric zeta-functions, we use a formula which describes η as a distribution. On a function φ, which we call a test function, η acts by

$$\langle\eta,\varphi\rangle=\int_0^\infty\varphi(x)\,\eta(\mathrm{d}x).$$

The kth primitive of this distribution is denoted by $P^{[k]}\eta$. More precisely, $P^{[k]}\eta$ is the distribution given for all test functions φ by

$$\langle P^{[k]}\eta,\varphi\rangle=\int_0^\infty\int_y^\infty\frac{(x-y)^{k-1}}{(k-1)!}\varphi(x)\mathrm{d}x\;\eta(\mathrm{d}y).$$

The geometric partition function $\theta_L(\tau)$ of an ordinary fractal string $L=(l_j)_{j=1}^\infty$ is usually defined, for $\tau>0$, by

$$\theta_L(\tau)=\sum_{j=1}^\infty\mathrm{e}^{-\tau l_j^{-1}}.$$

We deal with geometric partition function $\theta_\eta(t)$ for the generalized fractal string η. It is given by

$$\theta_\eta(t)=\int_0^\infty\mathrm{e}^{-xt}\eta(\mathrm{d}x)=\langle P^{[0]}\eta,\varphi_t\rangle,$$

where $\varphi_t(x) = \mathrm{e}^{-xt}$ for $t > 0$. Whereas we investigate some generalizations of the geometric partition function for the generalized fractal string.

First, let

$$p(x) = \frac{\mathrm{e}^{\pi\sqrt{\frac{2x}{3}}}}{4\sqrt{3}x},$$
$$q(x) = \frac{\mathrm{e}^{\pi\sqrt{\frac{x}{3}}}}{4\cdot 3^{\frac{1}{4}}x^{\frac{3}{4}}}.$$

We define geometric partition functions of the generalized fractal string η by

$$\theta_{\eta,p}(t) = \int\limits_0^\infty \frac{\mathrm{e}^{-xt+\pi\sqrt{\frac{2x}{3}}}}{4\sqrt{3}x}\eta(\mathrm{d}x) = \langle P^{[0]}\eta, \varphi_{t,p}\rangle$$

and

$$\theta_{\eta,q}(t) = \int\limits_0^\infty \frac{\mathrm{e}^{-xt+\pi\sqrt{\frac{x}{3}}}}{4\cdot 3^{\frac{1}{4}}x^{\frac{3}{4}}}\eta(\mathrm{d}x) = \langle P^{[0]}\eta, \varphi_{t,q}\rangle,$$

where $\varphi_{t,p}(x) = p(x)\mathrm{e}^{-xt}$ and $\varphi_{t,q}(x) = q(x)\mathrm{e}^{-xt}$, for $t > 0$.

Our purpose is to find distributional formulas for generalized geometric partition functions. Let us denote by ($\mathbf{H}_1$) a polynomial growth condition along horizontal lines of the complex plane (necessary by avoiding the poles of ζ_η), while let ($\mathbf{H}_2$) be a polynomial growth condition along the vertical direction of the screen. More precisely, there exists real constants $\kappa > 0$ and $C > 0$, and a sequence $\{T_n\}_{n\in\mathbb{Z}}$ of real numbers tending to $\pm\infty$ as $n \to \pm\infty$, with $T_{-n} < 0 < T_n$, for $n \geqslant 1$, and $\lim\limits_{n\to+\infty} \frac{T_n}{|T_{-n}|} = 1$, such that:

($\mathbf{H}_1$) for all $n \in \mathbb{Z}$ and all $\sigma \geqslant r(T_n)$,

$$|\zeta_\eta(\sigma + iT_n)| \leqslant C|T_n|^\kappa;$$

($\mathbf{H}_2$) for all $t \in \mathbb{R}$, $|t| \geqslant 1$,

$$|\zeta_\eta(r(t) + it)| \leqslant C|t|^\kappa,$$

where r is the Lipschitz continuous function, i.e., there exists a non-negative real number $\|r\|_{Lip}$ such that $|r(x) - r(y)| \leqslant \|r\|_{Lip}|x - y|$ for all $x, y \in \mathbb{R}$, which bounds the screen S.

Let, as usual, $\Gamma(s)$ denote the Euler gamma-function. Assume that a, b are complex numbers independent on the variable z. Then the differential equation

$$z(1-z)\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + (b - (a+1)z)\frac{\mathrm{d}u}{\mathrm{d}z} - au = 0$$

is called a hypergeometric equation.

If $b \neq -m$, $m \in \mathbb{N} \cup 0$, then the function

$$u = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b)}{\Gamma(a)\Gamma(b+m)m!} z^m \overset{def}{=} {}_1F_1(a;b;z)$$

is a regular solution of the hypergeometric equation at the point $z = 0$, and the function ${}_1F_1(a;b;z)$ is called the hypergeometric function with parameters a and b.

We shall denote by $\widetilde{\varphi}$ the Mellin transform of a (suitable) function φ on $(0,\infty)$, it is defined by

$$\widetilde{\varphi}(s) = \int_0^{\infty} \varphi(s) x^{s-1} \mathrm{d}x, \quad s \in \mathbb{C}.$$

Now we are ready to write distribution formulas for generalized geometric partition functions.

THEOREM 9. *Let η be a generalized fractal string satisfying* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *and let $\varphi_{\tau,q}$ be a test function given by $\varphi_{t,q}(x) = q(x)\mathrm{e}^{-tx}$. Then the geometric partition function $\theta_{\eta,q}(\tau)$ is given by*

$$\begin{aligned}
\theta_{\eta,q}(\tau) \;=\; & \sum_{\omega \in D_\eta(W)} \operatorname{Res}\left(\zeta_\eta(s)\widetilde{\varphi}_{\tau,q}(s);\omega\right) \\
& + \frac{1}{4 \cdot 3^{\frac{1}{4}}} \sum_{\substack{k=1 \\ \frac{3}{4}-k \in W \setminus D_\eta}}^{\infty} \frac{(-1)^k \tau^k}{k!} \zeta_\eta\left(\frac{3}{4} - k\right) {}_1F_1\left(-k; \frac{1}{2}; \frac{\pi^2}{12\tau}\right) \\
& + \frac{\pi}{4 \cdot 3^{\frac{3}{4}}} \sum_{\substack{l=1 \\ \frac{1}{4}-l \in W \setminus D_\eta}}^{\infty} \frac{(-1)^l \tau^l}{l!} \zeta_\eta\left(\frac{1}{4} - l\right) {}_1F_1\left(-l; \frac{3}{2}; \frac{\pi^2}{12\tau}\right) \\
& + \langle R_\eta^{[0]}, \varphi_{\tau,q} \rangle, \qquad (1)
\end{aligned}$$

where, for $\operatorname{Re} s > \frac{3}{4}$ *and* $\tau > 0$,

$$\begin{aligned}
\widetilde{\varphi}_{\tau,q}(s) \;=\; & \frac{\tau^{\frac{1}{4}-s}}{4 \cdot 3^{\frac{1}{4}}} \left\{ \sqrt{\tau}\,\Gamma\left(s - \frac{3}{4}\right) {}_1F_1\left(s - \frac{3}{4}; \frac{1}{2}; \frac{\pi^2}{12\tau}\right) \right. \\
& \left. + \frac{\pi}{\sqrt{3}} \Gamma\left(s - \frac{1}{4}\right) {}_1F_1\left(s - \frac{1}{4}; \frac{3}{2}; \frac{\pi^2}{12\tau}\right) \right\} \qquad (2)
\end{aligned}$$

and $\langle R_\eta^{[0]}, \varphi_{\tau,q} \rangle$ is the term given by

$$\langle R_\eta^{[0]}, \varphi_{\tau,q} \rangle = \frac{1}{2\pi i} \int_S \zeta_\eta(s) \widetilde{\varphi}_{\tau,q}(s) \mathrm{d}s.$$

The next important theorem is Theorem 10 which is an analogue of Theorem 9 for the geometric partition function $\theta_{\eta,p}(\tau)$.

THEOREM 10. *Let η be a generalized fractal string satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and let $\varphi_{\tau,p}$ be a test function given by*

$$\varphi_{t,p}(x) = p(x)\mathrm{e}^{-tx}.$$

Then the geometric partition function $\theta_{\eta,p}(\tau)$ is given by

$$\begin{aligned}\theta_{\eta,p}(\tau) &= \sum_{\omega\in D_\eta(W)} \operatorname{Res}\left(\zeta_\eta(s)\widetilde{\varphi}_{\tau,p}(s);\omega\right) &(3)\\ &+\frac{1}{4\sqrt{3}} \sum_{\substack{k=1\\ 1-k\in W\setminus D_\eta}}^{\infty} \frac{(-1)^k\tau^k}{k!}\zeta_\eta(1-k)\,{}_1F_1\left(-k;\frac{1}{2};\frac{\pi^2}{6\tau}\right)\\ &+\frac{\pi}{2\sqrt{6}} \sum_{\substack{l=1\\ \frac{1}{2}-l\in W\setminus D_\eta}}^{\infty} \frac{(-1)^l\tau^l}{l!}\zeta_\eta\left(\frac{1}{2}-l\right)\,{}_1F_1\left(-l;\frac{3}{2};\frac{\pi^2}{6\tau}\right)\\ &+\langle R_\eta^{[0]},\varphi_{\tau,p}\rangle, &(4)\end{aligned}$$

where, for $\operatorname{Re} s > 1$ *and* $\tau > 0$,

$$\begin{aligned}\widetilde{\varphi}_{t,p}(s) &= \frac{t^{\frac{1}{2}-s}}{4\sqrt{3}}\left\{\sqrt{t}\,\Gamma(s-1)\,{}_1F_1\left(s-1;\frac{1}{2};\frac{\pi^2}{6t}\right)\right.\\ &\quad\left.+\sqrt{\frac{2}{3}}\pi\Gamma\left(s-\frac{1}{2}\right)\,{}_1F_1\left(s-\frac{1}{2};\frac{3}{2};\frac{\pi^2}{6t}\right)\right\},\end{aligned}$$

and $\langle R_\eta^{[0]},\varphi_{\tau,p}\rangle$ is the term given by

$$\langle R_\eta^{[0]},\varphi_{\tau,p}\rangle = \frac{1}{2\pi i}\int\limits_S \zeta_\eta(s)\widetilde{\varphi}_{\tau,p}(s)\mathrm{d}s.$$

We end the paper with relations of distributions for different test functions.

THEOREM 11. *Let η be a generalized fractal string. Let $\tau > 0$ and $k \in \mathbb{N}$, then there exist the following relations between test functions $\varphi_\tau(x)$, $\varphi_{\tau,q}(x)$ and $\varphi_{\tau,p}(x)$:*

1° $\langle P^{[k]}\eta, \varphi_{\tau,q}(x)\rangle = \frac{1}{4\cdot 3^{\frac{1}{4}}}\langle P^{[k]}\eta, \varphi_\tau(x)\mathrm{e}^{\pi\sqrt{\frac{x}{3}}}\rangle$;

2° $\langle P^{[k]}\eta, \varphi_{\tau,p}(x)\rangle = \frac{1}{4\sqrt{3}}\langle P^{[k]}\eta, \varphi_\tau(x)\mathrm{e}^{\pi\sqrt{\frac{2x}{3}}}\rangle$;

3° $\langle P^{[k]}\eta, \varphi_{\tau,q}(x)\rangle = \langle P^{[k]}\eta, \varphi_{\tau,p}(x)(3x)^{\frac{1}{4}}\mathrm{e}^{\pi\sqrt{\frac{x}{3}}(1-\sqrt{2})}\rangle$;
4° $\langle P^{[k]}\eta, \varphi_{\tau,p}(x)\rangle = \langle P^{[k]}\eta, \varphi_{\tau,q}(x)(3x)^{-\frac{1}{4}}\mathrm{e}^{-\pi\sqrt{\frac{x}{3}}(1-\sqrt{2})}\rangle$.

The proofs of Theorems 9, 10 and 11 can be found in [5].

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