Šiauliai Math. Semin.,
3 (11), 2008, 141–152

ŠIAULIAI
MATHEMATICAL
SEMINAR
http://siauliaims.su.lt

ON DISCRETE UNIVERSALITY OF THE PERIODIC ZETA-FUNCTION

Roma KAČINSKAITĖ[1], **Algirdas JAVTOKAS**[2], **Darius ŠIAUČIŪNAS**[1]

[1]Šiauliai University, P. Višinskio 19, LT-77156 Šiauliai, Lithuania;
e-mail: r.kacinskaite@fm.su.lt, siauciunas@fm.su.lt

[2]Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania;
e-mail: ajavtokas@gmail.com

Abstract. In the paper, the discrete universality of the periodic zeta-function is considered. For this, a discrete limit theorem in the sense of the weak convergence of probability measures in the space of analytic functions is applied.

Key words and phrases: limit theorem, periodic zeta-function, universality, weak convergence.

2000 Mathematics Subject Classification: 11M41.

In honour of Professor Antanas Laurinčikas on the occasion of his 60th birthday.

1. Introduction

Let, as usual, $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all prime numbers, positive integers, integers, real and complex numbers, respectively, and let $s = \sigma + it$ be a complex variable. Moreover, let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic with the least period $k \in \mathbb{N}$ sequence of complex numbers, i.e., $a_{m+k} = a_m$ for all $m \in \mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma > 1$, by a Dirichlet series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Denote by $\zeta(s,\alpha)$, $0<\alpha\leqslant 1$, the Hurwitz zeta-function defined, for $\sigma>1$, by

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}.$$

It is well known that $\zeta(s,\alpha)$ is a meromorphic function, it has a simple pole at $s=1$ with residue 1. In virtue of the periodicity of $\mathfrak{a}$ it is not difficult to see that, for $\sigma>1$,

$$\zeta(s;\mathfrak{a})=\frac{1}{k^s}\sum_{l=1}^{k}a_l\zeta\left(s,\frac{l}{k}\right). \tag{1}$$

Therefore, the equality (1) gives an analytic continuation to the whole complex plane for the function $\zeta(s;\mathfrak{a})$, except, maybe, for the point $s=1$ with residue

$$a=\frac{1}{k}\sum_{m=1}^{k}a_m.$$

If $a=0$, then $\zeta(s;\mathfrak{a})$ is an entire function.

The first result on the universality of functions given by Dirichlet series was obtained by S. M. Voronin. In [13], he proved the universality of the Riemann zeta-function $\zeta(s)$ which is a case of $\zeta(s,\alpha)$ with $\alpha=1$. To state the last version of Voronin's theorem we need some notation. Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$, and define, for $T>0$,

$$\nu_T(\ldots)=\frac{1}{T}\text{meas}\{\tau\in[0,T]:\ldots\},$$

where in place of dots a condition satisfied by τ is to be written. Then in [3] the following statement was proved.

Theorem A. *Let K be a compact subset of the strip $D=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\nu_T\left(\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right)>0.$$

Later, the Voronin theorem was generalized for other zeta and L-functions, for references, see, [4], [5], [9].

In [11], J. Steuding began to study the universality of the function $\zeta(s;\mathfrak{a})$, and he proved the following assertion.

THEOREM B. *Suppose that k is odd prime, a_m is not a multiple of a character* mod k*, and $a_k = 0$. Let K be a compact subset of the strip D with connected complement and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right) > 0.$$

The proof of Theorem B is not direct, it uses the joint universality theorem for Dirichlet L-functions obtained by S. M. Voronin [12].

In [7], a limit theorem in the space of analytic functions was applied for the investigation of universality for the function $\zeta(s;\mathfrak{a})$. The aim of this note is to present a discrete version of results obtained in [7].

Let, for $N \in \mathbb{N}\bigcup\{0\}$,

$$\mu_N(\ldots) = \frac{1}{N+1}\sum_{\substack{m=0\\ \ldots}}^{N} 1,$$

where in place of dots a condition satisfied by m is to be written. We recall that the sequence $\mathfrak{a}$ is completely multiplicative if $a_1 = 1$ and, for all $m, n \in \mathbb{N}$,

$$a_{mn} = a_m a_n, \tag{2}$$

and multiplicative if equality (2) holds for all coprime $m, n \in \mathbb{N}$. We suppose that $h > 0$ is a fixed number such that $\exp\{\frac{2\pi m}{h}\}$ is irrational for all $m \in \mathbb{Z}\setminus\{0\}$. By p we denote a prime number.

THEOREM 1. *Suppose that the sequence $\mathfrak{a}$ is completely multiplicative. Let K and $f(s)$ be the same as in* Theorem A. *Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \mu_N\left(\sup_{s\in K} |\zeta(s+imh;\mathfrak{a}) - f(s)| < \varepsilon\right) > 0.$$

THEOREM 2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative and for all $p \in \mathbb{P}$*

$$\sum_{\alpha=1}^{\infty} \frac{|a_{p^\alpha}|}{p^{\frac{\alpha}{2}}} \leqslant c_2 < 1.$$

Let K and $f(s)$ be the same as in Theorem A. *Then the assertion of* Theorem 1 *is true.*

Note that a completely multiplicative periodic function coincides with a Dircihlet character. Thus, Theorem 1 means the discrete universality of Dirichlet L-functions.

Denote by γ the unit circle $\{s \in \mathbb{C} : |s| = 1\}$ and define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. The infinite-dimensional torus Ω is a compact topological Abelian group. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Then on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and put, for $m \in \mathbb{N}$,

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p),$$

where $p^\alpha \| m$ means that $p^\alpha | m$ but $p^{\alpha+1} \nmid m$. Thus, we obtain that $\omega(m)$ is a completely multiplicative function.

Denote by $H(D)$ the space of analytic on the strip D functions equipped with the topology of uniform convergence on compacta. For $s \in D$, define

$$\zeta(s, \omega; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}.$$

Since the sequence $\mathfrak{a}$ is periodic, it is bounded. Therefore, by standard method it can be proved similarly to the case of the Riemann zeta-function, see, [3], that $\zeta(s, \omega; \mathfrak{a})$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by P_ζ the distribution of $\zeta(s, \omega; \mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega; \mathfrak{a}) \in A), \quad A \in \mathcal{B}(H(D)).$$

Let S_{P_ζ} be the support of the measure P_ζ, i.e., S_{P_ζ} is a minimal closed subset of $H(D)$ such that $P_\zeta(S_{P_\zeta}) = 1$.

THEOREM 3. *Suppose that the sequence $\mathfrak{a}$ is not multiplicative. Let K be a compact subset of the strip D, and let $f(s)$ be an element of S_{P_ζ}. Then the assertion of* Theorem 1 *is true.*

2. A limit theorem

For the proof of Theorems 1–3 we will apply a limit theorem in the sense of weak convergence of probability measures in the space $H(D)$ for the function

$\zeta(s;\mathfrak{a})$. Let

$$P_N(A)=\mu_N(\zeta(s+imh;\mathfrak{a})\in A),\quad A\in\mathcal{B}(H(D)).$$

THEOREM 4. *The probability measure P_N converges weakly to P_ζ as $N\to\infty$.*

We will start with a limit theorem for one absolutely convergent Dirichlet series. Let $\sigma_1>\frac{1}{2}$ be fixed, and, for $m,n\in\mathbb{N}$,

$$v(m,n)=\exp\biggl\{-\Bigl(\frac{m}{n}\Bigr)^{\sigma_1}\biggr\}.$$

Define

$$\zeta_n(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m v(m,n)}{m^s}.$$

Then in [7] it is proved that the series for $\zeta_n(s;\mathfrak{a})$ absolutely converges for $\sigma>\frac{1}{2}$.

Now let

$$Q_N(A)=\mu_N\bigl(\{p^{-imh}:p\in\mathbb{P}\}\in A\bigr),\quad A\in\mathcal{B}(\Omega).$$

LEMMA 5. *The probability measure Q_N converges weakly to the Haar measure m_H as $N\to\infty$.*

PROOF. The dual group of Ω is $\bigoplus_p\mathbb{Z}_p$, where $\mathbb{Z}_p=\mathbb{Z}$ for all primes p. An element $\underline{k}=(k_2,k_3,\ldots)\in\bigoplus_p\mathbb{Z}_p$, where only a finite number of integers k_p are distinct from zero, acts on Ω by $\underline{k}\to\underline{x}^{\underline{k}}=x_2^{k_2}x_3^{k_3}\ldots$, $\underline{x}=(x_2,x_3,\ldots)\in\Omega$. Therefore, the Fourier transform $g_N(\underline{k})$ of the measure Q_N is of the form

$$\begin{aligned}g_N(\underline{k})=\int_\Omega\prod_p x_p^{k_p}\,\mathrm{d}Q_N &= \frac{1}{N+1}\sum_{m=0}^{N}\prod_p p^{-imhk_p}\\ &= \frac{1}{N+1}\sum_{m=0}^{N}\exp\biggl\{-imh\sum_p k_p\log p\biggr\},\end{aligned}\tag{3}$$

where only a finite number of integers k_p are distinct from zero. Since the system $\{\log p : p\in\mathbb{P}\}$ is linearly independent over the field of rational numbers, hence we obtain that, for $\underline{k}\neq\underline{0}$,

$$g_N(\underline{k})=\frac{1}{N+1}\,\frac{1-\exp\Bigl\{-i(N+1)h\sum_p k_p\log p\Bigr\}}{1-\exp\Bigl\{-ih\sum_p k_p\log p\Bigr\}}.\tag{4}$$

We observe that in this case

$$1 - \exp\left\{ -ih\sum_p k_p \log p\right\} \neq 0,$$

since $\exp\left\{\frac{2\pi m}{h}\right\}$ is irrational for all $m \in \mathbb{Z}\setminus\{0\}$. Thus, (3) and (4) show that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}, \end{cases}$$

and from this the lemma follows.

Define, for $\widehat{\omega} \in \Omega$,

$$\zeta_n(s, \widehat{\omega}; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \widehat{\omega}(m) v(m,n)}{m^s}.$$

Then the series for $\zeta_n(s, \widehat{\omega}; \mathfrak{a})$ also converges absolutely for $\sigma > \frac{1}{2}$. On $(H(D), \mathcal{B}(H(D)))$, define two probability measures

$$P_{N,n}(A) = \mu_N\big(\zeta_n(s+imh; \mathfrak{a}) \in A\big)$$

and

$$\widehat{P}_{N,n}(A) = \mu_N\big(\zeta_n(s+imh, \widehat{\omega}; \mathfrak{a}\big) \in A).$$

THEOREM 6. *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure P_n such that the probability measures $P_{N,n}$ and $\widehat{P}_{N,n}$ both converge weakly to P_n as $N \to \infty$.*

PROOF. Define a function $h_n : \Omega \to H(D)$ by the formula

$$h_n(\omega) = \sum_{m=1}^{\infty} \frac{a_m \omega(m) v(m,n)}{m^s}, \quad \omega \in \Omega.$$

The function h_n, clearly, is continuous, moreover,

$$h_n\big(\{p^{-imh} : p \in \mathbb{P}\}\big) = \zeta_n(s+imh; \mathfrak{a}).$$

Therefore, Lemma 5 together with Theorem 5.1 of [1] shows that the measure $P_{N,n}$ converges weakly to the measure $m_H h_n^{-1}$ as $N \to \infty$.

Now define a function $\widehat{h}_n : \Omega \to H(D)$ by the formula

$$\widehat{h}_n(\omega) = \sum_{m=1}^{\infty} \frac{a_m \widehat{\omega}(m) \omega(m) v(m,n)}{m^s}, \quad \omega \in \Omega.$$

Then similarly as above we obtain that the measure $\widehat{P}_{N,n}$ converges weakly to the measure $m_H\widehat{h}_n^{-1}$. However, $\widehat{h}_n(\omega) = h_n(h(\omega))$, where $h(\omega) = \widehat{\omega}\omega$. Since the Haar measure m_H is invariant, hence we find that

$$m_H\widehat{h}_n^{-1} = m_H(h_n h)^{-1} = (m_H h^{-1})h_n^{-1} = m_H h_n^{-1},$$

that is the measure $\widehat{P}_{N,n}$ also converges weakly to $m_H h_n^{-1}$ as $N \to \infty$.

To prove Theorem 4 it remains to pass from the function $\zeta_n(s;\mathfrak{a})$ to $\zeta(s;\mathfrak{a})$. For this, we need the following lemma.

Lemma 7. *Let K be a compact subset of the strip D. Then*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\zeta(s+imh;\mathfrak{a})-\zeta_n(s+imh;\mathfrak{a})| = 0$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\zeta(s+imh,\omega;\mathfrak{a})-\zeta_n(s+imh,\omega;\mathfrak{a})| = 0.$$

Proof. In [6], it was proved that, for $\sigma > \frac{1}{2}$,

$$\int_1^T |\zeta(\sigma+it;\mathfrak{a})|^2 \mathrm{d}t = \mathrm{O}(T), \quad T\to\infty. \tag{5}$$

From this, using the Cauchy integral formula, we obtain that also, for $\sigma > \frac{1}{2}$,

$$\int_1^T |\zeta'(\sigma+it;\mathfrak{a})|^2 \mathrm{d}t = \mathrm{O}(T), \quad T\to\infty.$$

This, (5) and the Gallagher lemma, see, [10], show that, for every fixed m_0 and $\sigma > \frac{1}{2}$,

$$\begin{aligned}\sum_{m=m_0}^{N}|\zeta(\sigma+imh;\mathfrak{a})|^2 &\leqslant \frac{1}{h}\int_{m_0h}^{Nh}|\zeta(\sigma+it;\mathfrak{a})|^2\mathrm{d}t \\ &\quad +\Big(\int_{m_0h}^{Nh}|\zeta(\sigma+it;\mathfrak{a})|^2\mathrm{d}t\int_{m_0h}^{Nh}|\zeta'(\sigma+it;\mathfrak{a})|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\ &= \mathrm{O}(N), \quad N\to\infty. \end{aligned}\tag{6}$$

Let $\sigma_1 > \frac{1}{2}$, and

$$l_n(s) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right)n^s.$$

Then the function $\zeta_n(s;\mathfrak{a})$ can be written, for $\sigma > \frac{1}{2}$, by

$$\zeta_n(s;\mathfrak{a}) = \frac{1}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \zeta(s+z;\mathfrak{a})l_n(z)\frac{\mathrm{d}z}{z}.$$

Therefore, the application of the residue theorem leads to the estimate

$$\begin{aligned}
&\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\zeta(s+imh;\mathfrak{a}) - \zeta_n(s+imh;\mathfrak{a})| \\
&= \mathrm{O}\biggl(\int\limits_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+it)| \\
&\quad \times\biggl(\frac{1}{N+1}\sum_{m=-t_0}^{2N+t_0}|\zeta(\sigma_2+imh;\mathfrak{a})|^2\biggr)^{\frac{1}{2}}\mathrm{d}t\biggr),
\end{aligned} \tag{7}$$

where $\sigma_2 > \frac{1}{2}$ and $\sigma_2 < \sigma$, and $t_0 = \left[\frac{|t|}{h}\right]$. Hence, in view of (6), we find that the right-hand side of (7) is bounded by

$$\mathrm{O}\biggl(\int\limits_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+it)|(1+|t|)\mathrm{d}t\biggr).$$

Since, for $\sigma < 0$,

$$\lim_{n\to\infty}\int\limits_{-\infty}^{\infty} |l_n(\sigma+it)|(1+|t|)\mathrm{d}t = 0,$$

from this the first assertion of the lemma follows.

The proof of the second statement of the lemma is similar to that of the first. We only observe that by standard method (application of the Birkhoff-Khinchine theorem) it can be obtained that, for $\sigma > \frac{1}{2}$ and almost all ω,

$$\int\limits_{1}^{T} |\zeta(\sigma+it,\omega;\mathfrak{a})|^2\mathrm{d}t = \mathrm{O}(T), \quad T\to\infty.$$

The further proof coincides with that of the first part.

Proof of Theorem 4. We define once one probability measure

$$\widehat{P}_N(A) = \mu_N\big(\zeta(s+imh,\omega;\mathfrak{a}) \in A\big), \quad A \in \mathcal{B}(H(D)).$$

First we will prove that the measures P_N and $\widehat{P}_N$ both converge weakly to the same probability measure P as $N \to \infty$.

By Theorem 6, the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ both converge weakly to the same measure P_n as $N \to \infty$. It is not difficult to prove that the family $\{P_n : n \in \mathbb{N}\}$ is tight, thus by the Prokhorov theorem it is relatively compact.

Let θ_N be a random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbf{P})$ with values mh, $m = 0, 1, \ldots, N$, and

$$\mathbf{P}(\theta_N = mh) = \frac{1}{N+1}, \quad m = 0, 1, \ldots, N.$$

Define

$$X_{N,n} = X_{N,n}(s) = \zeta_n(s + i\theta_N; \mathfrak{a}).$$

Then, by the above remark,

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{8}$$

where $X_n = X_n(s)$ is an $H(D)$-valued random element having the distribution P_n.

For $f, g \in H(D)$, let

$$\varrho(f,g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s\in K_l} |f(s)-g(s)|}{1 + \sup\limits_{s\in K} |f(s)-g(s)|},$$

where $\{K_l\}$ is a sequence of compact subsets of D such that $D = \bigcup\limits_{l=1}^{\infty} K_l$ and $K_l \subset K_{l+1}$, $l \in \mathbb{N}$. Then ϱ is a metric which induces the topology of $H(D)$. Define $X_N = X_N(s) = \zeta(s + i\theta_N; \mathfrak{a})$. Then from the first part of Lemma 7 we deduce that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mathbf{P}\big(\varrho(X_{N,n}, X_N) \geqslant \varepsilon\big) = 0. \tag{9}$$

Now let $\{P_{n_1}\} \subset \{P_n\}$ be such that P_{n_1} converges weakly to some measure P as $n_1 \to \infty$. Then $X_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} P$. This, (8), (9) and Theorem 4.2 of [1] show that

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{10}$$

and this implies the weak convergence of P_N to P as $N \to \infty$.

The relation (10) shows that the measure P is independent on the sequence n_1. Thus, $X_n \xrightarrow[n\to\infty]{\mathcal{D}} P$. Using this and repeating the above arguments for the random elements $Y_{N,n}(s) = \zeta_n(s + i\theta_N, \omega; \mathfrak{a})$ and $Y_N(s) = \zeta(s+i\theta_N, \omega; \mathfrak{a})$, $\omega \in \Omega$, we obtain that the measure $\widehat{P}_N$ also converges weakly to P.

Let $a_h = \{p^{-imh} : p \in \mathbb{P}\}$. Define $\varphi_h(\omega) = a_h\omega$, $\omega \in \Omega$. Then φ_h is a measurable measure preserving transformation on $(\Omega, \mathcal{B}(\Omega), m_H)$. It was proved in [2] that the transformation φ_h is ergodic.

Now we are able to complete the proof of Theorem 4. Let $A \in \mathcal{B}(H(D))$ be a continuity set of P, and define a random variable η on $(\Omega, \mathcal{B}(\Omega), m_H)$ by

$$\eta(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(s, \omega; \mathfrak{a}) \in A, \\ 0 & \text{if} \quad \zeta(s, \omega; \mathfrak{a}) \notin A. \end{cases}$$

Then we have that

$$E\eta = P_\zeta(A). \tag{11}$$

Since φ_h is ergodic, by the ergodic theorem and the definition of φ_h

$$\lim_{N\to\infty} \mu_N(\zeta(s + imh, \omega; \mathfrak{a}) \in A) = \lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \eta(\varphi_h^m(\omega)) = E\eta$$

for almost all $\omega \in \Omega$. Thus, by (11), $P(A) = P_\zeta(A)$. Hence we have that $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(H(D))$, and the theorem is proved.

3. Proof of Theorems 1–3

In [7], it was proved that, for $x \to \infty$,

$$\sum_{p\leqslant x} |a_p|^2 = \frac{1}{\varphi(k)} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} |a_l|^2 \frac{x}{\log x}(1 + \mathrm{o}(1)). \tag{12}$$

Proof of Theorem 1. Since the sequence $\mathfrak{a}$ is completely multiplicative, the random element $\zeta(s, \omega; \mathfrak{a})$ can be written in the form

$$\zeta(s, \omega; \mathfrak{a}) = \prod_p \left(1 - \frac{a_p\omega(p)}{p^s}\right)^{-1},$$

the product being uniformly convergent on compact subsets of D for almost all $\omega \in \Omega$. Hence, using the hypotheses of the theorem, Theorem 4, (12) and the method of [8], we obtain that the support of the measure P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Therefore, the further proof coincides with the case of the Riemann zeta-function, see [3].

PROOF OF THEOREM 2. Since the sequence $\mathfrak{a}$ is multiplicative,

$$\zeta(s, \omega; \mathfrak{a}) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{a_{p^\alpha} \omega^\alpha(p)}{p^{\alpha s}}\right),$$

the product being uniformly convergent on compact subsets of D for almost all $\omega \in \Omega$. From this, the hypotheses of the theorem, Theorem 4 and (12) we find again that the support of P_ζ is the set S, and the proof is completed in the same way as that of Theorem 1.

PROOF OF THEOREM 3. Since the sequence $\mathfrak{a}$ is not multiplicative, the random element has not the Euler product. However, the random variables $\omega(m)$ are not independent, therefore we cannot to apply the method used in the proof of Theorems 1 and 2 in the case of independent random variables $\omega(p)$. Hence, we have only a conditional version of universality for the function $\zeta(s; \mathfrak{a})$.

Define

$$G = \left\{g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon\right\}.$$

The set G, clearly, is open, and by the hypotheses of the theorem it is a neighbourhood of an element $f \in S_{P_\zeta}$. Therefore, $P_\zeta(G) > 0$. Theorem 4 and Theorem 2.1 of [1] yield

$$\liminf_{N \to \infty} \mu_N \left(\sup_{s \in K} |\zeta(s + imh; \mathfrak{a}) - f(s)| < \varepsilon \right) \geqslant P_\zeta(G) > 0.$$

The theorem is proved.

References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York, 1968.

[2] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function, *Liet. Matem. Rink.* **40**(4) (2000), 475–492 (in Russian) = *Lith. Math. J.* **40**(4) (2000), 364–378.

[3] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, 1996.

[4] A. Laurinčikas, The joint universality of Dirichlet series, *Fiz. Mat. Fak. Moksl. Semin. Darb.* **7** (2004), 33–44.

[5] A. Laurinčikas, The universality of zeta-functions, *Acta Appl. Math.* **78** (2003), 251–271.

[6] A. Laurinčikas, D. Šiaučiūnas, On the periodic zeta-function. II, *Liet. Matem. Rink.* **41**(4) (2001), 461–476 (in Russian) = *Lith. Math. J.* **41**(4) (2001), 361–372.

[7] A. Laurinčikas, D. Šiaučiūnas, Remarks on the universality of the periodic zeta-function, *Mat. Zametki* **80**(4) (2006), 561–568 (in Russian) = *Math. Notes* **80**(4) (2006), 532–538.

[8] A. Laurinčikas, R. Šleževičienė, The universality of zeta-functions with multiplicative coefficients, *Integral Transforms Spec. Funct.* **13** (2002), 243–257.

[9] K. Matsumoto, Probabilistic value-distribution theory of zeta-functions, *Sugaku Expo.* **17**, No. 1 (2004), 51–71.

[10] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer, Berlin, 1971.

[11] J. Steuding, *Value-distribution of L-functions and allied zeta-functions – with an emphasis on aspects of universality*, Habilitationschrift, J. W. Goethe-Universität Frankfurt, 2003.

[12] S. M. Voronin, On the functional independence of Dirichlet L-functions, *Acta Arith.* **27** (1975), 493–503 (in Russian).

[13] S. M. Voronin, Theorem on the "universality" of the Riemann zeta-function, *Izv. Akad. Nauk SSSR, Ser. Mat.* **39** (1975), 475–486 (in Russian) = *Math. USSR Izv.* **9** (1975), 443–453.

Received
3 December 2007