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A NOTE CONCERNING CERTAIN EXPONENTIAL SUMS RELATED TO CUSP FORMS

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Abstract. We consider certain specific exponential sums related to holomorphic and non-holomorphic cusp forms, give a reformulation for the Lehmer conjecture and note that the Fourier coefficients of holomorphic cusp forms are not very random.

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1. Introduction

Holomorphic cusp forms can be represented as Fourier series

$$F(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{\kappa-1}{2}} e(nz),$$

where $\operatorname{Im} z > 0$ and the numbers $a(n)$ are called normalized Fourier coefficients, and κ is the weight of the form. For Maass forms, this expansion is of the form

$$u(z) = u(x+iy) = \sqrt{y} \sum_{n \neq 0} t(n) K_{i\kappa}\left(2\pi |n| y\right) e\left(nz\right)$$

with the K-Bessel functions, where $\kappa > 0$ depends on the eigenvalue of the non-euclidean Laplacian connected to the form. For properties of cusp forms and closely related theory, see [1], [14] or [9]. It is of interest to consider exponential sums

$$A(M,\Delta,\alpha) \stackrel{def}{=} \sum_{M\leqslant n\leqslant M+\Delta} c(n)e(n\alpha)$$

with $0<\Delta\leqslant M$ and $\alpha\in\mathbb{R}$, and where $c(n)=a(n)$ or $t(n)$ or $d(n)=\sum_{d|n}1$. Wilton's estimate [16]

$$\sum_{n\leqslant M} a(n)e\left(n\alpha\right) \ll M^{\frac{1}{2}}\log M$$

from the year 1929 is a classical result. This estimate is nearly sharp, only the logarithm can be removed, and that was done by Jutila in 1987 [10]. Therefore, moving the focus to short sums was a logical next step. Karppinen and Ernvall-Hytönen [4] proved that, for $1\leqslant\Delta\ll M^{\frac{3}{4}}$,

$$A(M,\Delta,\alpha)\ll M^{\frac{1}{2}-f(\log_M \Delta)},$$

where f is positive when $\Delta\ll M^{\frac{3}{4}-\varepsilon}$ for any $\varepsilon>0$.

On the other hand, the case with $d(n)$ is different since $d(n)>0$. General upper bounds are much higher. However, some special cases behave very similarly to the cases with $a(n)$. There are several similarities between the functions $a(n)$ and $d(n)$, for instance, Deligne proved the so-called Ramanujan-Petterson conjecture for $a(n)$: $|a(n)|\leqslant d(n)$. This conjecture is not proved for $t(n)$, which makes considering exponential sums with $t(n)$ interesting. Common belief is that if $|t(n)|\leqslant d(n)$, then problems related to estimating general upper bounds for exponential sums with $t(n)$ are quite similar to the problems related to general upper bounds with $a(n)$.

Missing proofs can be found in [3].

2. Preliminaries

Let us state a lemma ([11], Lemma 6) which is crucial.

LEMMA 1. *Let A be a function which is compactly supported in a finite interval $[M_1,M_2]$ and at least $P\in\mathbb{Z}$ times differentiable for some $P\geqslant 0$. Assume also that there exist two natural numbers A_0 and A_1 such that, for any non-negative integer $\nu\leqslant P$ and for any $x\in[M_1,M_2]$,*

$$A^{(\nu)}(x)\ll A_0A_1^{-\nu}.$$

Also, let B be a function which is real-valued on $[M_1, M_2]$, and regular throughout the complex domain $\{x \in \mathbb{C} : \ \exists y \in [M_1, M_2] : |x - y| < \rho\}$; and assume that there exists a quantity B_1 such that

$$0 < B_1 \ll \left|B'(x)\right|$$

for any point x in the domain. Then we have

$$\int\limits_{-\infty}^{\infty} A(x) e\left(B(x)\right) \mathrm{d}x \ll A_0 \left(A_1 B_1\right)^{-P} \left(1 + \frac{A_1}{\varrho}\right)^{P} \left(M_2 - M_1\right).$$

3. Sharp estimates

Theorem 2. *Let $M^{\frac{1}{2}+\varepsilon} \ll \Delta \leqslant \lambda M^{\frac{3}{4}}$ and $0 \leqslant T \leqslant M^{\frac{3}{4}}$, where $0 < \lambda \leqslant \frac{1}{\sqrt{d}}$ is a constant, ε an arbitrarily small fixed positive number, d a positive integer, and let w be a smooth weight function on the interval $[M, M + \Delta]$ such that w is a constant function 1 on the interval $[a, b] \subset [M, M + \Delta]$ where $a - M$, $M + \Delta - b = \Delta^{1-\delta}$ with a sufficiently small fixed positive real number δ. Then*

$$\sum_{M+T \leqslant n \leqslant M+T+\Delta} c(n) w(n - T) e\left(\frac{\sqrt{d}n}{\sqrt{M}}\right)$$
$$= C c(d) d^{-\frac{1}{4}} \int\limits_{M+T}^{M+T+\Delta} x^{-\frac{1}{4}} w(x - T) e\left(\frac{\sqrt{d}}{\sqrt{M}} x - 2\sqrt{dx}\right) \mathrm{d}x + \mathrm{O}\left(1\right),$$

where $c(n) = a(n)$, $d(n)$ or $t(n)$, and C is a constant depending only whether $c(n)$ equals $d(n)$, $a(n)$ or $t(n)$ and on the weight of the form or the eigenvalue of the Laplacian.

Proof. The proof with $c(n) = a(n)$ or $d(n)$ is in [3]. Here we give the proof for the case $c(n) = t(n)$, which is similar to the proofs of the other cases. The integral on the right-hand side of the equality is $\asymp M^{-\frac{1}{4}} \Delta$.

Let us first use a Voronoi type summation formula for the exponential sum

$$\sum_{M+T \leqslant n \leqslant M+T+\Delta} t(n) w(n) e\left(\frac{n\sqrt{d}}{\sqrt{M}}\right)$$
$$= \frac{\pi i}{\sinh\left(\pi\kappa\right)} \sum_{n=1}^{\infty} t(n) \int\limits_{M+T}^{M+T+\Delta} \left(J_{2i\kappa}\left(4\pi\sqrt{nx}\right)\right.$$

$$-J_{-2i\kappa}\left(4\pi\sqrt{nx}\right)\right) w(x)e\left(\frac{\sqrt{d}x}{\sqrt{M}}\right)\mathrm{d}x$$
$$+4\omega\cosh\left(\pi\kappa\right)\sum_{n=1}^{\infty}t(n)\int\limits_{M+T}^{M+T+\Delta}K_{2i\kappa}\left(4\pi\sqrt{nx}\right)w(x)e\left(nx\right)\mathrm{d}x. \quad (1)$$

Let us first treat the sum containing the K-Bessel function. First use the asymptotic expansion for the K-Bessel function [13] (5.11.9)

$$K_{2i\kappa}(z)\sim\sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}, \quad (2)$$

when $z\to\infty$. Now

$$\mathrm{e}^{-\frac{4\pi\sqrt{nx}}{k}}\ll\left(\frac{k}{\sqrt{nx}}\right)^{3}.$$

This gives the estimate

$$\ll\int\limits_{M}^{M+\Delta}\left(\frac{k}{\sqrt{nx}}\right)^{3}\frac{\sqrt{k}}{(nx)^{\frac{1}{4}}}\mathrm{d}x\ll k^{\frac{7}{2}}\Delta M^{-\frac{7}{4}}n^{-\frac{7}{4}}$$

for the integral, and finally, this gives the estimate

$$\ll M^{-\frac{9}{4}}\Delta\ll 1$$

for the sum. We may now move to the J-Bessel functions. First write them as a sum of Hankel functions

$$J_{\nu}(z)=\frac{1}{2}\left(H_{\nu}^{(1)}(z)+H_{\nu}^{2}(z)\right).$$

Use the asymptotic expansion for the Hankel functions (see e.g., [13], (5.11.4) and (5.11.5))

$$H_{\nu}^{(j)}(z)=\sqrt{\frac{2}{\pi z}}e\left((-1)^{j}\left(\frac{z}{2\pi}-\frac{\nu}{4}+\frac{1}{8}\right)\right)\omega_{H}^{(j)}(z)+\mathrm{O}\left(|z|^{-\frac{3}{2}-H}\right),$$

where

$$\omega_{H}^{(j)}(z)=1+\sum_{h=1}^{H}(-1)^{jh}C_{h,\nu}z^{-h},$$

and $C_{h,\nu}$ are certain constants. Let us now choose $H = 1$. Treating the O-term is extremely simple (remembering that $|t(n)| \ll n^{\frac{5}{28}+\varepsilon}$):

$$\sum_{n=1}^{\infty} |t(n)|\, n^{-\frac{5}{4}} \int\limits_{M+T}^{M+T+\Delta} w(x)x^{-\frac{5}{4}}\mathrm{d}x \ll \sum_{n=1}^{\infty} n^{-1-\frac{1}{14}+\varepsilon}\Delta M^{-\frac{5}{4}} \ll \Delta M^{-\frac{5}{4}}.$$

Let us treat only the terms arising from the functions $H^{(1)}$ as the case with $j = 2$ is similar and even simpler. Let us now use Lemma 1 to treat the terms of the asymptotic expansion except in the case with $d = n$. Let us make the following choices $A(n) = w(x-T)\left(x^{-\frac{1}{4}}n^{-\frac{1}{4}} + cx^{-\frac{3}{4}}n^{-\frac{3}{4}}\right)$, $B(x) = \frac{\sqrt{d}}{\sqrt{M}}x - 2\sqrt{nx}$, $M_2 - M_1 = \Delta$, and $\varrho = \frac{1}{2}M$. Now $A_1 = \Delta^{1-\delta}$, $B_1 \asymp \frac{\sqrt{n}}{\sqrt{M}}$ and $A_0 = n^{-\frac{1}{4}}M^{-\frac{1}{4}}$. We obtain

$$\int\limits_{M+T}^{M+T+\Delta} A(x)e\left(B(x)\right)w(x-T)\mathrm{d}x \ll n^{-\frac{P}{2}-\frac{1}{4}}.$$

Therefore, the series over this integral converges. It is now sufficient to treat the two last terms, the cases with $d = n$. Let us start with the second term of the asymptotic expansion of the Hankel function. Using absolute values, we obtain

$$\left| \int\limits_{M+T}^{M+T+\Delta} e\left(\frac{\sqrt{d}}{\sqrt{M}}x - \sqrt{dx}\right) w(x)x^{-\frac{3}{4}}\mathrm{d}x \right| \ll 1.$$

This proves the theorem.

Corollary 3. *With the assumptions of the previous theorem and supposing, moreover, that $a(d) = 0$, we have*

$$\sum_{M\leqslant n\leqslant M+\Delta} a(n)w(n)e\left(\frac{\sqrt{d}}{\sqrt{M}}n\right) = \mathrm{O}(1).$$

On the other hand, if $a(d) \neq 0$, then

$$\sum_{M\leqslant n\leqslant M+\Delta} a(n)w(n)e\left(\frac{\sqrt{d}}{\sqrt{M}}n\right) \asymp M^{-\frac{1}{4}}\Delta.$$

Similar results can also be proved for exponential sums related to $d(n)$ and $t(n)$, and even for differences of exponential sums related to different cusp forms or exponential sums with coefficients of different type (e.g., $a(n)$ and $t(n)$). Of course, then one has to pay attention to choosing coefficients in front of the sums.

THEOREM 4. *With the assumptions of the* Theorem 2, *the following holds*

$$\sum_{M\leqslant n\leqslant M+\Delta} c(n)w(n)e\left(\frac{\sqrt{d}n}{\sqrt{M}}\right) - \sum_{M+T\leqslant n\leqslant M+T+\Delta} c(n)w(n-T)e\left(\frac{\sqrt{d}n}{\sqrt{M}}\right)$$
$$\ll \frac{T\Delta\left(T+\Delta\right)}{M^{\frac{7}{4}}} + \frac{\Delta\left(\Delta+T\right)}{M^{\frac{5}{4}}} + 1.$$

4. An Omega result

Other side of the problem is to prove Ω-results. An Ω-result is understood in the following way: $f = \Omega\left(g\right)$ if $f = o(g)$ does not hold.

Several mathematicians, e.g., Joris [8], Redmond [15], Corrádi and Kátai [2] have proved Ω-results for sums of Fourier coefficients of holomorphic cusp forms. In 1989, Ivić and Hafner [5] proved that there is a positive constant D such that

$$\sum_{n\leqslant M} a(n)n^{\frac{\kappa-1}{2}} = \Omega_{\pm}\left(M^{\frac{\kappa}{2}-\frac{1}{4}}\exp\left(D\frac{(\log\log M)^{\frac{1}{4}}}{(\log\log\log M)^{\frac{3}{4}}}\right)\right),$$

where $\Omega_{\pm}$ means the following: $f = \Omega_{\pm}\left(g\right)$ if $\limsup \frac{f}{g} > 0$ and $\liminf \frac{f}{g} < 0$. This is a so-called two-sided Ω-result. In 1990 appeared Ivić's paper [7] in which he proved the existence of $A, B, T_0 > 0$ such that, for $T \geqslant T_0$, every interval $\left[T, T + A\sqrt{T}\right]$ contains t_1 and t_2 for which

$$\sum_{n\leqslant t_1} a(n) > Bt_1^{\frac{1}{4}} \quad \text{and} \quad \sum_{n\leqslant t_2} a(n) < -Bt_2^{\frac{1}{4}}.$$

Very recently, Ivić [6] proved an Ω-result

$$\sum_{M\leqslant n\leqslant M+\Delta} a(n) = \Omega\left(\sqrt{\Delta}\right)$$

for very short sums, when $M^{\varepsilon} \leqslant \Delta \leqslant M^{\frac{1}{2}-\varepsilon}$. Personally, I think this result is extremely interesting as very little is known about very short sums. Compare to O-estimates by Karppinen and Ernvall-Hytönen [12] or [4]. The following result expanding Ivić's result by treating the "missing" case of $\Delta \asymp M^{\frac{1}{2}}$ is proved in [3].

THEOREM 5. *Let $c > 0$ be an arbitrary real number. Then*

$$\sum_{M \leqslant n \leqslant M + c\sqrt{M}} a(n) = \Omega\left(M^{\frac{1}{4}}\right).$$

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