INEXTENSIBLE FLOWS OF TIMELIKE CURVES WITH SABBAN FRAME IN $S^2_1$

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Abstract. In this paper, we study timelike curves according to Sabban frame in $S^2_1$. We research inextensible flows of timelike curves according to Sabban frame in $S^2_1$.

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1. Introduction

Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications. For example, both Chirikjian and Burdick [1] and Mochiyama et al. [6] study the shape control of hyper-redundant, or snake-like, robots. Inextensible curve and surface flows also arise in the context of many problems in computer vision [5] and computer animation [2], [4].

This paper is organized as follows. Firstly, we study timelike curves according to the Sabban frame in $S^2_1$. Finally, we research inextensible flows of timelike curves according to the Sabban frame in $S^2_1$.

2. Preliminaries

The Minkowski 3-space $E^3_1$ provided with the standard flat metric given by
\[ \langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2, \]

where \((x_1, x_2, x_3)\) is a rectangular coordinate system of \(\mathbb{E}^3_1\). Recall that, the norm of an arbitrary vector \(a \in \mathbb{E}^3_1\) is given by \(\|a\| = \sqrt{\langle a, a \rangle}\). \(\gamma\) is called a unit speed curve if velocity vector \(v\) of \(\gamma\) satisfies \(\|v\| = 1\).

Denote by \(\{T, N, B\}\) the moving Frenet-Serret frame along the timelike curve \(\gamma\) in the space \(\mathbb{E}^3_1\). For an arbitrary timelike curve \(\gamma\) with first and second curvature, \(\kappa\) and \(\tau\) in the space \(\mathbb{E}^3_1\), the following Frenet-Serret formulae are given

\[
\begin{align*}
T' &= \kappa N, \\
N' &= \kappa T + \tau B, \\
B' &= -\tau N,
\end{align*}
\]

where

\[
\begin{align*}
\langle T, T \rangle &= -1, \quad \langle N, N \rangle = \langle B, B \rangle = 1, \\
\langle T, N \rangle &= \langle T, B \rangle = \langle N, B \rangle = 0.
\end{align*}
\]

Here, the curvature functions are defined by \(\kappa = \kappa(s) = \|T'(s)\|\) and \(\tau(s) = -\langle N, B' \rangle\).

Torsion of the timelike curve \(\gamma\) is given by the aid of the mixed product

\[
\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.
\]

Now we give a new frame different from the Frenet frame. Let \(\alpha : I \to S^2_1\) be an unit speed spherical timelike curve. We denote \(\sigma\) as the arc-length parameter of \(\alpha\). Let us denote \(t(\sigma) = \alpha'(\sigma)\), and we call \(t(\sigma)\) a unit tangent vector of \(\alpha\). We now set a vector \(s(\sigma) = \alpha(\sigma) \times t(\sigma)\) along \(\alpha\). This frame is called the Sabban frame of \(\alpha\) on \(S^2_1\). Then we have the following spherical Frenet-Serret formulae of \(\alpha\) [3]:

\[
\begin{align*}
\alpha' &= t, \\
t' &= \alpha + \kappa_g s, \\
s' &= \kappa_g t,
\end{align*}
\]

where \(\kappa_g\) is the geodesic curvature of the timelike curve \(\alpha\) on the \(S^2_1\), and

\[
\begin{align*}
g(t, t) &= -1, \quad g(\alpha, \alpha) = 1, \quad g(s, s) = 1, \\
g(t, \alpha) &= g(t, s) = g(\alpha, s) = 0.
\end{align*}
\]
3. Inextensible flows of timelike curves according to the Sabban frame in $S^2_1$

Let $\alpha (u,t)$ is a one-parameter family of smooth timelike curves in $S^2_1$.

The arc length of $\alpha$ is given by

$$\sigma(u) = 0 \left| \frac{\partial \alpha}{\partial u} \right| du,$$

where

$$\left| \frac{\partial \alpha}{\partial u} \right| = \left( \frac{\partial \alpha \cdot \partial \alpha}{\partial u \cdot \partial u} \right)^{1/2}.$$

The operator $\frac{\partial}{\partial \sigma}$ is given in terms of $u$ by

$$\frac{\partial}{\partial \sigma} = \frac{1}{\nu} \frac{\partial}{\partial u},$$

where $v = \left| \frac{\partial \alpha}{\partial u} \right|$, and the arc length parameter is $d\sigma = vdu$.

Any flow of $\alpha$ can be represented as

$$\frac{\partial \alpha}{\partial t} = \int^S_0 \alpha + \int^S_2 t + \int^S_3 s. \quad (3.1)$$

Let the arc length variation be

$$\sigma(u,t) = \nu vdu.$$

In the $S^2_1$, the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} \sigma(u,t) = \nu \frac{\partial v}{\partial t} du = 0 \quad (3.2)$$

for all $u \in [0,l]$.

**Definition 3.1.** The flow $\frac{\partial \alpha}{\partial t}$ in $S^2_1$ is said to be inextensible if

$$\frac{\partial}{\partial t} \left| \frac{\partial \alpha}{\partial u} \right| = 0.$$
Lemma 3.2. Let \( \frac{\partial \alpha}{\partial t} = \beta_1^S \alpha + \beta_2^S t + \beta_3^S s \) be a smooth flow of the timelike curve \( \alpha \). The flow is inextensible if and only if

\[
\frac{\partial v}{\partial t} + \frac{\partial \beta_2^S}{\partial u} = -\beta_1^S v - \beta_3^S v \kappa_g.
\] (3.3)

Proof. Suppose that \( \frac{\partial \alpha}{\partial t} \) be a smooth flow of the timelike curve \( \alpha \). Using definition of \( \alpha \), we have

\[
v^2 = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle.
\] (3.4)

By differentiating of the formula (??), we get

\[
2v \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle.
\]

On the other hand, changing \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial t} \) we have

\[
v \frac{\partial v}{\partial t} = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial t} \left( \frac{\partial \alpha}{\partial u} \right) \right\rangle.
\]

From (3.1), we obtain

\[
\frac{\partial v}{\partial t} = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u} \left( \beta_1^S \alpha + \beta_2^S t + \beta_3^S s \right) \right\rangle.
\]

By the Sabban formula, we have

\[
\frac{\partial v}{\partial t} = \left\langle t, \left( \frac{\partial \beta_1^S}{\partial u} + \beta_2^S v \right) \alpha + \left( \beta_1^S v + \frac{\partial \beta_2^S}{\partial u} + \beta_3^S v \kappa_g \right) t + \left( \frac{\partial \beta_3^S}{\partial u} + \beta_2^S v \kappa_g \right) s \right\rangle.
\]

Making necessary calculations, from above equation we obtain (3.3) which proves the lemma.

Theorem 3.3. Let \( \frac{\partial \alpha}{\partial t} = \beta_1^S \alpha + \beta_2^S t + \beta_3^S s \) be a smooth flow of the timelike curve \( \alpha \). The flow is inextensible if and only if

\[
\frac{\partial \beta_1^S}{\partial u} = -\beta_3^S v \kappa_g - \beta_1^S v.
\]
Proof. Assume that $\frac{\partial \alpha}{\partial t}$ is inextensible. From (3.2), we have
\[
\frac{\partial}{\partial t} \sigma(u, t) = \frac{u}{\alpha} \frac{\partial v}{\partial t} du = \frac{u}{\alpha} \left( -f^S_1 v - \frac{\partial f^S_2}{\partial u} - f^S_3 v \kappa_g \right) du = 0. \tag{3.5}
\]
Substituting (3.3) in (3.5) completes the proof of the theorem.

Now we restrict ourselves to arc length parametrized curves. That is, $v = 1$ and the local coordinate $u$ corresponds to the curve arc length $\sigma$. We require the following lemma.

**Lemma 3.4.** The following relations hold
\[
\frac{\partial t}{\partial t} = \left( \frac{\partial f^S_1}{\partial \sigma} + f^S_2 \right) \alpha + \left( \frac{\partial f^S_3}{\partial \sigma} + f^S_2 \kappa_g \right) s,
\]
\[
\frac{\partial \alpha}{\partial t} = \left( \frac{\partial f^S_1}{\partial \sigma} + f^S_2 \right) t + \psi s,
\]
\[
\frac{\partial s}{\partial t} = \left( \frac{\partial f^S_3}{\partial \sigma} + f^S_2 \kappa_g \right) t - \psi \alpha,
\]
where $\psi = \left( \frac{\partial \alpha}{\partial t}, s \right)$.

**Proof.** Using definition of $\alpha$, we have
\[
\frac{\partial t}{\partial t} = \frac{\partial \alpha}{\partial t} = \frac{\partial}{\partial \sigma} \left( f^S_1 \alpha + f^S_2 t + f^S_3 s \right).\]
Using the Sabbian equations, we find that
\[
\frac{\partial t}{\partial t} = \left( \frac{\partial f^S_1}{\partial \sigma} + f^S_2 \right) \alpha + \left( f^S_1 + \frac{\partial f^S_2}{\partial \sigma} + f^S_3 \kappa_g \right) t + \left( \frac{\partial f^S_3}{\partial \sigma} + f^S_2 \kappa_g \right) s. \tag{3.6}
\]
Substituting (3.3) in (3.6), we get
\[
\frac{\partial t}{\partial t} = \left( \frac{\partial f^S_1}{\partial \sigma} + f^S_2 \right) \alpha + \left( \frac{\partial f^S_3}{\partial \sigma} + f^S_2 \kappa_g \right) s.
\]
Now differentiate the Sabbian frame by $t$:
\[
\frac{\partial f^S_1}{\partial \sigma} + f^S_2 + \left( t, \frac{\partial \alpha}{\partial t} \right) = 0,
\]
\[
\frac{\partial f^S_3}{\partial \sigma} + f^S_2 \kappa_g + \left( t, \frac{\partial s}{\partial t} \right) = 0,
\]
\[
\psi + \left( \alpha, \frac{\partial s}{\partial t} \right) = 0.
\]
Then, a straightforward computation using above system gives

\[
\frac{\partial \alpha}{\partial t} = \left( \frac{\partial \hat{f}^S}{\partial \sigma} + \hat{f}_2^S \right) \mathbf{t} + \psi \mathbf{s},
\]

\[
\frac{\partial \mathbf{s}}{\partial t} = \left( \frac{\partial \hat{f}^S}{\partial \sigma} + \hat{f}_2^S \kappa_g \right) \mathbf{t} - \psi \alpha,
\]

where \( \psi = \left( \frac{\partial \alpha}{\partial t} \cdot \mathbf{s} \right) \). Thus, we obtain the assertion of the theorem.

The following theorem states the conditions on the curvature and torsion for the flow to be inextensible.

**Theorem 3.5.** Let \( \frac{\partial \alpha}{\partial t} \) be inextensible. Then the system of partial differential equations

\[
\frac{\partial \kappa_g}{\partial \sigma} + \psi = \frac{\partial^2 \hat{f}^S_1}{\partial \sigma^2} + \frac{\partial}{\partial \sigma} \left( \hat{f}^S_2 \kappa_g \right),
\]

\[
\kappa_g \psi = - \left( \frac{\partial^2 \hat{f}^S_1}{\partial \sigma^2} + \frac{\partial \hat{f}^S_2}{\partial \sigma} \right)
\]

holds.

**Proof.** Assume that \( \frac{\partial \alpha}{\partial t} \) is inextensible. Then

\[
\frac{\partial}{\partial \sigma} \frac{\partial \mathbf{t}}{\partial t} = \frac{\partial}{\partial \sigma} \left[ \left( \frac{\partial \hat{f}^S_1}{\partial \sigma} + \hat{f}_2^S \right) \alpha + \left( \frac{\partial \hat{f}^S}{\partial \sigma} + \hat{f}_2^S \kappa_g \right) \right] \mathbf{s}
\]

\[
= \left( \frac{\partial^2 \hat{f}^S_1}{\partial \sigma^2} + \frac{\partial \hat{f}^S_2}{\partial \sigma} \right) \alpha + \left( \frac{\partial \hat{f}^S_1}{\partial \sigma} + \hat{f}_2^S \right) \kappa_g \left( \frac{\partial \hat{f}^S_3}{\partial \sigma} + \hat{f}_2^S \kappa_g \right) \mathbf{t}
\]

\[+ \left( \frac{\partial^2 \hat{f}^S_3}{\partial \sigma^2} + \frac{\partial \hat{f}^S_2}{\partial \sigma} \right) \mathbf{s}.
\]

From the Sabban frame, we have

\[
\frac{\partial}{\partial t} \frac{\partial \mathbf{t}}{\partial \sigma} = \frac{\partial}{\partial t} (\alpha + \kappa_g \mathbf{s}) \]

\[= \left[ \frac{\partial \kappa_g}{\partial \sigma} + \psi \right] \mathbf{s} + \left[ \left( \frac{\partial \hat{f}^S_1}{\partial \sigma} + \hat{f}_2^S \right) \kappa_g \left( \frac{\partial \hat{f}^S_3}{\partial \sigma} + \hat{f}_2^S \kappa_g \right) \right] \mathbf{t} - \kappa_g \psi \alpha.
\]

Therefore,

\[
\frac{\partial \kappa_g}{\partial \sigma} + \psi = \frac{\partial^2 \hat{f}^S_1}{\partial \sigma^2} + \frac{\partial}{\partial \sigma} \left( \hat{f}^S_2 \kappa_g \right)
\]
and
\[ \kappa_g \psi = - \left( \frac{\partial^2 \tilde{r}_1^S}{\partial \tau^2} + \frac{\partial \tilde{r}_2^S}{\partial \tau} \right). \]

Thus, we obtain the theorem.

**Corollary 3.6.**
\[ \kappa_g \left( \frac{\partial \tilde{r}_1^S}{\partial \tau} + \tilde{r}_2^S \right) = - \frac{\partial \tilde{r}_3^S}{\partial \tau} - \tilde{r}_2^S \kappa_g - \frac{\partial \psi}{\partial \tau}. \]

**Proof.** Similarly, we have
\[ \frac{\partial}{\partial \tau} \frac{\partial s}{\partial \tau} = \frac{\partial}{\partial \tau} \left[ \left( \frac{\partial \tilde{r}_1^S}{\partial \tau} + \tilde{r}_2^S \kappa_g \right) \frac{\partial t}{\partial \tau} - \psi \alpha \right] \]
\[ = \left[ \left( \frac{\partial^2 \tilde{r}_1^S}{\partial \tau^2} + \frac{\partial}{\partial \tau} \left( \tilde{r}_2^S \kappa_g \right) - \psi \right) \frac{\partial t}{\partial \tau} + \kappa_g \left( \frac{\partial \tilde{r}_3^S}{\partial \tau} + \tilde{r}_2^S \kappa_g \right) \right] s \]
\[ + \left[ \left( \frac{\partial \tilde{r}_3^S}{\partial \tau} + \tilde{r}_2^S \kappa_g \right) - \frac{\partial \psi}{\partial \tau} \right] \alpha. \]

On the other hand, a straightforward computation gives
\[ \frac{\partial}{\partial t} \frac{\partial s}{\partial \tau} = \frac{\partial}{\partial t} \kappa_g \left[ \left( \frac{\partial \tilde{r}_1^S}{\partial \tau} + \tilde{r}_2^S \right) \alpha + \left( \frac{\partial \tilde{r}_3^S}{\partial \tau} + \tilde{r}_2^S \kappa_g \right) s \right]. \]

Combining these equalities, we obtain the corollary.

In the light of Theorem 3.5, we express the following corollaries without proofs.

**Corollary 3.7.**
\[ \frac{\partial \kappa_g}{\partial t} = \frac{\partial \tilde{r}_2^S}{\partial \tau} + \frac{\partial}{\partial \tau} \left( \tilde{r}_2^S \kappa_g \right) - \psi. \]

**Corollary 3.8.**
\[ \frac{\partial \tilde{r}_3^S}{\partial \tau} + \tilde{r}_2^S \kappa_g - \frac{\partial \psi}{\partial \tau} = \frac{\partial \tilde{r}_1^S}{\partial \tau} + \tilde{r}_2^S. \]

**References**


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