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MELLIN TRANSFORM OF THE DIRICHLET L-FUNCTION WITH PRINCIPAL CHARACTER

Aidas BALČIŪNAS
Institute of Mathematics and Informatics, Vilnius University,
Akademijos 4, LT-08663 Vilnius, Lithuania;
e-mail: aidas.balciunas@mii.stud.vu.lt

Abstract. We obtain a meromorphic continuation for the modified Mellin transform of Dirichlet L-function with principal character. This extends the results of [19].

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1. Introduction

The classical Mellin transform $M_f(s)$, $s=\sigma+it$, of a function $f(x)$ is defined by

$$M_f(s)=\int_0^\infty f(x)x^{s-1}\mathrm{d}x$$

provided that the integral exists. Sometimes the modified Mellin transform

$$M_f^*(s)=\int_1^\infty f(x)x^{-s}\mathrm{d}x$$

is more convenient, because a possible convergence problem at the point $x=0$ does not occur.

The functions $M_f(s)$ and $M_f^*(s)$ are related by a simple relation. Let

$$\hat{f}(x)=\begin{cases} f\left(\frac{1}{x}\right) & \text{if} \quad 0<x\leqslant 1,\\ 0 & \text{otherwise.}\end{cases}$$

Then it is known [5] that

$$M_f^*(s) = M_{\hat{f}(x)/x}(s).$$

Modified Mellin transforms of powers of the Riemann zeta-function $\zeta(s)$

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} x^{-s} \mathrm{d}x$$

were introduced, studied and applied for investigation of the moments

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t, \quad T \to \infty,$$

in a series of works by A. Ivič, M. Jutila and Y. Motohashi [5]–[11], [15], [20], [21], and in Ph.D. Thesis of M. Lukkarinen [19]. The papers [14]–[17] are devoted to the Mellin transform

$$\int_1^\infty \left|\zeta\Big(\varrho + ix\Big)\right|^{2k} x^{-s} dx$$

with a fixed $\frac{1}{2} < \varrho < 1$.

Let χ be a Dirichlet character modulo q, and let $L(s,\chi)$ be the corresponding Dirichlet L-function. In the theory of L-functions, usually the modified Mellin transforms

$$\mathcal{Z}_k(s, L) \stackrel{def}{=} \sum_{\chi \bmod q} \int_1^\infty \left|L\left(\frac{1}{2} + ix, \chi\right)\right|^{2k} x^{-s} \mathrm{d}x$$

are considered. For example, in [19] a meromorphic continuation to the region $\sigma > 0$ for $\mathcal{Z}_1(s, L)$ has been obtained.

Our aim is to study an individual modified Mellin transform of a Dirichlet L-function

$$\mathcal{Z}_k(s, \chi) \stackrel{def}{=} \int_1^\infty \left|L\left(\frac{1}{2} + ix, \chi\right)\right|^{2k} x^{-s} \mathrm{d}x.$$

This note is devoted to a meromorphic continuation of the function $\mathcal{Z}_1(s, \chi_0)$, where, as usual, χ_0 is the principal character modulo q.

Denote by p a prime number. It is well known that

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Thus, the functions $\mathcal{Z}_1(s)$ and $\mathcal{Z}_1(s,\chi_0)$ should have similar analytic properties. A meromorphic continuation of the function $\mathcal{Z}_1(s)$ was considered in [9], and the final result was obtained in [19]. Let B_k stand for the kth Bernoulli number, and let γ be the Euler constant. Then, in [19], the following theorem has been proved.

THEOREM 1.1. *The function $\mathcal{Z}_1(s)$ has a meromorphic continuation to the whole complex plane. It has a double pole at the point $s=1$, and its Laurent expansion at this point is*

$$\mathcal{Z}_1(s)=\frac{1}{(s-1)^2}+\frac{2\gamma-\log 2\pi}{s-1}+\dots\ .$$

The other poles of $\mathcal{Z}_1(s)$ are the simple poles at the points $s=-(2k-1)$, $k\in\mathbb{N}$, and

$$\operatorname*{Res}_{s=-(2k-1)}\mathcal{Z}_1(s)=\frac{i^{-2k}(1-2^{-(2k-1)})}{2k}B_{2k}.$$

A meromorphic continuation to the half-plane $\sigma>0$ for the function $\mathcal{Z}_1(s,\chi_0)$ has been obtained in [19]. We extend this to the whole complex plane. Let

$$a(q)=\sum_{p|q}\frac{\log p}{p-1},$$

and $\varphi(q)$ denotes the Euler totient function.

THEOREM 1.2. *The function $\mathcal{Z}_1(s,\chi_0)$ has a meromorphic continuation to the whole complex plane. It has a double pole at the point $s=1$, and its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi_0)=\frac{\varphi(q)}{q}\left(\frac{1}{(s-1)^2}+\frac{2\gamma+2a(q)-\log 2\pi}{s-1}\right)+\dots\ .$$

The other poles of $\mathcal{Z}_1(s,\chi_0)$ are the simple poles at the points $s=-(2k-1)$, $k\in\mathbb{N}$, and

$$\operatorname*{Res}_{s=-(2k-1)}\mathcal{Z}_1(s,\chi_0)=\frac{\varphi(q)}{q}\frac{i^{-2k}(1-2^{-(2k-1)})}{2k}B_{2k}.$$

It is sufficient to prove only the second part of Theorem 1.2. However, for fullness, we follow the proof of Theorem 1.1 and present a full proof of Theorem 1.2.

2. The Laplace transform

The proof of Theorem 1.2 uses a formula for the modified Laplace transform of the Dirichlet L-function. In [2], such a formula has been obtained for

$$\mathfrak{L}(s,\chi) \overset{def}{=} \int_0^\infty \left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 \mathrm{e}^{-sx}\mathrm{d}x.$$

An analogous formula is also true for the modified Laplace transform

$$\int_1^\infty \left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 \mathrm{e}^{-sx}\mathrm{d}x.$$

We state this formula for $L(s,\chi_0)$. Denote by $\mu(m)$ the Möbius function.

LEMMA 2.1. *Suppose that* $0 < \operatorname{Re} w < \pi$. *Then*

$$\begin{aligned}\mathfrak{L}(w,\chi_0) &= i\mathrm{e}^{iw/2}\frac{\varphi(q)}{q}\sum_{m|q}\mu(m)\left(\gamma - \log 2\pi - \left(\frac{\pi}{2}-w\right)i\right.\\ &\quad \left. +a(q)+\log m\right)\\ &\quad +2\pi i\mathrm{e}^{-iw/2}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^\infty d(k)\exp\left\{-\frac{2\pi ikn}{m}\mathrm{e}^{-iw}\right\}\\ &\quad +\lambda(w,\chi_0),\end{aligned}$$

where the function $\lambda(w,\chi_0)$ *is analytic in the strip* $\{w \in \mathbb{C} : |\operatorname{Re} w| < \pi\}$, *and, for* $|\operatorname{Re} w| \le \theta, 0 < \theta < \pi$, *the estimate*

$$\lambda(w,\chi_0) = \mathrm{O}((1+|w|)^{-1})$$

is valid.

Now we express $\mathcal{Z}_1(s,\chi_0)$ by $\mathfrak{L}(w,\chi_0)$. For $\sigma > 1$, the definitions of $\mathcal{Z}_1(s,\chi_0)$, $\mathfrak{L}(w,\chi_0)$ and $\Gamma(s)$ imply the formula

$$\mathcal{Z}_1(s,\chi_0) = \frac{1}{\Gamma(s)}\int_0^\infty \mathfrak{L}(w,\chi_0)w^{s-1}\mathrm{d}w. \tag{1}$$

Further, we change the integration over the positive real axis in the above integral. For this, we use the following equalities which are easily obtained by partial integration. For $0 \leqslant \alpha < \frac{\pi}{2}$, let

$$L = \{w \in \mathbb{C} : w = R\mathrm{e}^{i\varphi}, 0 \leqslant \varphi \leqslant \alpha\}$$

and

$$L_1 = \{w \in \mathbb{C} : w = r\mathrm{e}^{i\varphi}, 0 \leqslant \varphi \leqslant \alpha\}.$$

Then

$$\lim_{R\to\infty} \int\limits_{L} \mathfrak{L}(w,\chi_0) w^{s-1} \mathrm{d}w = 0, \tag{2}$$

and, for $\sigma > 1$,

$$\lim_{r\to 0} \int\limits_{L_1} \mathfrak{L}(w,\chi_0) w^{s-1} \mathrm{d}w = 0. \tag{3}$$

LEMMA 2.2. *Suppose that $0 \leqslant \alpha < \frac{\pi}{2}$ and $\sigma > 1$. Then*

$$\mathcal{Z}_1(s,\chi_0) = \frac{1}{\Gamma(s)} \int_0^{\infty\mathrm{e}^{i\alpha}} \mathfrak{L}(w,\chi_0) w^{s-1} \mathrm{d}w.$$

Proof. The equality of the lemma is a consequence of the residue theorem and formulae (1)–(3). □

Now we fix a point $w_0 = |w_0|\mathrm{e}^{i\alpha}$ with $0 < \operatorname{Re} w_0 < \pi$, and define the functions

$$\begin{aligned}
\mathcal{Z}_{11}(s,\chi_0) &= \frac{i\varphi(q)}{\Gamma(s)} \sum_{m|q} \mu(m) \\
&\quad \times \int_0^{w_0} \left(\gamma - \log 2\pi - \left(\frac{\pi}{2} - w\right)i + a(q) + \log m\right) \mathrm{e}^{iw/2} w^{s-1} \mathrm{d}w, \\
\mathcal{Z}_{12}(s,\chi_0) &= \frac{1}{\Gamma(s)} \int_0^{w_0} \lambda(w,\chi_0) w^{s-1} \mathrm{d}w, \\
\mathcal{Z}_{13}(s,\chi_0) &= \frac{1}{\Gamma(s)} \int_{w_0}^{\infty e^{i\alpha}} \left(\int_1^\infty \left| L\left(\frac{1}{2} + ix, \chi_0\right)\right|^2 \mathrm{e}^{-wx} \mathrm{d}x \right) w^{s-1} \mathrm{d}w, \\
\mathcal{Z}_{14}(s,\chi_0) &= \frac{2\pi}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) \\
&\quad \times \int_0^{w_0} \mathrm{e}^{-iw/2} \exp\left\{ - \frac{2\pi i k n}{m} \mathrm{e}^{-iw} \right\} w^{s-1} \mathrm{d}w.
\end{aligned}$$

Then, taking into account Lemmas 2.1 and 2.2, we find that, for $\sigma > 1$,

$$\mathcal{Z}_1(s,\chi_0) = \sum_{j=1}^{4} \mathcal{Z}_{1j}(s,\chi_0). \tag{4}$$

LEMMA 2.3. *The functions $\mathcal{Z}_{1j}(s,\chi_0)$, $j = 1, 2, 3$, are entire.*

Proof. Let $k \in \mathbb{N}$ be arbitrary. Integrating by parts k times and using the functional equation $\Gamma(s+1) = s\Gamma(s)$, we find that

$$\begin{aligned}&\frac{1}{\Gamma(s)} \int_0^{w_0} \mathrm{e}^{iw/2} w^s \mathrm{d}w \\ &\quad = \frac{\left(-i/2\right)^k}{\Gamma(s+k+1)} \int_0^{w_0} \mathrm{e}^{iw/2} w^{s+k} \mathrm{d}w + \sum_{j=1}^{k} \frac{(-i/2)^{j-1} s \mathrm{e}^{iw_0/2} w_0^{s+j}}{\Gamma(s+j+1)},\end{aligned}$$

and the latter function is holomorphic for $\sigma > -k+1$. Since k is arbitrary, hence $\mathcal{Z}_{11}(s, \chi_0)$ is an entire function. Similarly, using the properties of the function $\lambda(w, \chi_0)$, we find that function $\mathcal{Z}_{12}(s, \chi_0)$ is also entire. Obviously, the function $\mathcal{Z}_{13}(s, \chi_0)$ is entire by its definition. □

In view of Lemma 2.3 and (4), it remains to study the function $\mathcal{Z}_{14}(s, \chi_0)$ which will give the poles of $\mathcal{Z}_1(s, \chi_0)$.

3. A transformation formula

For the investigation of the function $\mathcal{Z}_{14}(s, \chi_0)$, a certain transformation formula is needed. This formula involves the Estermann zeta-function.

Let $\alpha \in \mathbb{C}$, and

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha$$

be the generalized divisor function. We have that $\sigma_0(m) = d(m)$, where

$$d(m) = \sum_{d|m} 1$$

is the divisor function. Let $l \geqslant 1$ and $(k, l) = 1$. The Estermann zeta-function $E\big(s; \frac{k}{l}, \alpha\big)$, for $\sigma > \max(1 + \operatorname{Re}\alpha, 1)$, is defined by the series

$$E\left(s; \frac{k}{l}, \alpha\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

For $\lambda \in \mathbb{R}$ and $0 < \beta \leqslant 1$, denote by $L(\lambda, \beta, s)$ the Lerch zeta-function which, for $\sigma > 1$, is defined by

$$L(\lambda, \beta, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i m \lambda}}{(m+\beta)^s}.$$

It is well known, see, for example, [18], that $L(\lambda, \beta, s)$, for $\lambda \notin \mathbb{Z}$, continues analytically to an entire function, while for $\lambda \in \mathbb{Z}$, the function $L(\lambda, \beta, s)$ becomes the Hurwitz zeta-function which continues meromorphically to the whole complex plane with a unique simple pole at the point $s = 1$ with residue 1.

It is not difficult to see that, for $\sigma > \max(\operatorname{Re}\alpha + 1, 1)$

$$E\Big(s; \frac{k}{l}, \alpha\Big) = l^{\alpha - s} \sum_{v=1}^{l} \exp\Big\{2\pi i \frac{vk}{l}\Big\} L\Big(1, \frac{v}{l}, s - \alpha\Big) L\Big(\frac{v}{l}, 1, s\Big). \quad (5)$$

This and the above mentioned properties of the function $L(\lambda, \beta, s)$ show that the function $E\big(s; \frac{k}{l}, \alpha\big)$ is analytic in the whole complex plane, except for two simple poles at $s = 1$ and $s = 1 + \alpha$ if $\alpha \neq 0$, respectively double pole at $s = 1$ if $\alpha = 0$.

Now let $\overline{k}$ and k satisfy the congruence $k\overline{k} \equiv 1 (\bmod l)$. Then equality (5) together with the functional equation for the Lerch zeta-function, see, for example, [19], leads to the functional equation for $E\big(s; \frac{k}{l}, \alpha\big)$

$$\begin{aligned} E\Big(s; \frac{k}{l}, \alpha\Big) &= \frac{1}{\pi}\Big(\frac{2\pi}{l}\Big)^{2s-1-\alpha} \Gamma\Big(1 - s\Big)\Gamma\Big(1 + \alpha - s\Big) \\ &\times \Big(\cos\frac{\pi\alpha}{2} E\Big(1 + \alpha - s; \frac{k}{l}, \alpha\Big) \\ &\quad - \cos\Big(\pi s - \frac{\pi\alpha}{2}\Big) E\Big(1 + \alpha - s; -\frac{\overline{k}}{l}, \alpha\Big)\Big). \end{aligned}$$

The functions $L\big(1, \frac{v}{l}, s - \alpha\big)$ and $L\big(\frac{v}{l}, 1, s\big)$ are linear combinations of Hurwitz zeta-functions and the Riemann zeta-function. Therefore, it is not difficult to deduce from (5) that

$$E\Big(s; \frac{k}{l}, 0\Big) = \frac{1}{l}\Big(\frac{1}{(s-1)^2} + \frac{2\gamma - 2\log l}{s - 1} + c_0 + c_1(s - 1) + \ldots\Big). \quad (6)$$

The function $E\big(s; \frac{k}{l}, \alpha\big)$, for $\alpha = 0$, was introduced by T. Estermann in [3] for needs of the representation of numbers as a sum of two products. In [12], the extension for $\alpha \in [-1, 0]$ has been given.

Let k and l be coprime integers and $z \in \mathbb{C}\backslash\{0\}$. Define

$$\Phi\Big(z, \frac{k}{l}\Big) = \sum_{m=1}^{\infty} d(m) \mathrm{e}^{2\pi i km/l} \mathrm{e}^{-mz} - \frac{\gamma - 2\log l - \log z}{lz}.$$

In this section, we state a formula for $\Phi\big(z^{-1}, \frac{k}{l}\big)$ which will be used for a study of the function $\mathcal{Z}_{14}(s, \chi_0)$.

Let c_0^+ and c_0^- be the constant terms in (6) for $E\big(s; \frac{\overline{k}}{l}, 0\big)$ and $E\big(s; -\frac{\overline{k}}{l}, 0\big)$, respectively, where $k\overline{k} \equiv 1(\bmod l)$. Moreover, denote

$$\delta = \begin{cases} 1 & \text{if} \quad \operatorname{Im} z > 0, \\ -1 & \text{if} \quad \operatorname{Im} z < 0, \end{cases}$$

and, for $1 < b < 2$, define

$$\begin{aligned} I(z, b) &= \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \left(\frac{2\pi}{l}\right)^{1-2w} \Gamma(w) \Bigg(\bigg(\sin\Big(\pi w\Big)\bigg)^{-1} E\bigg(w; \frac{\overline{k}}{l}, 0\bigg) \\ &\quad + \bigg(\cot\Big(\pi w\Big) + \delta i\bigg) E\bigg(w; -\frac{\overline{k}}{l}, 0\bigg)\Bigg) z^{1-w} \mathrm{d}w. \end{aligned}$$

Lemma 3.1. *Suppose that* $\operatorname{Re} z > 0$ *and* $\operatorname{Im} z \neq 0$. *Then*

$$\begin{aligned} \Phi\bigg(z^{-1}, \frac{k}{l}\bigg) &= -\frac{2\pi i \delta z}{l} \sum_{m=1}^{\infty} d(m) \mathrm{e}^{-2\pi i m \overline{k}/l} \mathrm{e}^{-4\pi^2 m z/l^2} + \frac{l}{2\pi^2}\bigg(c_0^+ - c_0^-\bigg) \\ &\quad + \frac{1}{4} + I(z, b). \end{aligned}$$

Proof of the lemma has been sketched in [19], and in the extended form has been given in [1].

Transformation formulae for

$$\Phi(z) = \sum_{m=1}^{\infty} d(m) \mathrm{e}^{-mz} - \frac{\gamma - \log z}{z}$$

and

$$\Phi_\varrho(z) = \sum_{m=1}^{\infty} \frac{\sigma_{2\varrho-1}(m)}{m^{2\varrho-1}} \mathrm{e}^{-mz} - \Gamma(2-2\rho)\zeta(2-2\rho) z^{2\rho-2} - \zeta(2\rho) z^{-1},$$

where $\frac{1}{2} < \varrho < 1$ is a fixed number, have been obtained in [19] and [13], respectively.

4. A formula for $\mathcal{Z}_{14}(s,\chi_0)$

In the definition of the function $\mathcal{Z}_{14}(s,\chi_0)$, we take $\mathrm{e}^{-iw}=1+\frac{1}{z}$, and let $z_0=(\mathrm{e}^{-iw_0}-1)^{-1}$. This leads to the formula

$$\begin{aligned}
&\mathcal{Z}_{14}(s,\chi_0)\\
&\quad=\frac{2\pi i^s}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty}d(k)\mathrm{e}^{-2\pi ikn/m}\\
&\quad\times\int_{z_0}^{\infty}z^{-2}\Big(1+\frac{1}{z}\Big)^{-1/2}\Big(\log\Big(1+\frac{1}{z}\Big)\Big)^{s-1}\mathrm{e}^{-(2\pi ikn)/(zm)}\mathrm{d}z, \quad (7)
\end{aligned}$$

where the integral is taken over the curve $z=(\mathrm{e}^{-ire^{i\alpha}}-1)^{-1}$.

Suppose that $|z_0|<1$, and apply Lemma 3.1 for the function $\mathcal{Z}_{14}(s,\chi_0)$ in formula (7). In view of the definition of $\Phi\big(z,\frac{n}{m}\big)$, we find that

$$\begin{aligned}
&\sum_{k=1}^{\infty}d(k)\mathrm{e}^{-2\pi ikn/m}\mathrm{e}^{-2\pi ikn/(zm)}\\
&\quad=\Phi\Big(\frac{2\pi in}{zm},-\frac{n}{m}\Big)+\frac{\gamma-2\log\big(m/(m,n)\big)-\log\big((2\pi in)(mz)\big)}{(2\pi in)/(z(m,n))}, \quad (8)
\end{aligned}$$

because in the definition of $\Phi\big(z,\frac{n}{m}\big)$ the numbers n and m are coprime.

First let $|z|>1$. We apply Lemma 3.1 with $\frac{mz}{2\pi in}$ in place of z. In this case, $\mathrm{Im}\,(\frac{mz}{2\pi in})<0$, therefore $\delta=-1$, and Lemma 3.1 yields

$$\begin{aligned}
\Phi\Big(\frac{2\pi in}{zm},-\frac{n}{m}\Big) &= \frac{z(m,n)}{n}\sum_{k=1}^{\infty}d(k)\mathrm{e}^{2\pi ik\overline{n}/m}\mathrm{e}^{2\pi ikz(m,n)^2/(mn)}\\
&\quad+\frac{m}{2\pi^2(m,n)}(c_0^+-c_0^-)+\frac{1}{4}+I\Big(\frac{mz}{2\pi in},b\Big).
\end{aligned}$$

This, together with (8), shows that

$$\begin{aligned}
&\sum_{k=1}^{\infty}d(k)\mathrm{e}^{-2\pi ikn/m}\mathrm{e}^{-(2\pi ikn)/(zm)}\\
&\quad=\frac{z(m,n)}{n}\sum_{k=1}^{\infty}d(k)\mathrm{e}^{2\pi ik\overline{n}/m}\mathrm{e}^{(2\pi ikz(m,n)^2)/(mn)}\\
&\quad\quad+\frac{m}{2\pi^2(m,n)}(c_0^+-c_0^-)+\frac{1}{4}\\
&\quad\quad+\frac{z(m,n)}{2\pi in}\Big(\gamma-2\log\frac{m}{(m,n)}-\log\frac{2\pi in}{mz}\Big)+I\Big(\frac{mz}{2\pi in},b\Big). \quad (9)
\end{aligned}$$

Now let $|z| \leqslant 1$. We apply Lemma 3.1 with $\frac{imn}{2\pi z(m,n)^2}$ in place of z. In this case $\mathrm{Im}\,(\frac{imn}{2\pi z(m,n)^2}) > 0$, therefore, $\delta = 1$, and, by Lemma 3.1, we find that

$$\begin{aligned}\Phi\Big(\frac{2\pi z(m,n)^2}{imn}, \frac{\overline{n}}{m}\Big) &= -\frac{n}{z(m,n)}\sum_{k=1}^{\infty} d(k)\mathrm{e}^{-2\pi ikn/m}\mathrm{e}^{-(2\pi ikn)/(zm)} \\ &\quad + \frac{m}{2\pi^2(m,n)}(c_0^+ - c_0^-) + \frac{1}{4} + I\Big(\frac{imn}{2\pi z}, b\Big).\end{aligned}$$

Hence, by the definition of $\Phi\big(z, \frac{n}{m}\big)$,

$$\begin{aligned}&\sum_{k=1}^{\infty} d(k)\mathrm{e}^{-2\pi ikn/m}\mathrm{e}^{-(2\pi ikn)/(zm)} \\ &\quad = \frac{z(m,n)}{n}\Phi\Big(\frac{2\pi z(m,n)^2}{imn}, \frac{\overline{n}}{m}\Big) - \frac{mz}{2\pi^2 n}(c_0^+ - c_0^-) - \frac{z(m,n)}{4n} \\ &\quad\quad - \frac{z(m,n)}{n} I\Big(\frac{imn}{2\pi z(m,n)^2}, b\Big) \\ &\quad = \frac{z(m,n)}{n}\sum_{k=1}^{\infty} d(k)\mathrm{e}^{2\pi ik\overline{n}/m}\mathrm{e}^{(2\pi ikz(m,n)^2)/(mn)} \\ &\quad\quad - \Big(\gamma - 2\log\frac{m}{(m,n)} - \log\frac{2\pi z(m,n)^2}{imn}\Big)\frac{i}{2\pi} \\ &\quad\quad - \frac{mz}{2\pi^2 n}(c_0^+ - c_0^-) - \frac{z(m,n)}{4n} - \frac{z(m,n)}{n} I\Big(\frac{imn}{2\pi z(m,n)^2}, b\Big). \quad (10)\end{aligned}$$

Now let $\hat{z}$ be a point on the path of integration in formula (7) such that $|\hat{z}| = 1$. Then, in view of (9) and (10), we have that

$$\mathcal{Z}_{14}(s, \chi_0) = \sum_{j=1}^{9} I_{j4}(s, \chi_0),$$

where

$$\begin{aligned}&I_{14}(s, \chi_0) \\ &\quad = \frac{2\pi i^s}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\sum_{k=1}^{\infty} d(k)\mathrm{e}^{2\pi ik\overline{n}/m} \\ &\quad\quad \times \int_{z_0}^{\infty} z^{-1}\Big(1+\frac{1}{z}\Big)^{-1/2}\Big(\log\Big(1+\frac{1}{z}\Big)\Big)^{s-1}\mathrm{e}^{(2\pi ikz(m,n)^2)/(mn)}\mathrm{d}z,\end{aligned}$$

$$I_{24}(s,\chi_0) = -\frac{i^s}{\pi\Gamma(s)} \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)(c_0^+ - c_0^-)}{n} \times \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-1/2}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \mathrm{d}z,$$

$$I_{34}(s,\chi_0) = -\frac{2\pi i^s}{4\Gamma(s)} \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \times \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-1/2}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \mathrm{d}z,$$

$$I_{44}(s,\chi_0) = -\frac{i^{s+1}}{\Gamma(s)} \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)}{m} \int_{z_0}^{\hat{z}} z^{-2}\left(1+\frac{1}{z}\right)^{-1/2} \times\left(\gamma - 2\log\frac{m}{(m,n)} - \log\frac{2\pi z(m,n)^2}{imn}\right)\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \mathrm{d}z,$$

$$I_{54}(s,\chi_0) = -\frac{2\pi i^s}{\Gamma(s)} \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \times \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-1/2} I\left(\frac{imn}{2\pi z(m,n)^2}, b\right)\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \mathrm{d}z,$$

$$I_{64}(s,\chi_0) = \frac{i^s}{\pi\Gamma(s)} \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)(c_0^+ - c_0^-)}{(m,n)} \times \int_{\hat{z}}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-1/2}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \mathrm{d}z,$$

$$I_{74}(s,\chi_0) = \frac{2\pi i^s}{4\Gamma(s)} \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)}{m} \times \int_{\hat{z}}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-1/2}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \mathrm{d}z,$$

$$\begin{aligned}
&I_{84}(s,\chi_0)\\
&\quad=\frac{i^{s-1}}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\\
&\qquad\times\int_{\hat z}^{\infty} z^{-1}\Big(1+\frac{1}{z}\Big)^{-1/2}\Big(\gamma-2\log\frac{m}{(m,n)}-\log\frac{2\pi in}{zm}\Big)\\
&\qquad\quad\times\Big(\log\Big(1+\frac{1}{z}\Big)\Big)^{s-1}\mathrm{d}z,\\
&I_{94}(s,\chi_0)\\
&\quad=\frac{2\pi i^{s}}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\\
&\qquad\times\int_{\hat z}^{\infty} z^{-2}\Big(1+\frac{1}{z}\Big)^{-1/2} I\Big(\frac{mz}{2\pi in},b\Big)\Big(\log\Big(1+\frac{1}{z}\Big)\Big)^{s-1}\mathrm{d}z.
\end{aligned}$$

Since $0<r\cos\alpha<\pi$, we have that

$$\begin{aligned}
\operatorname{Im} z &= \operatorname{Im}\frac{1}{\mathrm{e}^{-ire^{i\alpha}}-1}=\operatorname{Im}\frac{1}{\mathrm{e}^{r\sin\alpha}(\cos(r\cos\alpha)-i\sin(r\cos\alpha))-1}\\
&= \frac{\mathrm{e}^{r\sin\alpha}\sin(r\cos\alpha)}{(\mathrm{e}^{r\sin\alpha}\cos(r\cos\alpha)-1)^2+(\mathrm{e}^{r\sin\alpha}\sin(r\cos\alpha))^2}>0.
\end{aligned}$$

Therefore, $\operatorname{Re}(2\pi iz)<0$ in the integral $I_{14}(s,\chi_0)$. Hence, the series and integral both converge absolutely and uniformly in s on compact subsets of $\mathbb{C}$. Thus the function $I_{14}(s,\chi_0)$ is entire.

The functions $I_{24}(s,\chi_0)-I_{44}(s,\chi_0)$ are entire by their definitions.

Since $1<b<2$, the integrand in the definition of $I(z,b)$ has simple poles at $w=j+1$, $j\in\mathbb{N}$. If we move to the right the integration line in the integral for $I(z,b)$, we get residues a_jz^{-j}, where

$$a_j=\Big(\frac{2\pi}{l}\Big)^{-2j-1}\frac{j!}{\pi}\Big(-E\Big(j+1;\frac{\overline{k}}{l},0\Big)+(-1)^{j+1}E\Big(j+1;-\frac{\overline{k}}{l},0\Big)\Big).$$

Thus, by the residue theorem,

$$I(z,b)=a_1z^{-1}+a_2z^{-2}+...+a_jz^{-j}+I(z,b_1),$$

where $I(z,b_1)$ is defined by the same formula as $I(z,b)$ with $j+1<b_1<j+2$. By the latter formula, we have that

$$I\Big(\frac{imn}{2\pi z(m,n)^2},b\Big)=\hat a_1z+\hat a_2z^2+...+\hat a_jz^j+I(z,b_1)+I\Big(\frac{imn}{2\pi z(m,n)^2},b_1\Big)$$

with

$$\hat{a}_j = \left(\frac{2(m,n)^2\pi}{imn}\right)^j a_j.$$

Therefore, $I_{54}(s,\chi_0)$ is also an entire function of s.

The functions $I_{64}(s,\chi_0)$–$I_{94}(s,\chi_0)$ can produce poles.

5. Proof of Theorem 1.2

Using the Taylor series expansion, we find that

$$\begin{aligned}
&\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \\
&= \left(1-\frac{1}{2z}+\frac{3}{4\cdot 2!z^2}-...\right)\left(\frac{1}{z}-\frac{1}{2z^2}+\frac{1}{3z^3}-...\right)^{s-1} \\
&= z^{-s+1}\left(1-\frac{s}{2z}+b_2(s)\frac{1}{z^2}+...\right).
\end{aligned} \tag{11}$$

This and the definition of $I_{64}(s,\chi_0)$, for $\sigma > 0$, yield

$$I_{64}(s,\chi_0) = \frac{i^s}{\pi\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(c_0^+ - c_0^-)}{(m,n)}\sum_{k=0}^{\infty} b_k(s)\int_{\hat{z}}^{\infty} z^{-s-1-k}\mathrm{d}z.$$

Obviously,

$$\int_{\hat{z}}^{\infty} z^{-s-1-k}\mathrm{d}z = \frac{\hat{z}^{-s-k}}{s+k}.$$

Thus, the poles at $s=-k$, $k\in\mathbb{N}\bigcup\{0\}$, are canceled by the zeros of the function $\Gamma^{-1}(s)$, and the function $I_{64}(s,\chi_0)$ is entire. The same arguments show that the function $I_{74}(s,\chi_0)$ is entire, too.

Similarly to the case of the function $I_{54}(s,\chi_0)$, we obtain that the function $I_{94}(s,\chi_0)$ is entire. Thus, to obtain the poles of the function $\mathcal{Z}_1(s,\chi_0)$, it remains to consider the function $I_{84}(s,\chi_0)$. Using expansion (11), we can write $I_{84}(s,\chi_0)$ in the form

$$\begin{aligned}
I_{84}(s,\chi_0) \;=\; & \frac{i^{s-1}}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\sum_{k=0}^{\infty} b_k(s) \\
& \times\int_{\hat{z}}^{\infty} z^{-k-s}\left(\gamma-\log\frac{2\pi imn}{(m,n)^2}+\log z\right)\mathrm{d}z.
\end{aligned} \tag{12}$$

Suppose that $\sigma > 1$. Then, clearly,

$$\int_{\hat{z}}^{\infty} z^{-s} \log z \mathrm{d}z = \frac{1}{s-1}\hat{z}^{-s+1}\log\hat{z} + \frac{\hat{z}^{-s+1}}{(s-1)^2}.$$

Therefore,

$$\int_{\hat{z}}^{\infty} \left(\gamma - \log\frac{2\pi imn}{(m,n)^2} + \log z\right) z^{-s}\mathrm{d}z$$
$$= \frac{\hat{z}^{-s+1}}{(s-1)^2} + \frac{\big(\gamma - \log(\big(2\pi imn)/((m,n)^2\hat{z})\big)\big)\hat{z}^{-s+1}}{s-1}.$$

Since $b_0(s) = 1$, hence, the first term in (12) is

$$\frac{i^{s-1}}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}$$
$$\times\left(\frac{\hat{z}^{-s+1}}{(s-1)^2} + \frac{\big(\gamma - \log(\big(2\pi imn)/((m,n)^2\hat{z})\big)\big)\hat{z}^{-s+1}}{s-1}\right).$$

This shows that the function $I_{84}(s,\chi_0)$ has a double pole at the point $s = 1$. The properties of the gamma-function imply

$$\begin{aligned}
\frac{i^{s-1}\hat{z}^{1-s}}{\Gamma(s)} &= \mathrm{e}^{(s-1)\log i}\mathrm{e}^{-(s-1)\log\hat{z}}\Gamma^{-1}(s)\\
&= (1+\log(s-1)i+...)(1-\log(s-1)\hat{z}+...)(1+\gamma(s-1)+...)\\
&= 1+(\gamma+\log i-\log\hat{z})(s-1)+... .
\end{aligned}$$

Therefore, the Laurent expansion of the function $I_{84}(s,\chi_0)$, and thus, of the function $\mathcal{Z}_1(s,\chi_0)$, is

$$\begin{aligned}
&\mathcal{Z}_1(s,\chi_0)\\
&\quad= \sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\frac{1}{(s-1)^2}\\
&\qquad+\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\\
&\qquad\times\frac{\big(2\gamma - \log\big((mn)/((m,n)^2) - \log 2\pi\big)}{s-1} + ... \qquad (13)
\end{aligned}$$

From this, the first part of the theorem follows.

Suppose that $\sigma > 1-k$ with $k \in \mathbb{N}$. Then we have that

$$\int_{\hat{z}}^{\infty} z^{-k-s}\mathrm{d}z = \frac{\hat{z}^{-s-k+1}}{s+k-1}.$$

The poles at the points $s=-k$, $k\in\mathbb{N}_0$, are canceled by the zeros of $\Gamma^{-1}(s)$.

For the same σ and k as above, we have that

$$\int_{\hat{z}}^{\infty} z^{-s-k}\log z\mathrm{d}z=\frac{\hat{z}^{-s-k+1}}{s+k-1}\log\hat{z}+\frac{\hat{z}^{-s-k+1}}{(s+k-1)^2}.$$

In virtue of $\Gamma^{-1}(s)$, this shows that the points $s=-k$, $k\in\mathbb{N}_0$, are the possible poles of $I_{84}(s,\chi_0)$. However, equality (12) involves the coefficients $b_k(s)$, and some of them can cancel the above poles. Indeed, in view of (11), $b_1(s)=-\frac{s}{2}$. Thus, the pole at $s=0$ is canceled by $b_1(s)$.

We return to the initial definition of the function $\mathcal{Z}_{14}(s,\chi_0)$,

$$\begin{aligned}\mathcal{Z}_{14}(s,\chi_0) &= \frac{2\pi}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty}d(k)\\ &\quad\times\int_0^{w_0}\mathrm{e}^{-iw/2}\exp\left\{-\frac{2\pi ikn}{m}\mathrm{e}^{-iw}\right\}w^{s-1}\mathrm{d}w,\end{aligned}$$

where $w_0=|w_0|\mathrm{e}^{i\alpha}$, $0<|\mathrm{Re}\,w_0|<\pi$, is a fixed point. We put $\mathrm{e}^{-iw}=1-iz$ in the integral of the above formula, and let z_0 correspond the point w_0. Then we obtain that, for $\sigma>1$,

$$\begin{aligned}\mathcal{Z}_{14}(s,\chi_0) &= \frac{2\pi}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty}d(k)\\ &\quad\times\int_0^{z_0}(1-iz)^{-1/2}\mathrm{e}^{-2\pi ikn/m}\mathrm{e}^{-2\pi knz/m}\\ &\quad\times(i\log(1-iz))^{s-1}\mathrm{d}z. \qquad (14)\end{aligned}$$

By the residue theorem, we write

$$\begin{aligned}&\int_0^{z_0}(1-iz)^{-\frac{1}{2}}\mathrm{e}^{-2\pi knz/m}(i\log(1-iz))^{s-1}\mathrm{d}z\\ &=\int_0^{|z_0|}(1-iz)^{-1/2}\mathrm{e}^{-2\pi knz/m}(i\log(1-iz))^{s-1}\mathrm{d}z\\ &\quad+\int_{|z_0|}^{z_0}(1-iz)^{-1/2}\mathrm{e}^{-2\pi knz/m}(i\log(1-iz))^{s-1}\mathrm{d}z, \qquad (15)\end{aligned}$$

where the second integral in the right-hand side of (15) is taken along the path connecting the points $|z_0|$ and z_0, and is an entire function. Thus, it remains to consider the integral over $(0,z_0)$ in (15). Suppose that $|z|$ is small enough. Then, for the part of the integrand of this integral, we have

$$(1-iz)^{-1/2}(i\log(1-iz))^{s-1}$$

$$
\begin{aligned}
&= \Big(1 - \frac{1}{2}(-iz) + \frac{1/2 \cdot 3/2}{2!}(-iz)^2 + \ldots\Big) \\
&\quad \times \bigg(i\Big(- iz + \frac{(iz)^2}{2} - \frac{(iz)^3}{3} + \ldots\Big)\bigg)^{s-1} \\
&= z^{s-1}\Big(1 + \frac{iz}{2} - \frac{z^3}{3} + \ldots\Big) \\
&\quad \times \Big(1 - \frac{1}{2}(-iz) + \frac{3}{4 \cdot 2!}(-iz)^2 + \ldots\Big) \\
&= z^{s-1}\Big(1 + iz\frac{s}{2} + b_2(s)(-iz)^2 + \ldots\Big). \qquad (16)
\end{aligned}
$$

Here the polynomials $b_k(s)$ are the same as above.

For $\sigma > -l$, the integral

$$
\int_0^{|z_0|} z^{s-1} \sum_{j=l+1}^{\infty} \mathrm{e}^{-2\pi knz/m} b_j(s)(-iz)^j \,\mathrm{d}z
$$

is an analytic function. Therefore, for $\sigma > -l$, we consider only the integral

$$
\begin{aligned}
&\int_0^{|z_0|} \mathrm{e}^{-2\pi knz/m} z^{s-1} \sum_{j=0}^{l} b_j(s)(-iz)^j \,\mathrm{d}z \\
&= \int_0^{\infty} \mathrm{e}^{-2\pi knz/m} z^{s-1} \sum_{j=0}^{l} b_j(s)(-iz)^j \,\mathrm{d}z \\
&\quad - \int_{|z_0|}^{\infty} \mathrm{e}^{-2\pi knz/m} z^{s-1} \sum_{j=0}^{l} b_j(s)(-iz)^j \,\mathrm{d}z. \qquad (17)
\end{aligned}
$$

The second integral in the above formula reduces to the incomplete gamma-function and defines an analytic function. Thus, in view of (14)–(17), we obtain that, for $\sigma > -l$, the function

$$
\begin{aligned}
J(s) \overset{def}{=} &\frac{2\pi}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k)\mathrm{e}^{-2\pi ikn/m} \\
&\times \int_0^{\infty} \mathrm{e}^{-2\pi knz/m} \Big(\sum_{j=0}^{l} b_j(s)(-iz)^j\Big) z^{s-1} \mathrm{d}z \qquad (18)
\end{aligned}
$$

gives the main part of the function $\mathcal{Z}_{14}(s, \chi_0)$.

We have that, for $\sigma > 1 - j$,

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-2\pi knz/m} z^{s+j-1} \mathrm{d}z &= (2\pi)^{-s-j} \left(\frac{m}{n}\right)^{s+j} \Gamma(s+j), \\ \sum_{k=1}^\infty \frac{d(k)}{k^{s+j}} \mathrm{e}^{-2\pi ikn/m} &= E\left(s+j; \frac{\big(n/(m,n)\big)}{\big(m/(m,n)\big)}, 0\right), \end{aligned}$$

and

$$\frac{\Gamma(s+j)}{\Gamma(s)} = (s+j-1)...(s+1)s.$$

Therefore, for $\sigma > 1$, by (18),

$$\begin{aligned} J(s) &= \sum_{j=0}^{l} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} (-i)^j b_j(s)(2\pi)^{1-s-j} s(s+1)...(s+j-1) \\ &\quad \times \left(\frac{m}{n}\right)^{s+j} E\left(s+j; \frac{\big(n/(m,n)\big)}{\big(m/(m,n)\big)}, 0\right). \end{aligned} \tag{19}$$

Since the function $E\big(s+j; \frac{m}{n}, 0\big)$ is meromorphic with a double pole at $s = 1 - j$, and

$$\begin{aligned} &E\left(s+j; \frac{\big(n/(m,n)\big)}{\big(m/(m,n)\big)}, 0\right) \\ &\quad = \frac{(m,n)}{m}\left(\frac{1}{(s-(1-j))^2} + \frac{2\gamma - 2\log(m/(m,n))}{s-(1-j)} + ...\right), \end{aligned}$$

equality (19) gives a meromorphic continuation for $J(s)$, and thus, for $\mathcal{Z}_1(s, \chi_0)$, to the whole complex plane, and

$$\begin{aligned} J(s) &= \sum_{j=1}^{l} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} (-i)^j b_j(s)(2\pi)^{1-s-j} s(s+1)...(s+j-1) \\ &\qquad \times \left(\frac{m}{n}\right)^{s+j} E\left(s+j; \frac{\big(n/(m,n)\big)}{\big(m/(m,n)\big)}, 0\right) \\ &\quad + \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{mn} \\ &\qquad \times \left(\frac{1}{(s-1)^2} + \frac{2\gamma - 2\log\big((mn)/((m,n)^2)\big) - \log 2\pi}{s-1} + ...\right). \end{aligned} \tag{20}$$

Moreover, we conclude that $J(s)$ has a simple pole at $s = -j$, $j \in \mathbb{N}$, if $b_j(1-j) \neq 0$.

We take $s = 1 - j$ in formula (16). This gives

$$z^j(1-iz)^{-1/2}(i\log(1-iz))^{-j} = \sum_{k=0}^{\infty} b_k(1-j)(-iz)^k.$$

Hence, by the formula for Taylor's coefficients,

$$b_j(1-j) = \frac{1}{2\pi}\int_{|w|=r}\frac{\mathrm{d}w}{w(1+w)^{1/2}(\log(1+w))^j},$$

where $r > 0$ is sufficiently small. Let $\log(1+w) = -z$ in the above integral. Then some calculations lead to the formula

$$b_j(1-j) = \frac{(-1)^j}{2\pi i}\int_{|z|=r}\frac{\mathrm{d}z}{z^j 2\sinh(z/2)},$$

where

$$\sinh\frac{z}{2} = \frac{\mathrm{e}^{z/2} - \mathrm{e}^{-z/2}}{2}.$$

Therefore, $(-1)^j b_j(1-j)$ is the $(j-1)$th coefficient of the Laurent expansion for the function $f(z) = \left(2\sinh\frac{z}{2}\right)^{-1}$ at the point $z = 0$. However, for k odd,

$$f^{(k)}(z)|_{z=0} = 0.$$

This shows that $b_j(1-j) = 0$ for $j \in 2k$, $k \in \mathbb{N}_0$. This means that $J(s)$ has not a simple pole at $s = -2k$, $k \in \mathbb{N}_0$. Thus, it remains to consider the points $s = -(2k-1)$, $k \in \mathbb{N}$. We use the formula

$$\frac{1}{2\sinh(z/2)} = \frac{1}{z} - \sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)B_{2k}}{2^{2k-1}(2k)!}z^{2k-1},$$

where B_{2k} denotes the $2k$th Bernoulli number. Since $B_{2k} \neq 0$, we have that $b_j(1-j) \neq 0$ for odd j. Thus, the function $J(s)$ has poles at $s = -(2k-1)$, $k \in \mathbb{N}$. Properties of the Möbus function show that

$$\begin{aligned}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{mn}(m,n) &= \frac{\varphi(q)}{q},\\ \sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\log\left(\frac{mn}{(m,n)^2}\right) &= -\frac{2\varphi(q)a(q)}{q}.\end{aligned}$$

Therefore, Theorem 1.2 follows from (20).

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