THE UNIFORM DISTRIBUTION $\text{mod } 1$ OF SEQUENCES INVOLVING THE LARGEST PRIME FACTOR FUNCTION

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Dedicated to Professor Antanas Laurinčikas on his 65th anniversary

Abstract. Let $P(n)$ stand for the largest prime factor of $n$, and let $f$ be a real valued function satisfying certain conditions. We prove that the sequence $(f(P(n)))_{n \geq 2}$ is uniformly distributed modulo 1. We also show an analogous result if $P(n)$ is replaced by $P_k(n)$, the $k$th largest prime factor of $n$.

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1. Introduction

Given a real number $x$, let $\{x\}$ denote its fractional part, that is $\{x\} = x - \lfloor x \rfloor$. A sequence of real numbers $(x_n)_{n \geq 1}$ is said to be uniformly distributed modulo 1 if, given any real numbers $0 < a < b < 1$, then

$$\lim_{N \to \infty} \frac{1}{b-a} \# \{n \leq N : a < \{x_n\} < b\} = 1.$$ 

For instance, given an irrational number $\alpha > 0$, one can show that the
sequence \((a_n)_{n \geq 1}\) is uniformly distributed modulo 1. Various other examples of such sequences are given in the book of Kuipers and Niederreiter [3].

On the other hand, it is known that if \(2 = p_1 < p_2 < \ldots\) stands for the sequence of primes, then the sequence \((\log p_n)_{n \geq 1}\) is not uniformly distributed modulo 1.

In this paper, we prove that if \(P(n)\) stands for the largest prime factor of \(n\) and if \(f\) is a real valued function satisfying certain conditions, then the sequence \((f(P(n)))_{n \geq 2}\) is uniformly distributed modulo 1. We also prove that if \(P_k(n)\) stands for the \(k\)th largest prime factor of \(n\) and if \(\tau\) stands for an arbitrary non zero real number, then the sequence \((\tau \log P_k(n))_{n \geq 1}\) is uniformly distributed modulo 1.

2. Main results

**Theorem 1.** Let \(g : [1, \infty) \to \mathbb{R}\) be a differentiable function and let \(f : [0, \infty) \to \mathbb{R}\) be defined by \(f(u) = g(\log u)\). Assume that the function \(vg'(v)\) is increasing and tends to infinity. For \(x \geq 2\), let \(R(x) := \pi(x) - \text{li}(x)\) be the error term in Prime Number Theorem and further assume that, for any given real number \(d > 0\),

\[
\lim_{y \to \infty} \int_y^{y+d} \frac{|R(u)|}{u} \left| f'(u) \right| \, du = 0. \tag{2.1}
\]

Then the sequence \((f(P(n)))_{n \geq 1}\) is uniformly distributed modulo 1.

**Remark 1.** Observe that it is clear that (2.1) holds for \(f(u) := \tau \log u\), where \(\tau \neq 0\) is some fixed constant, thus implying that, as a consequence of Theorem 1, the sequence \((\tau \log P(n))_{n \geq 1}\) is uniformly distributed modulo 1.

For a positive integer \(n\) which is not a prime, we let \(P_2(n) = P\left(\frac{n}{P(n)}\right)\) and more generally, given an arbitrary integer \(k \geq 2\) and an integer \(n\) with at least \(k\) prime divisors counting their multiplicity, let \(P_k(n) = P_{k-1}\left(\frac{n}{P(n)}\right)\).

**Theorem 2.** Given an arbitrary real number \(\tau \neq 0\) and an arbitrary integer \(k \geq 1\), the sequence \((\tau \log P_k(n))_{n \geq 2}\) is uniformly distributed modulo 1.

3. Notation

We will use the standard notation \(e(y) := \exp\{2\pi iy\}\).

For \(2 \leq y \leq x\), let

\[
\Psi(x, y) = \#\{n \leq x : P(n) \leq y\}.
\]
Let $\rho(u)$ stand for the Dickman function, that is the unique continuous solution of the differential equation $u\rho'(u) + \rho(u - 1) = 0$ with initial condition $\rho(0) = 1$ for $0 \leq u \leq 1$.

The letter $c$, with or without subscript, always denotes a positive constant, but not necessarily the same at each occurrence.

4. Preliminary results

The following two results are well known and their proofs can be found in the book of Tenenbaum [4].

**Lemma 1.** For all $2 \leq y \leq x$, letting $u = \frac{\log x}{\log y}$,

$$\Psi(x, y) \ll x \exp \left\{ - \frac{u}{2} \right\}.$$ 

**Lemma 2.** Given a fixed $\varepsilon > 0$, then uniformly for $u = \frac{\log x}{\log y} \leq \frac{1}{\varepsilon}$,

$$\Psi(x, y) = (1 + o(1))\rho(u)x, \quad x \to \infty.$$ 

Finally, the following result, a proof of which can be found in the book of De Koninck and Luca [2], has the advantage of being true for all $2 \leq y \leq x$.

**Lemma 3.** Uniformly for $2 \leq y \leq x$,

$$\Psi(x, y) = x\rho(u) + O \left( \frac{x}{\log y} \right).$$

**Lemma 4.** Let $(f(p))_{p \in \mathbb{P}}$ be a sequence of real numbers which is such that, for any given real number $d > 0$ and integer $k \geq 1$,

$$\lim_{y \to \infty} \sum_{y < p < y^{1+d}} \frac{e(kf(p))}{p} = 0. \tag{4.1}$$

Then the sequence $(f(P(n)))_{n \geq 1}$ is uniformly distributed mod 1.

**Proof.** We will be using the well known result of H. Weyl [5] which asserts that a sequence $(x_n)_{n \geq 1}$ is uniformly distributed mod 1 if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(kx_n) = 0$$

for each positive integer $k$. 

Let $k$ be an arbitrary positive integer and let $\varepsilon > 0$ be a small number. We then write

\[
S(N) = \frac{1}{N} \sum_{n \leq N} e(kf(P(n)))
= \frac{1}{N} \sum_{n \leq N} e(kf(P(n))) + \frac{1}{N} \sum_{n \leq N \atop P(n) \leq N^{\varepsilon}} e(kf(P(n)))
= S_1(N) + S_2(N),
\]

say. Hence, in view of Weyl's criterion, it will be sufficient to show that

\[
\limsup_{N \to \infty} |S(N)| = 0. \tag{4.3}
\]

Now, it follows from Lemma 1 that

\[
S_1(N) \leq \frac{1}{N} \Psi(N, N^{\varepsilon}) \leq \exp \left\{-\frac{1}{2\varepsilon}\right\}. \tag{4.4}
\]

On the other hand, using Lemma 3, we have

\[
S_2(N) = \frac{1}{N} \sum_{N^{\varepsilon} < p \leq N} e(kf(p)) \psi \left(\frac{N}{p}, p\right)
= \sum_{N^{\varepsilon} < p \leq N} \left(\frac{e(kf(p))}{p} \rho \left(\frac{\log N/p}{\log p}\right) + O \left(\frac{1}{(p \log p)}\right)\right)
= \sum_{N^{\varepsilon} < p \leq N} \left(\frac{e(kf(p))}{p} \rho \left(\frac{\log N/p}{\log p}\right) + O \left(\frac{1}{\varepsilon \log N}\right)\right)
= S_3(N) + O \left(\frac{1}{\varepsilon \log N}\right), \tag{4.5}
\]

say.

It remains to show that

\[
|S_3(N)| < c \varepsilon \quad \text{if} \quad N \quad \text{is large enough.}
\]

To do so, we proceed as follows. Since $\rho(u)$ is uniformly continuous in the closed interval $[1, \frac{1}{\varepsilon} - 1]$, there exists a suitable constant $\Delta > 0$ (depending only on $\varepsilon$) such that $|\rho(u_1) - \rho(u_2)| \leq \varepsilon$ for all $u_1, u_2 \in [1, \frac{1}{\varepsilon} - 1]$ satisfying $|u_1 - u_2| \leq \Delta$. Let us now consider the sequence $\ell_0, \ell_1, \ldots, \ell_{j_0}$ defined by $\ell_0 = 1$ and $\ell_j = 1 + j\Delta$ for each $j = 1, \ldots, j_0$, where $j_0$ is the smallest integer
such that $j_0 \Delta > \varepsilon$. Further set $J_j = [\ell_j, \ell_{j+1})$ and then split the sum $S_3(N)$ as follows

$$S_3(N) = \sum_{\sqrt{N} < p \leq N} \frac{e(kf(p))}{p} + \sum_{j=0}^{j_0-1} T_j,$$

where

$$T_j = \sum_{\log N \leq p \in J_j} \frac{e(kf(p))}{p} \rho \left( \frac{\log(N/p)}{\log p} \right), \quad j = 0, 1, \ldots, j_0 - 1.$$

Since in each $T_j$, we have

$$\left| \rho \left( \frac{\log(N/p)}{\log p} \right) - \rho(\ell_j - 1) \right| \leq \varepsilon,$$

it follows, recalling that $\rho(\ell_j - 1) = \rho(j\Delta)$,

$$|T_j| \leq \rho(j\Delta) \left| \sum_{\log N \leq p \in J_j} \frac{e(kf(p))}{p} \right| + \varepsilon \sum_{\log N \leq p \in J_j} \frac{1}{p}.$$

Now observe that

$$\frac{\log N}{\log p} \in J_j \iff \ell_j \leq \frac{\log N}{\log p} < \ell_{j+1} \iff p \in (N^{1/\ell_j+1}, N^{1/\ell_j}].$$

We then have

$$|S_3(N)| \leq \left| \sum_{\sqrt{N} < p \leq N} \frac{e(kf(p))}{p} \right| + \sum_{j=0}^{j_0-1} \rho(j\Delta) \left| \sum_{\log N \leq p \in J_j} \frac{e(kf(p))}{p} \right|$$

$$+ \varepsilon \sum_{j=0}^{j_0-1} \sum_{\log N \leq p \in J_j} \frac{1}{p}.$$

Again, observe that since in $S_3(N)$, we sum only over those primes $p$ for which $\frac{\log N}{\log p} \in [2, \frac{1}{\varepsilon}]$, it follows that the number of intervals $J_j$’s necessary to cover the interval $[2, \frac{1}{\varepsilon}]$ is finite. This is why we can allow $j$ to vary only from 0 to $j_0 - 1$. 

It follows from the condition (4.1) that the first two terms on the right hand side of (4.7) tend to 0 as \( N \to \infty \), implying that
\[
\limsup_{N \to \infty} |S_3(N)| \leq \varepsilon \limsup_{N \to \infty} \sum_{\log \frac{N}{p} \in [\frac{1}{2}, 1]} \frac{1}{p} \\
= \varepsilon \limsup_{N \to \infty} \sum_{N^{1-\varepsilon} \leq p \leq N} \frac{1}{p} \\
< c \varepsilon \log \frac{1}{\varepsilon}. \tag{4.8}
\]

Hence, using estimates (4.4), (4.5) and (4.8) in (4.2), relation (4.3) follows immediately, thus completing the proof of Lemma 4. \( \square \)

**Lemma 5.** Given a fixed positive number \( \varepsilon < 1 \),
\[
\# \{ n \leq N : P(n) > N^{1-\varepsilon} \} \leq c \varepsilon N.
\]

**Proof.** This follows from the fact
\[
\sum_{n = mp \leq N \atop N^{1-\varepsilon} < p \leq N} 1 \leq N \sum_{N^{1-\varepsilon} < p \leq N} \frac{1}{p} \leq c \varepsilon N.
\]
\( \square \)

**Lemma 6.** For any fixed number \( \lambda \neq 0 \), consider the function
\[
\kappa(y, z) := \sum_{y < p < z} \frac{e(\lambda \log p)}{p}.
\]
Then
\[
\sup_{z > y} |\kappa(y, z)| \to 0 \quad \text{as} \quad y \to \infty.
\]

**Proof.** This result follows essentially from the fact that \( \sum_{p} \frac{1}{p^{\lambda}} \) is convergent for all \( \lambda \neq 0 \). \( \square \)

5. **Proof of Theorem 1**

With \( \pi(u) = \int_{2}^{u} \frac{dt}{\log t} + R(u) \), we get that
\[
d \pi(u) = \frac{du}{\log u} + dR(u).
\]
Hence,
\[
\sum_{y<p<y^{1+d}} \frac{e(kf(p))}{p} = \int_y^{y^{1+d}} \frac{e(kf(u))}{u} \, d\pi(u) = \int_y^{y^{1+d}} \frac{e(kf(u))}{u \log u} \, du + \int_y^{y^{1+d}} \frac{e(kf(u))}{u} \, dR(u) = \mathcal{J}_1(y) + \mathcal{J}_2(y),
\]
say. With the change of variable \( v = \log u \), so that \( f(u) = f(e^v) = g(v) = g(\log u) \), we get
\[
\mathcal{J}_1(y) = \int_{\log y}^{(1+d)\log y} \frac{e(kg(v))}{v} \, dv. \tag{5.2}
\]
Since
\[
(e(kg(v)))' = e(kg(v)) \cdot 2\pi ik \cdot g'(v),
\]
it follows from (5.2) that
\[
\mathcal{J}_1(y) = \int_{\log y}^{(1+d)\log y} \frac{(e(kg(v)))'}{2\pi ik vg'(v)} \, dv = \frac{e(kg(v))}{2\pi ik vg'(v)} \bigg|_{\log y}^{(1+d)\log y} - \frac{1}{2\pi ik} \int_{\log y}^{(1+d)\log y} e(kg(v)) \left( \frac{1}{vg'(v)} \right)' \, dv = \mathcal{J}_1^{(1)}(y) - \mathcal{J}_1^{(2)}(y), \tag{5.3}
\]
say.

First of all, since by hypothesis, \( vg'(v) \to \infty \) as \( v \to \infty \), it is clear that
\[
\left| \mathcal{J}_1^{(1)}(y) \right| \to 0 \text{ as } y \to \infty. \tag{5.4}
\]
On the other hand, since by hypothesis, \( vg'(v) \to \infty \) monotonically as \( v \to \infty \), it follows that \( \left( \frac{1}{vg'(v)} \right)' \) is negative for all real \( v \geq 1 \), and therefore, that
\[
\left| \mathcal{J}_1^{(2)}(y) \right| \leq \frac{1}{2\pi k} \int_{\log y}^{(1+d)\log y} \left( -\frac{1}{vg'(v)} \right)' \, dv = \frac{1}{2\pi k} \frac{-1}{vg'(v)} \bigg|_{\log y}^{(1+d)\log y} \to 0 \tag{5.5}
\]
as $y \to \infty$.

On the other hand,

$$J_2(y) = \int_y^{y^{1+d}} \frac{e(kf(u))}{u} dR(u)$$

$$= \frac{R(u)e(kf(u))}{u}\bigg|_y^{y^{1+d}} - \int_y^{y^{1+d}} R(u) \left( \frac{e(kf(u))}{u} \right)' du$$

$$= J_2^{(1)}(y) - J_2^{(2)}(y), \quad (5.6)$$

say.

Now, on the one part, we clearly have

$$\left| J_2^{(1)}(y) \right| \to 0 \quad \text{as} \quad y \to \infty, \quad (5.7)$$

while on the other hand, since

$$J_2^{(2)}(y) = \int_y^{y^{1+d}} R(u) \frac{e(kf(u)) \cdot 2\pi ik \cdot uf'(u) - e(kf(u))}{u^2} du,$$

it follows that

$$\left| J_2^{(2)}(y) \right| \leq 2\pi k \int_y^{y^{1+d}} \frac{|R(u)|}{u} |f'(u)| du + \int_y^{y^{1+d}} \frac{|R(u)|}{u^2} du \to 0 \quad (5.8)$$

as $y \to \infty$, where we used (2.1).

Combining estimates (5.4), (5.5), (5.7) and (5.8), then, in view of (5.3) and (5.6), it follows from (5.1) that

$$\sum_{y^{1+d} < p < y^{1+d}} \frac{e(kf(p))}{p} \to 0 \quad \text{as} \quad y \to \infty.$$

Thus, using Lemma 4, the theorem is proved.

6. Proof of Theorem 2

We will only consider the case $k = 2$, the general case being similar.

Now, given real numbers $0 < \alpha < \beta \leq 1$, let us introduce the two expressions

$$\omega_{\alpha,\beta}(n) = \omega_{\alpha,\beta,N}(n) = \sum_{p|n} 1, \quad E_{\alpha,\beta}(N) = \sum_{N^\alpha < p < N^\beta} \frac{1}{p}.$$
Since
\[ E_{\alpha, \beta}(N) = \log \frac{\beta}{\alpha} + o(1) \quad \text{as} \quad N \to \infty, \]

it follows from the Turán-Kubilius inequality that
\[
\sum_{n \leq N} (\omega_{\alpha, \beta}(n) - E_{\alpha, \beta}(N))^2 \leq cN E_{\alpha, \beta}(N) < c_1 N \log \frac{\beta}{\alpha}. \quad (6.1)
\]

Let \( \varepsilon > 0 \) be an arbitrary small number. Let \( \delta \in (0, 1) \), to be determined later. Since \( \omega_{\delta, 1}(n) = 0, 1 \) or \( 2 \) when \( P_3(n) < N^\delta \), we have that
\[
\frac{(\omega_{\delta, 1}(n) - E_{\delta, 1}(N))^2}{(E_{\delta, 1}(N) - 2)^2} \geq 1,
\]
thus implying, using (6.1), that
\[
\#\{n \leq N : P_3(n) < N^\delta\} \leq \frac{1}{(E_{\delta, 1}(N) - 2)^2} \sum_{n \leq N} (\omega_{\delta, 1}(n) - E_{\delta, 1}(N))^2
\]
\[
\leq c_1 N \frac{1}{E_{\delta, 1}(N)} \quad \text{provided} \quad E_{\delta, 1}(N) \geq 4,
\]
\[
\leq c_1 \frac{N}{\log(1/\delta)} \leq c_2 \varepsilon N, \quad (6.2)
\]

provided we choose \( \delta = \exp\{-\frac{1}{\varepsilon}\} \).

Let \( \delta_1, \delta_2 \) be two small positive numbers, to be determined later. We will now count the number \( T(N) \) of positive integers \( n \leq N \) for which there exist two prime numbers \( p, q \) satisfying \( pq | n \) and \( N^{\delta_1} < p < q < p^{1+\delta_2} \). It is clear that
\[
T(N) \leq N \sum_{N^{\delta_1} < p < q < p^{1+\delta_2}} \frac{1}{pq} \leq N \sum_{N^{\delta_1} < p < p^{1+\delta_2}} \frac{1}{p} \sum \frac{1}{q}
\]
\[
\leq c N \cdot \log \frac{1}{\delta_1} \cdot \log(1 + \delta_2)
\]
\[
\leq c_1 \varepsilon N, \quad (6.3)
\]

provided we choose \( \delta_1 \) and \( \delta_2 \) so that \( \delta_2 \log \frac{1}{\delta_1} \leq \varepsilon \); for instance, the choice
\( \delta_1 = \delta \) and \( \delta_2 = \varepsilon^2 \) is appropriate.

Note that our main goal will be to prove that
\[
A(N) := \sum_{n \leq N} e(\ell \log P_2(n)) = o(N)
\]
for every non-zero real number $\ell$. Indeed, choosing $\ell = k\tau$, with $0 \neq k \in \mathbb{Z}$, we will then be guaranteed by Weyl’s criterion that $(\tau \log P_2(n))_{n \geq 1}$ is uniformly distributed modulo 1, as claimed.

So, let us write each number $n \leq N$ as $n = np_1^2p_2$, where $P(\nu) < p_2 < p_1$. Choose $\varepsilon, \delta_1, \delta_2$ as above. We first drop those positive integers $n \leq N$ for which

$$p_1 > N^{1-\varepsilon} \quad \text{or} \quad P(\nu) < N^{\delta} \quad \text{or} \quad p_2 \leq p_1 < p_2^{1+\delta_2}.$$ 

Indeed, we can do this in view of Lemma 5, (6.2) and (6.3).

We have thus established that

$$A(N) = A_1(N) + O(\varepsilon N),$$

where $A_1(N)$ counts the number of positive integers $n \leq N$ counted by $A(N)$ but that were not dropped in the above process.

With the above notation, we may therefore write

$$|A_1(N)| \ll \left| \sum_{p_1, p_2 \leq N^{1-\delta_1}} e(\ell \log p_2)\Psi \left( \frac{N}{p_1p_2} \right) \right| = |A_2(N)| + |A_3(N)|, \quad (6.4)$$

where in $A_2(N)$, the summation runs over those primes $p_1, p_2$ such that $\frac{N}{p_1p_2} < p_2$, while in $A_3(N)$ it runs over those such that $\frac{N}{p_1p_2} \geq p_2$.

Observe that for those $p_1, p_2$ running in $A_2(N)$, we have

$$\Psi \left( \frac{N}{p_1p_2} \right) = \left\lfloor \frac{N}{p_1p_2} \right\rfloor = \frac{N}{p_1p_2} + O(1),$$

thus implying that

$$A_2(N) = N \sum_{p_1, p_2 \leq N^{1-\delta_1}} e(\ell \log p_2) \frac{1}{p_1p_2} + O \left( \sum_{p_1, p_2 \leq N} 1 \right)$$

$$= N \sum_{p_1, p_2 \leq N^{1-\delta_1}} e(\ell \log p_2) \frac{1}{p_1p_2} + O \left( \frac{N \log \log N}{\log N} \right). \quad (6.5)$$

We have

$$N \sum_{p_1, p_2 > N} e(\ell \log p_2) \frac{1}{p_1p_2} = N \sum_{p_1 > N^\delta} \frac{1}{p_1} \sum_{\sqrt{p_1} < p_2 < p_1} e(\ell \log p_2) \frac{1}{p_2}$$

$$= N \sum_{N^{\delta_1} < p_1 < N^{1-\delta_1}} \frac{1}{p_1} \sum_{\sqrt{p_1} < p_2 < p_1} e(\ell \log p_2) \frac{1}{p_2} \sqrt{p_1} \cdot \sqrt{p_2} \cdot \sqrt{p_1} \cdot \sqrt{p_2} = N \sum_{N^{\delta_1} < p_1 < N^{1-\delta_1}} \frac{1}{p_1} \sum_{\sqrt{p_1} < p_2 < p_1} e(\ell \log p_2) \frac{1}{p_2}. \quad (6.6)$$
In view of Lemma 6, we have that \(|\kappa(\sqrt[3]{\frac{N}{p_1}}, p_1)| < \varepsilon\) if \(N\) is sufficiently large, and therefore, that it follows from (6.5) and (6.6) that
\[
|A_2(N)| \ll \varepsilon N \sum_{N^{\delta_1} < p_1 < N^{1 - \delta_1}} \frac{1}{p_1} < \varepsilon N \log \frac{1}{\delta_1}.
\] (6.7)

On the other hand, using Lemma 2, we have, as \(N \to \infty\),
\[
A_3(N) &= \sum_{p_2 < p_1, \frac{p_1 p_2}{N^{1 - 2\delta_1}} \not\in \mathbb{Z}} e(\ell \log p_2) \left( \frac{N}{p_1 p_2} \right) \\
&= (1 + o(1)) \sum_{p_2 < p_1, \frac{p_1 p_2}{N^{1 - 2\delta_1}} \not\in \mathbb{Z}} e(\ell \log p_2) \frac{N}{p_1 p_2} \rho \left( \frac{\log(N/p_1 p_2)}{\log p_2} \right) \\
&= \sum_{p_2 < p_1, \frac{p_1 p_2}{N^{1 - 2\delta_1}} \not\in \mathbb{Z}} e(\ell \log p_2) \frac{N}{p_1 p_2} \rho \left( \frac{\log(N/p_1 p_2)}{\log p_2} \right) \\
&\quad + o \left( \sum_{p_2 < p_1, \frac{p_1 p_2}{N^{1 - 2\delta_1}} \not\in \mathbb{Z}} \frac{N}{p_1 p_2} \right) \\
&= B_1(N) + o(B_2(N)),
\] (6.8)
say. It is clear that
\[
B_2(N) < N \sum_{N^{\delta_1} < p_1 < N^{1 - \delta_1}} \frac{1}{p_1} \cdot \sum_{N^{\delta_1} < p_2 < N^{1 - \delta_1}} \frac{1}{p_2} \ll N \left( \log \frac{1}{\delta_1} \right)^2.
\] (6.9)

To estimate \(B_1(N)\), let us denote the interval \([N^{\delta_1}, \min \left( \sqrt[3]{\frac{N}{p_1}}, p_1 \right)]\) by \(I(p_1)\) and then, using the same approach as that used in Theorem 1, we subdivide each interval \(I(p_1)\) into subintervals \(J_j - 1 = [\ell_j - 1, \ell_j + 1 - 1]\) depending if
\[
\ell_j - 1 \leq \nu_{p_1, p_2} := \frac{\log(N/p_1 p_2)}{\log p_2} = \frac{\log(N/p_1)}{\log p_2} - 1 \leq \ell_{j+1} - 1.
\]
This is useful because the function \(\rho(\nu_{p_1, p_2})\) varies very little in each interval \(J_j - 1 = [\ell_j - 1, \ell_{j+1} - 1]\) in the sense that
\[
|\rho(\nu_{p_1, p_2}) - \rho(\nu_{p_1, p_2}^*)| < \varepsilon_1 \quad \text{for all} \quad \nu_{p_1, p_2}, \nu_{p_1, p_2}^* \in J_j - 1.
\]
Taking into account the fact that the number of such intervals \( J_j \) is bounded for each \( p_1 \) by a constant \( c(p_1) \), and moreover, that we can conclude to the existence of a universal constant \( C > 0 \) such that \( \sup_{p_1} c(p_1) \leq C \), it follows from (6.8) and since \( |\rho(\nu p_1, p_2) - \rho(j \Delta)| \leq \varepsilon_1 \), that

\[
|B_1(N)| \ll \varepsilon_1 \cdot N \sum_{p_1 > N^{\varepsilon_1}} \frac{1}{p_1} \sum_{j=0}^{j_0-1} \rho(j \Delta) \left| \sum_{\nu p_1, p_2 \in J_j-1} e(\ell \log p_2) \right| .
\]  

(6.10)

Hence, it follows from (6.10), (6.9) and (6.8) that

\[
|A_3(N)| \ll o \left( N \left( \log \frac{1}{\delta_1} \right)^2 \right) + \varepsilon_1 \kappa_0(N) N \sum_{N^{\varepsilon_1} < p_1 < N^{1-\delta_1}} \frac{1}{p_1} \sum_{j} \rho(j \Delta),
\]  

(6.11)

where

\[
\kappa_0(N) := \sup_{j,p_1} \left| \sum_{\nu p_1, p_2 \in -1+j_0} e(\ell \log p_2) \right|.
\]

tends to 0 uniformly in \( p_1 \) and in \( j \) as \( N \to \infty \). Hence, (6.11) yields

\[
\limsup_{N \to \infty} \frac{|A_3(N)|}{N} \leq \varepsilon_1 \left( \log \frac{1}{\delta_1} \right)^2 .
\]

Since \( \varepsilon_1 \) can be chosen arbitrarily small, we conclude that

\[
\limsup_{N \to \infty} \frac{|A_3(N)|}{N} = 0.
\]  

(6.12)

Gathering estimates (6.7) and (6.12) in (6.4), the proof of Theorem 2 (in the case \( k = 2 \)) is complete.

7. Example

Given a real number \( \alpha > 0 \), set \( f(u) = (\log u)^\alpha \). Setting \( v = \log u \), we get

\[
g(v) = v^\alpha,
\]

so that \( v g'(v) = \alpha v^\alpha \to \infty \) as \( v \to \infty \). It remains to check that condition (2.1) is satisfied. On the one hand,

\[
f'(u) = \alpha (\log u)^{\alpha-1} \frac{1}{u},
\]

(7.1)
while on the other hand, it is known since de la Vallée-Poussin (see [1]) that, for some constant $C > 0$,

$$R(u) \ll u \exp\{-C\sqrt{\log u}\}. \quad (7.2)$$

Using (7.1) and (7.2), we get, by setting $v = \log u$,

$$\int_1^{\log y} \frac{|R(u)|}{u} |f'(u)| du \leq \alpha \int_1^{\log y} \frac{(\log u)^{\alpha-1}}{u} e^{-C\sqrt{\log u}} du$$

$$\leq \alpha \int_1^{\log y} (1+\delta \log y) e^{\alpha-1} e^{-C\sqrt{v}} dv. \quad (7.3)$$

Since this last quantity clearly tends to 0 as $y \to \infty$, condition (2.1) of Theorem 1 is satisfied, thereby implying that, if $f(m) = (\log m)^\alpha$, then the sequence $f(P(n))_{n\geq 1}$ is uniformly distributed mod 1.

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[1] Ch.-J. de la Vallée-Poussin, Sur la fonction $\zeta(s)$ de Riemann et le nombre de nombres premiers inférieurs à une limite donnée, Belg. Mém. cour. in 8°, 59, 74 S (1899).


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