CONTINUITY OF THE QUENCHING TIME IN A SEMILINEAR PARABOLIC EQUATION WITH VARIABLE EXPONENT

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Abstract. This paper concerns the study of a semilinear parabolic equation with variable exponent subject to Neumann boundary conditions and positive initial datum. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the initial datum. Finally, we give some numerical results to illustrate our analysis.

Key words and phrases: numerical quenching time, quenching, semilinear parabolic equation.

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1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). Consider the following initial-boundary value problem

\[
 u_t = \Delta u - u^{-p(x)} \quad \text{in} \quad \Omega \times (0, T),
\]  

(1.1)
\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, T),
\]
\[
u(x, 0) = u_0(x) \quad \text{in} \quad \overline{\Omega},
\]

where \( \Delta \) is the Laplacian, and \( \nu \) is the exterior normal unit vector on \( \partial \Omega \).

The exponent \( p \in C^0(\overline{\Omega}) \) and \( 0 < p_0 = \inf_{x \in \overline{\Omega}} p(x) \leq \sup_{x \in \overline{\Omega}} p(x) = p_+ \).

The initial datum \( u_0 \in C^2(\overline{\Omega}) \), \( u_0(x) > 0 \) in \( \overline{\Omega} \), and there exists a positive constant \( A \) such that

\[
\Delta u_0(x) - (u_0(x))^{-p(x)} \leq -A(u_0(x))^{-p_0} \quad \text{in} \quad \Omega.
\]

Here \((0, T)\) is the maximal time interval of existence of the solution \( u \), and by a solution we mean the following.

**Definition 1.1.** A solution of (1.1)–(1.3) is a function \( u(x, t) \) continuous in \( \overline{\Omega} \times [0, T) \), \( u(x, t) > 0 \) in \( \overline{\Omega} \times [0, T) \), and twice continuously differentiable in \( x \) and once in \( t \) in \( \overline{\Omega} \times (0, T) \).

The time \( T \) may be finite or infinite. When \( T \) is infinite, then we say that the solution \( u \) exists globally. When \( T \) is finite, then the solution \( u \) develops a singularity in a finite time, namely,

\[
\lim_{t \to T^-} u_{\min}(t) = 0,
\]

where \( u_{\min}(t) = \min_{x \in \overline{\Omega}} u(x, t) \). In this last case, we say that the solution \( u \) quenches in a finite time, and the time \( T \) is called the quenching time of the solution \( u \). The first paper concerning the phenomenon of quenching has been written by Kawarada in 1975 (see, [25]). A year later, Acker and Walter published a paper about the same phenomenon (see, [3]). Since these works, the study of solutions for semilinear heat equations which quench in a finite time has been the subject of investigations of several authors (see, [2], [4], [6], [8], [11]–[12], [14]–[15], [22], [26], [29]–[30], [34], and the references cited therein). For the initial-boundary value problem described in this paper, the earlier papers treated only the case where the exponent is constant (see, [8]). Thus, this paper is the first attempt of studying the phenomenon of quenching with variable exponent. Using standard methods, the local in time existence and uniqueness of solutions can be easily proved (see, [6], [16]).

Our aim in this paper consists in showing that, under some hypotheses, the solution of (1.1)–(1.3) quenches in a finite time, and its quenching time as a function of the initial datum is continuous. In order to reach our objective,
we consider the following initial-boundary value problem
\begin{align}
v_t &= \Delta v - v^{-p(x)} \quad \text{in} \quad \Omega \times (0, T_h), \quad (1.5) \\
\frac{\partial v}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega \times (0, T_h), \quad (1.6) \\
v(x, 0) &= u_0^h(x) \quad \text{in} \quad \overline{\Omega}, \quad (1.7)
\end{align}

where \( u_0^h \in C^2(\Omega) \), \( u_0^h(x) \geq u_0(x) \) in \( \overline{\Omega} \), and \( \lim_{h \to 0} \| u_0^h - u_0 \|_{\infty} = 0 \). Here \((0, T_h)\) is the maximal time interval on which the solution \( v \) of (1.5)-(1.7) exists. Let us notice that, setting \( g(x, u) = u^{-p(x)} \), one observes that the function \( g \) is continuous in both variables and locally Lipschitz in the second one. Consequently, due to the fact that the initial data of the different problems considered are sufficiently regular, it is well known that the solutions of these problems exist and are regular. In addition, one may apply without difficulties the maximum principle (see, for instance, [16], [27], [33]). For a good understanding of our paper, we recall that, in [8], Boni and N’gohisse studied the same problem in the case where \( p(x) = p, p \) being a positive constant. They proved that, if the initial datum verifies the conditions given in the introduction of the present paper, then the solution quenches in a finite time. In addition, they exhibited the continuity of the quenching time as a function of the initial datum but they did not provide an upper bound of \( |T_h - T| \). Motivated by their paper, we handle a similar study considering the same problem with variable exponent. Under suitable hypotheses, we prove that the solution \( v \) of (1.5)-(1.7) quenches in a finite time and estimate its quenching time \( T_h \). We also prove that \( T_h \) goes to \( T \) as \( h \) goes to zero, and we provide an upper bound of \( |T_h - T| \) in terms of \( \| u_0^h - u_0 \|_{\infty} \) for \( h \) small enough.

The remainder of the paper is written in the following manner. In the next section, we prove that the solution \( v \) of (1.5)-(1.7) quenches in a finite time and estimate its quenching time. In the third section, we analyze the continuity of the quenching time as a function of the initial datum, and finally, in the last section, we illustrate our analysis with some computational results.

2. Quenching time

In this section, we study the quenching time. We prove that, under some assumptions, the solution \( v \) of (1.5)-(1.7) quenches in a finite time and estimate its quenching time.

The result contained in the following theorem proceeds from an idea of Friedman and McLeod in [17].
Theorem 2.1. Suppose that there exists a constant $B \in (0, 1]$ such that the initial datum at (1.7) satisfies
\[ \Delta u_0^h(x) - (u_0^h(x))^{p(x)} \leq -B(u_0^h(x))^{-p_0} \text{ in } \Omega. \quad (2.1) \]
Then, the solution $v$ of (1.5)-(1.7) quenches in a finite time $T_h$ which verifies the following estimate
\[ T_h \leq \frac{(u_{0\text{min}}^h)^{1+p_0}}{B(1 + p_0)}. \]

Proof. We know that $(0, T_h)$ is the maximal time interval of existence of the solution $v$. Consequently, it remains to prove that $T_h$ is finite and satisfies the above inequality. To this end, let us introduce the function $J(x, t)$ defined as follows
\[ J(x, t) = v_t(x, t) + B(v(x, t))^{-p_0} \text{ in } \Omega \times [0, T_h). \]
By a straightforward computation, we discover that
\[ J_t - \Delta J = (v_t - \Delta v)_t - Bp_0v^{-p_0-1}v_t - B\Delta v^{-p_0} \text{ in } \Omega \times (0, T_h). \quad (2.2) \]
Again, a simple calculation gives $\Delta v^{-p_0} = p_0(p_0+1)v^{-p_0-2}||\nabla v||^2 - p_0v^{-p_0-1}\Delta v$ in $\Omega \times (0, T_h)$. It is clear that $\Delta v^{-p_0} \geq -p_0v^{-p_0-1}\Delta v$ in $\Omega \times (0, T_h)$. This inequality and (2.2) yield
\[ J_t - \Delta J \leq (v_t - \Delta v)_t - Bp_0v^{-p_0-1}(v_t - \Delta v) \text{ in } \Omega \times (0, T_h). \quad (2.3) \]
Considering (1.5) and (2.3), we find that
\[ J_t - \Delta J \leq p(x)v^{-p(x)-1}v_t + Bp_0v^{-p_0-1}v^{-p(x)} \text{ in } \Omega \times (0, T_h), \]
which leads us to
\[ J_t - \Delta J \leq p(x)v^{-p(x)-1}(v_t + Bv^{-p_0}) \text{ in } \Omega \times (0, T_h). \]
Exploiting the expression of $J$, we get
\[ J_t - \Delta J \leq p(x)v^{-p(x)-1}J \text{ in } \Omega \times (0, T_h). \]
Also from (1.6), we derive the following equalities
\[ \frac{\partial J}{\partial v} = (\frac{\partial v}{\partial v})_t - Bp_0v^{-p_0-1}\frac{\partial v}{\partial v} = 0 \text{ on } \partial \Omega \times (0, T_h), \]
and due to (2.1), we discover that
\[ J(x,0) = \Delta u^h_0(x) - (u^h_0(x))^{-p(x)} + B(u^h_0(x))^{-p_0} \leq 0 \quad \text{in} \quad \Omega. \]

It follows from the maximum principle that \( J(x,t) \leq 0 \) in \( \Omega \times (0,T_h) \), or equivalently \( v_t + Bv^{-p_0} \leq 0 \) in \( \Omega \times (0,T_h) \). This estimate can be rewritten as follows
\[ v^{p_0} dv \leq -B dt \quad \text{in} \quad \Omega \times (0,T_h). \]

Integrating the above inequality over \( (0,T_h) \), we obtain \( T_h \leq \frac{(v(x,0))^{1+p_0}}{B(1+p_0)} \) for \( x \in \Omega \). We deduce that
\[ T_h \leq \frac{(v^h_{0\min})^{1+p_0}}{B(1+p_0)}. \]

Since the quantity on the right-hand side of the above inequality is finite, we conclude that \( T_h \) is finite. Consequently, \( v \) quenches in a finite time. This ends the proof.

**Remark 2.1.** Let \( t \in (0,T_h) \). Then an integration of (2.4) from \( t \) to \( T_h \) yields
\[ T_h - t \leq \frac{(v(x,t))^{1+p_0}}{B(1+p_0)}, \quad x \in \Omega. \]

We deduce from (2.5) that
\[ T_h - t \leq \frac{(v^h_{\min}(t))^{1+p_0}}{B(1+p_0)}. \]

The remark below provides a lower bound of \( u_{\min}(t) \) for \( t \in (0,T) \).

**Remark 2.2.** In view of the condition (1.4) and making the same reasoning as in the proof of the above theorem, it is easy to obtain
\[ u_{\min}(t) \geq C(T-t)^{1/(1+p_0)} \quad \text{for} \quad t \in (0,T), \]
where \( C = (A(1+p_0))^{1/(1+p_0)}. \)

Now, let us give an upper bound of \( u_{\min}(t) \) for \( t \in (0,T) \).

**Theorem 2.2.** Let \( u \) be the solution of (1.1)-(1.3). Then there exists a positive constant \( D \) such that the following estimate holds
\[ u_{\min}(t) \leq D(T-t)^{1/(1+p_+)} \quad \text{for} \quad t \in (0,T). \]
Proof. The proof of this theorem is articulated around the following reasoning. Let \( w(t) \) be the function defined as follows

\[
w(t) = \frac{u_{\min}(t)}{||u_0||_\infty} \quad \text{for} \quad t \in [0, T).
\]

For each \( t_1, t_2 \in [0, T) \), there exist \( x_1, x_2 \in \Omega \) such that \( w(t_1) = \frac{u(x_1, t_1)}{||u_0||_\infty} \) and \( w(t_2) = \frac{u(x_2, t_2)}{||u_0||_\infty} \). From Taylor’s expansion, we have

\[
w(t_2) - w(t_1) \geq \frac{u(x_2, t_2) - u(x_2, t_1)}{||u_0||_\infty} = (t_2 - t_1) \frac{u_t(x_2, t_2)}{||u_0||_\infty} + o(t_2 - t_1),
\]

\[
w(t_2) - w(t_1) \leq \frac{u(x_1, t_2) - u(x_1, t_1)}{||u_0||_\infty} = (t_2 - t_1) \frac{u_t(x_1, t_1)}{||u_0||_\infty} + o(t_2 - t_1),
\]

which implies that \( w(t) \) is Lipschitz continuous. In view of the first inequality and (1.1), it is not hard to see that, for \( t_2 > t_1 \),

\[
\frac{w(t_2) - w(t_1)}{t_2 - t_1} \geq \frac{\Delta u(x_2, t_2)}{||u_0||_\infty} - ||u_0||_\infty^{-p(x_2)-1} \left( \frac{u_t(x_2, t_2)}{||u_0||_\infty} \right)^{-p(x_2)} + o(1).
\]

Obviously, we know that \( \Delta u(x_2, t_2) \geq 0 \). Consequently, letting \( t_1 \rightarrow t_2 \), we find that \( w'(t) \geq -||u_0||_\infty^{-p(x_2)-1}(w(t))^{-p(x_2)} \) a.e. for \( t \in (0, T) \). Exploiting the maximum principle, it is clear that \( u(x, t) \leq ||u_0||_\infty \) in \( \Omega \times (0, T) \). This implies that, for \( t \in [0, T) \), \( w(t) \) belongs to the interval \((0, 1)\). Due to the fact that the function \( x \mapsto E^{-x} \) with \( E \in (0, 1) \) is non-decreasing, we deduce that \( w'(t) \geq -\lambda w(t))^{-p_+} \) a.e. for \( t \in (0, T) \), where \( \lambda = \max(||u_0||_\infty^{-p_0-1}, ||u_0||_\infty^{-p_+ -1}) \).

Integrating the above inequality over \((t, T)\) and taking into account the fact that \( w(t) = \frac{u_{\min}(t)}{||u_0||_\infty} \), we obtain \( u_{\min}(t) \leq D(T - t)^{(1/(1+p_+))} \) for \( t \in (0, T) \), where \( D = ||u_0||_\infty (\lambda(1 + p_+))^{1/(1+p_+)} \). This ends the proof.

3. Continuity of the quenching time

In this section, we study the continuity of the quenching time. Our aim consists in proving that if \( h \) is small enough, then the solution \( v \) of (1.5)–(1.7) quenches in a finite time and its quenching time \( T_h \) goes to \( T \) as \( h \) tends to zero. We also provide an upper bound of \(|T_h - T|\) in terms of \( ||u_0^h - u_0||_\infty \).

Theorem 3.1. Suppose that the problem (1.1)–(1.3) has a solution \( u \) which quenches at the time \( T \). Then, under the assumption of Theorem 2.1, the solution \( v \) of (1.5)–(1.7) quenches in a finite time \( T_h \), and there exist positive
constants $\beta$, $\delta$ and $\gamma$ such that, for $h$ sufficiently small, the following estimate holds

$$|T - T_h| \leq \beta \left( \log \left( \frac{\delta}{\|u_0^h - u_0\|_\infty} \right) \right)^{-\gamma}.$$ 

**Proof.** It follows from Theorem 2.1 that the solution $v$ of (1.5)–(1.7) quenches in a finite time $T_h$. Now, to achieve our objective, it remains to establish the above estimate. We recall that $u_0^h(x) \geq u_0(x)$ in $\Omega$. Consequently, making use of the maximum principle, we see that $v \geq u$ as long as all of them are defined. Due to this fact, we conclude that $T_h \geq T$, and, obviously, $T_h - T = |T_h - T|$. For the remainder of the proof, let us introduce the error function $e(x,t)$ defined as follows

$$e(x,t) = v(x,t) - u(x,t) \quad \text{in} \quad \Omega \times [0,T).$$

For any positive quantity $t_0$ such that $t_0 < T$, a minute computation reveals that

$$e_t - \Delta e = p(x)\|u_0^h\|_{\infty}^{-p(x)-1} \left( \frac{\theta}{\|u_0^h\|_{\infty}} \right)^{-p(x)-1} e \quad \text{in} \quad \Omega \times (0,t_0),$$

where $\theta$ is an intermediate value between $u$ and $v$. Since $v(x,t) \geq \theta(x,t) \geq u(x,t) \geq \Omega \times (0,t_0)$, we have $\frac{\theta(x,t)}{\|u_0^h\|_{\infty}} \leq 1$ in $\Omega \times (0,t_0)$. Consequently, the above equation becomes

$$e_t - \Delta e \leq N \left( \frac{\theta}{\|u_0^h\|_{\infty}} \right)^{-p+1} e \quad \text{in} \quad \Omega \times (0,t_0),$$

where $N = p_+ \max(\|u_0^h\|_{\infty}^{-p_0+1}, \|u_0^h\|_{\infty}^{-p+1})$. Due to the fact that $v(x,t) \geq u(x,t) \geq \Omega \times (0,t_0)$, then making use of Remark 2.2, it is easy to check that

$$\theta(x,t) \geq u_{\min}(t) \geq C(T - t)^{1/(1+p_0)} \quad \text{in} \quad \Omega \times (0,t_0),$$

which implies that

$$e_t \leq \Delta e + K(T-t)^{-\alpha}e \quad \text{in} \quad \Omega \times (0,t_0),$$

where $K = N\left( \frac{C}{\|u_0^h\|_{\infty}} \right)^{-p+1}$ and $\alpha = \frac{p_0+1}{p_0+T} > 1$. We also have

$$\frac{\partial e}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0,t_0), \quad e(x,0) = u_0^h(x) - u_0(x) \quad \text{in} \quad \overline{\Omega}.$$
Let $Z(t)$ be the solution of the following ODE

$$Z'(t) = K(T - t)^{-\alpha} Z(t) \quad \text{for} \quad t \in (0, t_0), \quad Z(0) = \|u_0^h - u_0\|_\infty.$$ 

The solution $Z(t)$ of the above ODE is given explicitly by

$$Z(t) = L\|u_0^h - u_0\|_\infty e^{K/(\alpha-1)(T-t)^{1-\alpha}} \quad \text{for} \quad t \in [0, t_0),$$

where $L = e^{K/(1-\alpha)T^{1-\alpha}}$. On the other hand, the maximum principle allows us to write

$$e(x, t) \leq Z(t) = L\|u_0^h - u_0\|_\infty e^{K/(\alpha-1)(T-t)^{1-\alpha}} \quad \text{in} \quad \Omega \times [0, t_0).$$

Fix a positive constant and let $t_1 \in (0, T)$ be a time such that $\|e(\cdot, t_1)\|_\infty \leq L\|u_0^h - u_0\|_\infty e^{K/(\alpha-1)(T-t_1)^{1-\alpha}} = a$ for $h$ sufficiently small. This implies that

$$T - t_1 = \left(\frac{\alpha - 1}{K} \frac{\alpha}{\log \left(L\|u_0^h - u_0\|_\infty\right)}\right)^{1/(1-\alpha)}.$$  \hspace{1cm} (3.1)

The triangle inequality and Remark 2.1 allow us to write

$$|T_h - t_1| \leq \frac{(\varepsilon_{\min}(t_1))^{1+p_0}}{B(1 + p_0)} \leq \frac{(\varepsilon_{\min}(t_1) + \|e(\cdot, t_1)\|_\infty)^{1+p_0}}{B(1 + p_0)}.$$

Since $\|e(\cdot, t_1)\|_\infty \leq a$, then taking into account the fact that the function $x \mapsto x^{1+p_0}$ is non-decreasing for $x \in (0, \infty)$, we deduce from Theorem 2.2 that

$$|T_h - t_1| \leq \frac{D(T - t_1)^{1/(1+p_0)} + a)^{1+p_0}}{B(1 + p_0)}.$$ \hspace{1cm} (3.2)

It is not hard to see that there exists a positive constant $C_2$ such that

$$D(T - t_1)^{1/(1+p_0)} + a \leq C_2(T - t_1)^{1/(1+p_0)}.$$

Having in mind the above inequality and (3.2), we derive the following estimate

$$|T_h - t_1| \leq M |T - t_1|^{(1+p_0)/(1+p_+)}.$$ \hspace{1cm} (3.3)

where $M = \frac{C_2^{1+p_0}}{B(1 + p_0)}$. From (3.3) and the triangle inequality, we discover that

$$|T - T_h| \leq |T - t_1| + |T_h - t_1| \leq |T - t_1| + M |T - t_1|^{(1+p_0)/(1+p_+)}.$$

This implies that there exists a positive constant $Q$ such that

$$|T - T_h| \leq Q |T - t_1|^{(1+p_0)/(1+p_+)}.$$ 

Use the equality (3.1) to complete the rest of the proof.
4. Numerical results

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

\[
\begin{aligned}
    u_t &= \Delta u - u^{-p(x)} \quad \text{in} \quad B \times (0, T), \\
    \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad S \times (0, T), \\
    u(x, 0) &= u_0(x) \quad \text{in} \quad \bar{B},
\end{aligned}
\]

where \( p(x) = \frac{1}{2} + \|x\|_\infty, \ u_0(x) = \frac{2 + \cos(\pi \|x\|_\infty)}{4} + \varepsilon(1 + \cos(\pi \|x\|_\infty)) \), with \( \varepsilon \) a non-negative parameter, \( B = \{ x \in \mathbb{R}^N ; \|x\| < 1 \}, \ S = \{ x \in \mathbb{R}^N ; \|x\| = 1 \}. \)

The above problem may be rewritten in the following form

\[
\begin{aligned}
    u_t &= u_{rr} + \frac{N - 1}{r} u_r - u^{-p(r)}, \quad r \in (0, 1), \quad t \in (0, T), \\
    u_r(0, t) &= 0, \quad u_r(1, t) = 0, \quad t \in (0, T), \\
    u(r, 0) &= \varphi(r), \quad r \in [0, 1],
\end{aligned}
\]

(4.1)

where we take \( 1 \leq N \leq 3, \ \varphi(r) = \frac{2 + \cos(\pi r)}{4} + \varepsilon(1 + \cos(\pi r)) \) and \( p(r) = \frac{1}{2} + r \). We start by the construction of an adaptive scheme as follows. Let \( I \) be a positive integer and let \( h = \frac{1}{I} \). Define the grid \( x_i = ih, \ 0 \leq i \leq I \), and approximate the solution \( u \) of (4.1)–(4.3) by the solution \( U_h^{(n)} = (U_h^{(n)}(0), \ldots, U_h^{(n)}(I))^T \) of the following explicit scheme

\[
\begin{aligned}
    \frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= N \frac{U_1^{(n)} - U_0^{(n)}}{h^2} - (U_0^{(n)})^{-\psi_0}, \\
    \frac{U_{i+1}^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \frac{U_{i+1}^{(n)} - U_{i}^{(n)} + U_{i+1}^{(n)} - (N - 1) U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \\
    \frac{U_{i}^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= -U_i^{(n)} - \psi_i, \quad 1 \leq i \leq I - 1, \\
    U_i^{(0)} &= \varphi_i, \quad 0 \leq i \leq I,
\end{aligned}
\]

where \( \varphi_i = \frac{2 + \cos(i \pi h)}{4} + \varepsilon(1 + \cos(i \pi h)) \) and \( \psi_i = \frac{1}{2} + ih \) for \( 0 \leq i \leq I \).

An important fact concerning the phenomenon of quenching is that, if the solution \( u \) quenches at the time \( T \), then when the time \( t \) approaches the quenching \( T \), the solution \( u \) decreases to zero rapidly. Thus, in order to permit the discrete solution to reproduce the properties of the continuous one...
when the time $t$ approaches the quenching time $T$, we need to adapt the size of the time step so that we take $\Delta t_n = \min \left\{ \frac{(1-h^2)k^2}{2N}, h^2(\frac{U_{h_{min}}^{(n)}}{\|\psi_h\|_{\infty}}) \right\}$ with $U_{h_{min}}^{(n)} = \min_{0 \leq i \leq I} U_i^{(n)}$. For this fact, our explicit scheme becomes an adaptive scheme which is one of suitable schemes for problems whose solutions develop singularities in a finite time. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution $u$ of (4.1)-(4.3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$
\begin{align*}
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - (U_0^{(n)})^{-\psi_0-1}U_0^{(n+1)}, \\
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)} + \frac{(N-1)U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} - (U_i^{(n)})^{-\psi_i-1}U_i^{(n+1)}, \\
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= N \frac{2U_i^{(n+1)} - 2U_i^{(n+1)}}{h^2} - (U_i^{(n)})^{-\psi_i-1}U_i^{(n+1)}, \\
U_i^{(0)} &= \varphi_i, \quad 0 \leq i \leq I.
\end{align*}
$$

As in the case of the explicit scheme, here again, $\varphi_i = \frac{2+\cos(i\pi h)}{4} + \varepsilon(1 + \cos(i\pi h))$, $\psi_i = \frac{1}{2} + ih$ for $0 \leq i \leq I$, and we transform our scheme to an adaptive scheme by choosing $\Delta t_n = h^2(U_{h_{min}}^{(n)})^{\|\psi_h\|_{\infty}} + 1$.

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution are also guaranteed using standard methods (see [7]). It is not hard to see that $u_{rr}(1, t) = \lim_{r \to 1} \frac{u_r(x, \ell)}{r}$ and $u_{rr}(0, t) = \lim_{r \to 0} \frac{u_r(x, \ell)}{r}$. Hence, if $r = 0$ and $r = 1$, then we see that

$$
\begin{align*}
u_t(0, t) &= Nu_{rr}(0, t) - (u(0, t))^{-p(r)}, \quad t \in (0, T), \\
u_t(1, t) &= Nu_{rr}(1, t) - (u(1, t))^{-p(r)}, \quad t \in (0, T).
\end{align*}
$$

These observations have been taken into account in the construction of our schemes at the first and last nodes. We need the following definition.

**Definition 4.1.** We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n \to \infty} U_{h_{min}}^{(n)} = 0$, and the series $\sum_{n=0}^\infty \Delta t_n$ converges. The quantity $\sum_{n=0}^\infty \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical
quenching time \( t_n = \sum_{j=0}^{n-1} \Delta t_j \) which is computed at the first time when

\[
\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.
\]

The order \( s \) of the method is computed from

\[
s = \frac{\log((t_{2h} - t_h)/(t_{4h} - t_{2h}))}{\log 2}.
\]

**Numerical experiments**

First case. \( \varepsilon = 0 \).

<table>
<thead>
<tr>
<th>( I )</th>
<th>( t_n )</th>
<th>( n )</th>
<th>CPU time</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.021148</td>
<td>3176</td>
<td>8</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.020292</td>
<td>12221</td>
<td>38</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.019860</td>
<td>46949</td>
<td>301</td>
<td>0.98</td>
</tr>
<tr>
<td>128</td>
<td>0.019643</td>
<td>179956</td>
<td>4685</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

<table>
<thead>
<tr>
<th>( I )</th>
<th>( t_n )</th>
<th>( n )</th>
<th>CPU time</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.021126</td>
<td>3176</td>
<td>12</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.020288</td>
<td>12222</td>
<td>67</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.019859</td>
<td>46950</td>
<td>825</td>
<td>0.96</td>
</tr>
<tr>
<td>128</td>
<td>0.019643</td>
<td>179957</td>
<td>14280</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Second case. \( \varepsilon = \frac{1}{100} \).

<table>
<thead>
<tr>
<th>( I )</th>
<th>( t_n )</th>
<th>( n )</th>
<th>CPU time</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.021536</td>
<td>3185</td>
<td>7</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.020631</td>
<td>12256</td>
<td>37</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.020175</td>
<td>47089</td>
<td>473</td>
<td>0.99</td>
</tr>
<tr>
<td>128</td>
<td>0.019947</td>
<td>180517</td>
<td>5125</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.
<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.021512</td>
<td>3485</td>
<td>11</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.020926</td>
<td>12256</td>
<td>64</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.020174</td>
<td>47089</td>
<td>703</td>
<td>0.97</td>
</tr>
<tr>
<td>128</td>
<td>0.019947</td>
<td>180518</td>
<td>15084</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Third case. $\varepsilon = \frac{1}{10000}$.

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.021152</td>
<td>3176</td>
<td>7</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.020296</td>
<td>12221</td>
<td>37</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.019863</td>
<td>46957</td>
<td>312</td>
<td>0.98</td>
</tr>
<tr>
<td>128</td>
<td>0.019645</td>
<td>179953</td>
<td>6684</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

<table>
<thead>
<tr>
<th>$I$</th>
<th>$t_n$</th>
<th>$n$</th>
<th>CPU time</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.021129</td>
<td>3177</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.020291</td>
<td>12222</td>
<td>61</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.019862</td>
<td>46952</td>
<td>695</td>
<td>0.96</td>
</tr>
<tr>
<td>128</td>
<td>0.019646</td>
<td>179963</td>
<td>14091</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Remark 4.2. If we consider the problem (4.1)–(4.3) in the case where $p(r) = \frac{1}{r} + r$ and the initial datum $\varphi(r) = \frac{2 + \cos(\pi r)}{4} + \varepsilon(1 + \cos(\pi r))$, then we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is approximately equal to that in which the initial datum increases slightly, that is when $\varepsilon$ is a small positive real (see, Tables 1–6 for an illustration). This result confirms the theory regarding the continuity of the quenching time as a function of the initial datum.

In the following, we also give some plots to illustrate our analysis. In Figures 1–6, we can appreciate the evolution of the discrete solution which quenches in a finite time.
Continuity of the quenching time...

Figure 1: Evolution of the explicit discrete solution: $\varepsilon = 0$.

Figure 2: Evolution of the implicit discrete solution: $\varepsilon = 0$.

Figure 3: Evolution of the explicit discrete solution: $\varepsilon = \frac{1}{100}$.

Figure 4: Evolution of the implicit discrete solution: $\varepsilon = \frac{1}{100}$.

Figure 5: Evolution of the explicit discrete solution: $\varepsilon = \frac{1}{10000}$.

Figure 6: Evolution of the implicit discrete solution: $\varepsilon = \frac{1}{10000}$.
References


Continuity of the quenching time...


[18] V. A. Galaktionov, Boundary value problems for the nonlinear parabolic equation \( u_t = \Delta u^{\sigma+1} + u^\beta \), Diff. Equat., 17, 551–555 (1981).


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