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RESTRICTED AND WEIGHTED SUM FORMULAS FOR DOUBLE ZETA VALUES OF EVEN WEIGHT

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Abstract. In this paper, we give a simple proof for the sum formula of the even- and odd-argument double zeta values of even weight, which has already been proved by H. Gangl, M. Kaneko and D. Zagier. Moreover, we show restricted and weighted sum formulas for double zeta values of even weight.

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1. Introduction and main results

The double zeta values are defined, for integers $a \geqslant 2$ and $b \geqslant 1$, by

$$\zeta(a,b) := \sum_{m>n>0} \frac{1}{m^a n^b}, \tag{1.1}$$

and are subject to a lot of relations. Already Euler found that, when weight $c = a + b$ is odd, the double zeta values can be reduced to products of usual zeta values. After Euler's research, there came very long inactive period in the study of multiple zeta values. In the twentieth century, this series regained the interest of many mathematicians (see Hoffman's web page). At first some sporadic research was done for its own sake, or under the motivation of analytic number theory. In 1990s, it turned out that the multiple zeta values are closely related to many other branch of mathematics, for example, arithmetic geometry, Galois representations, invariants for knots, quantum groups, etc. In [2], H. Gangl, M. Kaneko and D. Zagier proved the following theorem.

Theorem A ([2, Theorem 1]). *For $n \geqslant 2$, one has*

$$\sum_{m=1}^{n-1}\zeta(2m,2n-2m)=\frac{3}{4}\zeta(2n),\quad \sum_{m=1}^{n-1}\zeta(2m+1,2n-2m-1)=\frac{1}{4}\zeta(2n).$$

Moreover, Y. Ohno and W. Zudlin [3] showed the following weighted sum formula by using double shuffle relations.

Theorem B ([3, Theorem 3]). *For any $n \geqslant 3$, we have*

$$\sum_{m=2}^{n-1}2^m\zeta(m,n-m)=(n+1)\zeta(n).$$

In this paper, we give a new proof of Theorem A. Note that, for the proof of Theorem A, we only use the equations proved by Euler (see Lemma 2.1). Next we show the following restricted and weighted sum formulas, which are analogues of Theorems A and B. The key for the proof of Theorem 1.1 is the property (2.8) of the Bernoulli polynomials.

Theorem 1.1. *For $n \geqslant 2$, one has*

$$\sum_{m=1}^{n-1}\left(4^m+4^{n-m}\right)\zeta(2m,2n-2m)=\left(n+\frac{4}{3}+\frac{2}{3}4^{n-1}\right)\zeta(2n). \tag{1.2}$$

Theorem 1.2. *For $n \geqslant 4$, we have*

$$\sum_{m=2}^{n-2}(2m-1)(2n-2m-1)\zeta(2m,2n-2m)=\frac{3}{4}(n-3)\zeta(2n). \tag{1.3}$$

2. Proof of Theorems

We denote by $B_n(x)$ the nth Bernoulli polynomial defined by

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<2\pi,$$

and the nth Bernoulli number B_n by $B_n := B_n(0)$. Euler proved the following equalities.

LEMMA 2.1 (Euler, [1, p. 17, 119], and [2, (2)]). *We have*

$$\zeta(2n) = -\frac{(-4\pi^2)^n}{2(2n)!}B_{2n}, \quad n \geqslant 1, \tag{2.1}$$

$$(2n+1)B_{2n} = -\sum_{m=1}^{n-1}\binom{2n}{2m}B_{2m}B_{2n-2m}, \quad n \geqslant 2, \tag{2.2}$$

and

$$\sum_{m=2}^{n-1}\zeta(m, n-m) = \zeta(n), \quad n \geqslant 3. \tag{2.3}$$

PROOF OF THEOREM A. By (2.1) and (2.2), we have

$$\begin{aligned}\sum_{m=1}^{n-1}\zeta(2m)\zeta(2n-2m) &= \frac{(-4\pi^2)^n}{4}\sum_{m=1}^{n-1}\frac{B_{2m}B_{2n-2m}}{(2m)!(2n-2m)!}\\ &= \frac{(-4\pi^2)^n}{4(2n)!}\sum_{m=1}^{n-1}\binom{2n}{2m}B_{2m}B_{2n-2m}\\ &= -\frac{(-4\pi^2)^n}{4(2n)!}(2n+1)B_{2n}\\ &= \frac{2n+1}{2}\zeta(2n).\end{aligned} \tag{2.4}$$

By the harmonic product formula, we find that

$$\zeta(2m)\zeta(2n-2m) = \zeta(2m, 2n-2m) + \zeta(2n-2m, 2m) + \zeta(2n). \tag{2.5}$$

Using (2.4) and summing the above formula over m from 1 up to $n-1$, we deduce that

$$\frac{2n+1}{2}\zeta(2n) = 2\sum_{m=1}^{n-1}\zeta(2m, 2n-2m) + (n-1)\zeta(2n).$$

Hence, we obtain Theorem A by the above formula and (2.3).

In order to prove Theorem 1.1, we need the following lemma. It should be noted that we can obtain (2.2) by putting $x = y = 0$ in (2.8).

LEMMA 2.2 ([1, p. 4, 6, 119]). *We have the relations:*

$$B_{2n-1} = 0, \quad n \geqslant 2, \tag{2.6}$$

$$B_n\left(\frac{1}{2}\right) = -(1-2^{1-n})B_n, \tag{2.7}$$

and

$$\sum_{m=0}^{n} \binom{n}{m} B_m(x)B_{n-m}(y) = n(x+y-1)B_{n-1}(x+y) -(n-1)B_n(x+y). \tag{2.8}$$

PROOF OF THEOREM 1.1. By using (2.6) and $B_0 = 1$, and putting $x = \frac{1}{2}$ and $y = 0$ in (2.8) with $2n$ instead of n, we have

$$\sum_{m=2}^{2n-2} \binom{2n}{m} B_m\left(\frac{1}{2}\right) B_{2n-m} = -2nB_{2n}\left(\frac{1}{2}\right) - B_0\left(\frac{1}{2}\right)B_{2n}.$$

Applying (2.6) and (2.7) to the above formula, we obtain

$$\sum_{m=1}^{n-1} \binom{2n}{2m} (2^{1-2m}-1)B_{2m}B_{2n-2m} = 2n(1-2^{1-2n})B_{2n} - B_{2n}.$$

The preceding equation is rewritten in the form

$$\sum_{m=1}^{n-1} \binom{2n}{2m} 2^{1-2m} B_{2m}B_{2n-2m} = -(n4^{1-n}+2)B_{2n}$$

because of (2.2). By using (2.1), we have

$$\sum_{m=1}^{n-1} 4^{1-m}\zeta(2m)\zeta(2n-2m) = (n4^{1-n}+2)\zeta(2n).$$

By applying (2.5) and $\sum_{m=1}^{n-1} 2^{2-2m} = \frac{4(1-4^{1-n})}{3}$ to the above formula, we obtain

$$\sum_{m=1}^{n-1} 4^{1-m}\big(\zeta(2m,2n-2m) + \zeta(2n-2m,2m)\big) = \Big(n4^{1-n} + \frac{2}{3} + \frac{4}{3}4^{1-n}\Big)\zeta(2n).$$

On the other hand, changing the order of summation, we have

$$\sum_{m=1}^{n-1} 4^{1-m}\zeta(2n-2m,2m) = \sum_{m=1}^{n-1} 4^{1-n+m}\zeta(2m,2n-2m).$$

This gives (1.2).

Remark 2.3. By modifying the proof of Theorem 1.1, and using $B_{2m}(\frac{1}{3}) = \frac{(3^{1-2m}-1)B_{2m}}{2}$ and (2.8) with $x = \frac{1}{3}$ and $y = 0$, we obtain a new restricted and weighted sum formula. However, the numbers $B_{2m+1}(\frac{1}{3}) \neq 0$ appear in this case.

The next lemma is proved by the Fourier expansions of Eisenstein series.

Lemma 2.4 ([4, (11.10)]). *For $n \geqslant 4$, we have*

$$6\sum_{m=2}^{n-2}(2m-1)(2n-2m-1)\zeta(2m)\zeta(2n-2m) = (n-3)(4n^2-1)\zeta(2n). \quad (2.9)$$

Proof of Theorem 1.2. We obtain (1.3) by using (2.9), (2.5) and $\sum\limits_{m=2}^{n-2}(2m-1)(2n-2m-1) = \frac{(n-3)(2n^2-5)}{3}$.

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