THE WEBER-FECHNER LAW

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Abstract. In this note, we shall elucidate the reason why the degree of disturbance by noise is measured in logarithmic scale by the Weber-Fechner law in psychometrics. This in turn relies on the solution to the Schröder equation in the theory of functional equations. We shall solve it by the method of [3].

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1. Introduction and statement of results

1.1. The logarithmic scale

Noise pollution is one of the main annoying issues in the modern society (e.g., [7]). Since the sound is a wave, it is essential to make frequency analysis to prevent noise. As will be explained in Section 3, the degree of disturbances by noise is measured in phon, which is a logarithmic scale. We have taken it for granted and usually never pay attention to the reason why the logarithmic
scale is used. Since it is related to psychological sensation, there must be a reason in psychology as to why it is used. The reason seems to lie in the Weber-Fechner law to the effect that the intensity of sound is perceived in logarithmic scale (as is explained in [2]).

In this paper, we elucidate this law as a solution to the Abel functional equation, which has a rather intriguing property that it involves the iterates of a function, i.e., discrete dynamical systematic aspect. This aspect will be studied elsewhere, and we concentrate on solving the equation by the method of [3].

1.2. Weber-Fechner law

On [6, pp. 7–8] as well as on [2, pp. 188–191], the Weber-Fechner law is explained as follows. Let $I$ denote the intensity of a stimulus given on a subject. In an experiment of giving the stimulus by gradually enhancing it by $\Delta I$ to a subject, the minimum bound $\Delta I$ at which the subject can feel the difference between $I$ and $I + \Delta I$ is called the discrimination threshold. Letting the intensity of emotion (sensation level) at stimulus $I$ by $E$, the empirical Weber law may be expressed as $\frac{\Delta I}{I} = k \Delta E$. Fechner, viewing this as a differential equation $\frac{dI}{I} = kdE$, solved it to get $E = k^{-1} \log I + C$, which is known as the Weber-Fechner law.

We note that on [4, p. 212] the generalized Weber law is referred to as the equation for the Weber function $\Delta(s, \pi)$, $0 < \pi < 1$,

$$\Delta(s, \pi) = K(\pi)s + C(\pi), \quad (1.1)$$

where $K(\pi)$ is the Weber fraction computed for several stimuli [4, p. 205].

More important remark would be, however, that as is stressed on [4, p. 213], the Weber law and the Fechner law are independent of each other and the former refers to (1.1) and the latter (2) below.

The revised Fechner problem is stated on [4, p. 210] as follows. For every cutoff probability $\pi$, $0 < \pi < 1$, find the smooth strictly monotone solutions $u = u(s)$ that are independent of $\pi$ to the functional equation

$$u(s + \Delta(s, \pi)) - u(s) = g(\pi), \quad s \geq s_0, \quad (1.2)$$

where $g(\pi)$ is a strictly monotone function in $\pi$ and independent of $s$.

1.3. Weber-Fechner law revised, revisited

Note that equation (1.2) is a simple type Abel equation

$$u(f(x)) - u(x) = c, \quad c \neq 0, \quad (1.3)$$
where \( c \) is arbitrary. This changes into the Schröder form under the transformation \( \varphi(x) = s^u(x)/c; \):
\[
\varphi(f(x)) = s\varphi(x), \quad s \neq 0, 1. \tag{1.4}
\]

This correspondence is similar to that between additive and multiplicative arithmetical functions among arithmetical functions in number theory.

We shall solve (1.4) with \( f(x) = ax + b, a > 1, b > 0, \) where \( a - 1 = K(\pi). \)

**Theorem 1.** The unique strictly increasing solution to (1.4) with \( f(x) = ax + b, a > 1, b > 0, \) and \( 0 < s = \frac{1}{a} < 1 \) is given by
\[
\varphi(x) = \frac{\eta(a - 1)}{(a - 1)x + b},
\]
where \( \varphi'(0) = \eta > 0, \) and that of (1.3) is given by
\[
u(x) = A\log(x + \gamma) + B,\tag{1.5}\]
where \( \gamma = \frac{b}{a-1}, A = -\frac{c}{\log s}, \) and \( B = c\frac{\log \eta}{\log s} \).

For comparison’s sake, we state Theorem 4 from [4, p. 212].

**Theorem 2.** Under the assumption of the generalized Weber law (1.1) and that
\[
\frac{C(\pi)}{K(\pi)} = \gamma,
\]
where \( \gamma \) is a constant, the “smooth” strictly monotone solutions to (1.2) are given by
\[
u(s) = A\log(s + \gamma) + B,
\]
where \( A > 0, B \) are constants.

2. **Proof of Theorem 1**

For the proof of Theorem 1, we need some auxiliary results from [3].

**Lemma 1** (Lemma 6.1, [3]). Let \( f(x) \) be defined on a set \( E \subset \mathbb{R} \) such that \( f(E) \subset E \), let \( g(x) \) be a \( 1 : 1 \) mapping from \( E \) onto a set \( E_1 \), and let
\[
h = g \circ f \circ g^{-1}; \quad h(x) = g(f(g^{-1}(x))). \tag{2.1}
\]

Then if \( \psi(x) \) is a solution to the equation
\[
\psi(h(x)) = s\psi(x), \quad s \neq 1, 0, \tag{2.2}
\]
in $E_1$, then the function
\[ \varphi(x) = \psi(g(x)) \]
is a solution to the Schröder equation (1.4).

Proof follows from the observation
\[ \varphi(f(x)) = \psi(g \circ f(x)) = \psi(h \circ g(x)) = s\psi(g(x)) = s\varphi(x). \]

**Lemma 2 (Theorem 6.1, [3]).** Suppose $h(x)$ is a $C^r$-class strictly increasing function satisfying the $R_0$-condition with $r \geq 2$ on [3, p. 20]. Then, for every $\eta \in \mathbb{R}$, there exists a unique $C^r$-class solution $\varphi$ to
\[ \psi(h(x)) = s\psi(x), \quad s \neq 0, 1, \]
satisfying
\[ \psi'(0) = \eta, \]
the solution to (1.4) being given by
\[ \psi(x) = \eta \lim_{n \to \infty} s^{-n}h^n(x). \]

**Proof of Theorem 1.** Lemma 1 is used for several purposes as stated on [3, Lemma 6.1, p. 136] and here we use it to transfer the fixed point $\xi = \infty$ of $f$ to the origin by putting
\[ g(x) = \frac{1}{x}. \]

Then, by (2.1),
\[ h(x) = \frac{x}{bx + a} = M \begin{pmatrix} x \\ 1 \end{pmatrix}, \]
where $M = \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix}$, the last equality being an abbreviation of the linear fractional transformation. Hence, 0 is the fixed point of $h(x)$, and we are to solve (2.2). Since
\[ h'(0) = \frac{1}{a}, \]
our assumption $a > 1$ assures the regularity condition $0 < s < 1$ in view of
\[ \psi'(0)h'(0) = \psi'(h(0))h'(0) = s\psi'(0), \]
whence $\psi'(0) = 0$ or $h'(0) = s$, the former case being excluded as in [3], whence we conclude that

$$s = h'(0) = \frac{1}{a}.$$ 

By a standard technique, we may obtain a closed form for $M^n$:

$$M^n = \begin{pmatrix} b/(a-1)(a^n - 1) & 0 \\ a^n & a^n \end{pmatrix},$$

so that

$$h^n(x) = \frac{x}{b/(a-1)(a^n - 1)x + a^n}.$$ 

Applying Lemma 2, we find that

$$\psi(x) = \eta \lim_{n \to \infty} a^n h^n(x) = \eta \lim_{n \to \infty} a^n \frac{x}{b/(a-1)(a^n - 1)x + a^n}$$

$$\frac{(a-1)x}{bx + a - 1},$$

and therefore the solution to (1.4) is

$$\varphi(x) = \psi(g(x)) = \frac{\eta}{x + b/(a - 1)},$$

whence

$$\log \varphi(x) = \log \eta - \log \left( x + \frac{b}{a - 1} \right).$$

Hence the solution to the Abel equation (1.3) is given by

$$u(x) = -\frac{c}{\log s} \log \left( x + \frac{b}{a - 1} \right) + \frac{c \log \eta}{\log s},$$

which is tantamount to (1.5), completing the proof.

**Remark 2.1.**

(i) Although we applied Lemma 1 with $g(x) = \frac{1}{x}$, we could do well with $g(x) = \frac{1}{ax + 1}$. For since $g^{-1}(x) = \frac{1-bx}{ax}$, it follows that $f \circ g^{-1}(x) = \frac{1}{x}$, and we still have the same $h(x)$. Hence, there is no problem on the inclusion of the fixed point $\xi = 0$.

(ii) We may apply [3, Theorem 6.6, pp. 141-142] equally well to prove Theorem 1.

(iii) If $b = 0$, then $\xi = 0$ is a fixed point of $f$, which however is to be excluded by the remark on [3, p. 136].
3. Scales for measuring noise

The pitch of sound (note) is defined by its frequency measured in cycles/second (Herz) (c/sec = Hz), e.g., the middle C (do) is 260 Hz, while the middle A (la) is 440 Hz.

On [6, p. 12], the sensational unit dB (deci Bel-deci = \( \frac{1}{10} \)) is introduced to measure intensity of sound (or sound level) \( S \) by

\[
\text{dB} = 20 \log \frac{S}{S_0},
\]

where the standard sound pressure \( S_0 \) is 0.0002\( \mu \)bar, corresponding to the minimum audible threshold by a young adult subject of pure sound with frequency 1000 Hz. Note that two octaves above middle C has frequency \( 260 \times 2 \times 2 = 1040 \) Hz.

Similarly, another more familiar sensational unit phon is introduced as the intensity dB of the sound of 1000 Hz that is observed as the same intensity as that of the given sound, i.e., phon is the unit by which the psychologically same intensity of sound and thus used in measuring the degree of disturbance. The Fletcher-Mason curve on [6, p. 121] indicates that the audible sensitivity is dull at sound of lower frequency and sharp at higher frequency.

In addition to the Weber-Fechner law, another reason why the logarithm is used for measuring the sensational sound intensity, can be found on [1, p. 429]. There the interval \( aI_b \) of two (musical) notes of frequencies \( a \) and \( b \), \( b > a \), with \( b \) being a higher note, \( a \) lower note. This gives an ordering relation. More commonly accepted word is “key”; key word! \( aI_b = \frac{b}{a} > 1 \). The interval is not the distance function because its value for \( a = a \) (unison) is \( aI_a \), which is 1. Thus taking the logarithm of \( aI_b \), \( \log aI_b \), we have the logarithmic interval, which defines a distance function \( d(a, b) = |\log aI_b| \). Hence it makes the space of all notes a metric space and makes it possible to apply rich results from the theory of metric spaces, possibly opening a new field of topological theory of music. If the base of the logarithm is 2, we understand the intervals in octaves while if it is \( 2^{1/12} \), they are interpreted in tempered semitones.

References


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