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ON UNIVERSALITY OF PERIODIC ZETA-FUNCTIONS

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Abstract. We obtain universality theorems for $F(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$, where $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ are periodic and periodic Hurwitz zeta-functions, respectively, and the operator F satisfies certain Lipschitz type conditions.

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1. Introduction

Let $\mathfrak{a} = \{a_m : \ m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be two periodic sequences of complex numbers with minimal periods $k \in \mathbb{N}$ and $l \in \mathbb{N}$, respectively. Moreover, let α, $0 < \alpha \leqslant 1$, be a fixed parameter. Then the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$, $s = \sigma + it$, defined, for $\sigma > 1$, by the series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}$$

are called the periodic zeta-function and the periodic Hurwitz zeta-function, respectively. By periodicity of $\mathfrak{a}$ and $\mathfrak{b}$, we have that, for $\sigma > 1$,

$$\zeta(s;\mathfrak{a}) = \frac{1}{k^s}\sum_{m=1}^{k} a_m \zeta\left(s, \tfrac{m}{k}\right) \tag{1}$$

and

$$\zeta(s,\alpha;\mathfrak{b}) = \frac{1}{l^s}\sum_{m=0}^{l-1} b_m \zeta\left(s, \tfrac{m+\alpha}{l}\right), \tag{2}$$

where $\zeta(s,\alpha)$ is the classical Hurwitz zeta-function defined, for $\sigma > 1$, by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

The function $\zeta(s,\alpha)$ continues to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. This together with equalities (1) and (2) shows that the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ are meromorphic functions with possible simple pole at $s = 1$ with residues

$$a = \frac{1}{k}\sum_{m=1}^{k} a_m \quad \text{and} \quad b = \frac{1}{l}\sum_{m=0}^{l-1} b_m,$$

respectively. If $a = 0$, then the function $\zeta(s;\mathfrak{a})$ is entire, and if $b = 0$, then the function $\zeta(s,\alpha;\mathfrak{b})$ is entire. For $a_m \equiv 1$, the function $\zeta(s;\mathfrak{a})$ reduces to the Riemann zeta-function $\zeta(s)$.

In [1] and [7], universality of the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ was obtained, respectively. Roughly speaking, this means that any analytic function can be approximated by shifts $\zeta(s+i\tau;\mathfrak{a})$ or $\zeta(s+i\tau,\alpha;\mathfrak{b})$. Suppose that, for each prime p, the inequality

$$\sum_{m=1}^{\infty} \frac{|a_{p^m}|}{p^{m/2}} \leqslant c < 1 \tag{3}$$

holds. Also, we recall that the sequence $\mathfrak{a}$ is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all coprimes $m, n \in \mathbb{N}$. Moreover, let $\mathrm{meas}A$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Then the following statement is the main theorem of [7].

THEOREM 1. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative and inequality* (3) *is satisfied. Let* $K \subset D$ *be a compact subset with connected complement, and*

$f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Universality theorem for the function $\zeta(s;\mathfrak{a})$ with arbitrary periodic sequence $\mathfrak{a}$ was proved in [2].

Now we state the theorem from [1] on universality of the function $\zeta(s,\alpha;\mathfrak{b})$.

Theorem 2. *Suppose that the number α is transcendental. Let $K \subset D$ be a compact subset with connected complement, and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f(s)| < \varepsilon\right\} > 0.$$

The joint universality of the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ was obtained in [3], and is expressed by the following theorem.

Theorem 3. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, inequality* (3) *is satisfied, and that the number α is transcendental. Let K_1 and K_2 be compact subsets of the strip D with connected complements, $f_1(s)$ be a continuous non-vanishing function on K_1 which is analytic in the interior of K_1, and let $f_2(s)$ be a continuous function on K_2 which is analytic in the interior of K_2. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{a}) - f_1(s)| < \varepsilon\ ,\right.$$

$$\left.\sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{b}) - f_2(s)| < \varepsilon\right\} > 0.$$

The aim of this note is to obtain the universality of the function $F(\zeta(s;\mathfrak{a}), \zeta(s,\alpha;\mathfrak{b}))$, for some class of operators $F: H^2(D) \to H(D)$, where $H(D)$ denotes the space of analytic on D functions equipped with the topology of uniform convergence on compacta, and $H^2(D) = H(D) \times H(D)$. Let $\beta_1 > 0$, $\beta_2 > 0$, and $\beta = \min(\beta_1, \beta_2)$. The operator $F: H^2(D) \to H(D)$ belongs to the class $Lip(\beta_1, \beta_2)$ if it satisfies the following hypotheses:

1° For each polynomial $p = p(s)$, and any compact subset $K \subset D$ with connected complement, there exists an element $(g_1, g_2) \in F^{-1}\{p\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on K.

2° For any compact subset $K \subset D$ with connected complement, there exist a positive constant c, and compact subsets K_1, K_2 of D with connected complements such that

$$\sup_{s \in K} |F(g_{11}(s), g_{12}(s)) - F(g_{21}(s), g_{22}(s))| \leqslant c \sup_{1 \leqslant j \leqslant 2} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{r1}, g_{r2}) \in H^2(D)$, $r = 1, 2$.

THEOREM 4. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, inequality* (3) *is satisfied, the number α is transcendental, and that $F \in Lip(\beta_1, \beta_2)$. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s \in K} \big|F\big(\zeta(s + i\tau; \mathfrak{a}), \zeta(s + i\tau, \alpha; \mathfrak{b})\big) - f(s)\big| < \varepsilon\Big\} > 0.$$

Theorem 4 also has a joint version. Let $F_r: H^2(D) \to H(D)$, $r = 1, \dots, n$. The operator $F_{2,n} = (F_1, \dots, F_n) : H^2(D) \to H^n(D)$ belongs to the class $Lip_n(\beta_1, \beta_2)$, $\beta_1 > 0$, $\beta_2 > 0$, if the following hypotheses are satisfied:

1° For all polynomials $p_1(s), \dots, p_n(s)$, and any compact subset $K \subset D$ with connected complement, there exists an element $(g_1, g_2) \in F_{2,n}^{-1}\{p_1, \dots, p_n\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on K.

2° For all compact subsets $K_1, \dots, K_n \subset D$ with connected complements, there exist a positive constant c, and compact subsets $\hat{K}_1, \hat{K}_2$ of D with connected complements such that

$$\sup_{1 \leqslant j \leqslant n} \sup_{s \in K_j} |F_j(g_{11}(s), g_{12}(s)) - F_j(g_{21}(s), g_{22}(s))| \leqslant c \sup_{1 \leqslant j \leqslant 2} \sup_{s \in \hat{K}_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{r1}, g_{r2}) \in H^2(D)$, $r = 1, 2$.

THEOREM 5. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, inequality* (3) *is satisfied, the number α is transcendental, and that $\hat{F} = (F_1, F_2) : H^2(D) \to$*

$H^2(D)$ belongs to the class $Lip_2(\beta_1, \beta_2)$. For $j = 1, 2$, let K_j be a compact subset of the strip D with connected complement, and let $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant 2}\sup_{s\in K_j} \left|F_j\left(\zeta(s+i\tau;\mathfrak{a}), \zeta(s+i\tau,\alpha;\mathfrak{b})\right) - f_j(s)\right| < \varepsilon\Big\} > 0.$$

Universality of composite functions $F(\zeta(s))$ and $F(\zeta(s,\alpha))$ with transcendental and rational parameter α, for some classes of operators $F : H(D) \to H(D)$, has been considered in [4] and [6], respectively.

Let $L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r\}$. Universality of $F(\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r))$ with linearly independent set $L(\alpha_1, \dots, \alpha_r)$ over the field of rational numbers, for some operators $F : H^r(D) \to H(D)$, has been studied in [5].

2. Proofs

We start with the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 6. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a function continuous on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof of the lemma is given in [8], see also [9].

Theorems 4 and 5 follow from Theorem 3 and the definition of the classes $Lip(\beta_1, \beta_2)$ and $Lip_n(\beta_1, \beta_2)$.

Proof of Theorem 4. The set K and function $f(s)$ in the statement of theorem satisfy the hypotheses of Lemma 6. Therefore, for every sufficiently small $\varepsilon > 0$, we can find a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4}$$

Let K_1 and K_2 be compact subsets of D with connected complements corresponding the set K in hypothesis 2° of the class $Lip(\beta_1, \beta_2)$. In view of hypothesis 1° of the class $Lip(\beta_1, \beta_2)$, there exists an element $(g_1, g_2) \in$

$F^{-1}\{p\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on K. Let $\hat{c} = \max(1, c)$, and let $G(g_1, g_2)$ be the set of all $\tau \in \mathbb{R}$ satisfy the inequalities

$$\sup_{s \in K_1} |\zeta(s + i\tau; \mathfrak{a}) - g_1(s)| < \hat{c}^{-1/\beta} \left(\tfrac{\varepsilon}{2}\right)^{1/\beta}$$

and

$$\sup_{s \in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{b}) - g_2(s)| < \hat{c}^{-1/\beta} \left(\tfrac{\varepsilon}{2}\right)^{1/\beta}.$$

Then, by Theorem 3, we have that

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\,\{\tau \in [0, T] \cap G(g_1, g_2)\} > 0. \tag{5}$$

In view of hypothesis 2° of the class $Lip(\beta_1, \beta_2)$, for $\tau \in G(g_1, g_2)$,

$$\begin{aligned}
&\sup_{s \in K} |F(\zeta(s + i\tau; \mathfrak{a}), \zeta(s + i\tau, \alpha; \mathfrak{b})) - p(s)| \\
&\quad \leqslant c \max\left(\sup_{s \in K_1} |\zeta(s + i\tau; \mathfrak{a}) - g_1(s)|^{\beta_1}, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{b}) - g_2(s)|^{\beta_2}\right) \\
&\quad < c\, \hat{c}^{-\beta/\beta} \left(\frac{\varepsilon}{2}\right)^{\beta/\beta} \leqslant \frac{\varepsilon}{2}.
\end{aligned}$$

This, together with (5) shows that

$$\begin{aligned}
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] :& \\
&\sup_{s \in K} |F(\zeta(s + i\tau; \mathfrak{a}), \zeta(s + i\tau, \alpha; \mathfrak{b})) - p(s)| < \frac{\varepsilon}{2}\Big\} > 0. \quad (6)
\end{aligned}$$

Taking into account (4), we find that

$$\begin{aligned}
&\left\{\tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau; \mathfrak{a}), \zeta(s + i\tau, \alpha; \mathfrak{b})) - p(s)| < \frac{\varepsilon}{2}\right\} \\
&\quad \subset \left\{\tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau; \mathfrak{a}), \zeta(s + i\tau, \alpha; \mathfrak{b})) - f(s)| < \varepsilon\right\}.
\end{aligned}$$

Combining this with (6) gives the theorem. □

Proof of Theorem 5. We use the arguments similar to those used in the proof of Theorem 4. Lemma 6 implies the existence of polynomials $p_1 = p_1(s), \ldots, p_n = p_n(s)$ such that

$$\sup_{1 \leqslant j \leqslant n} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{7}$$

Let $\hat{K}_1$ and $\hat{K}_2$ be the sets corresponding the sets $K_1, \dots, K_n$ in hypothesis 2° of the class $Lip_n(\beta_1, \beta_2)$. Then, in virtue of hypothesis 1° of the class $Lip_n(\beta_1, \beta_2)$, there exists an element $(g_1, g_2) \in F_{2,n}^{-1}\{p_1, \dots, p_n\} \subset H^2(D)$ such that $g_1(s) \neq 0$ on $\hat{K}_1$. For the set $G(g_1, g_2)$, we preserve the notation of the proof of Theorem 4. Applying hypothesis 2° of the class $Lip_n(\beta_1, \beta_2)$, we find that, for $\tau \in G(g_1, g_2)$,

$$\begin{aligned}&\sup_{1\leqslant j\leqslant n}\sup_{s\in K_j}\big|F_j\big(\zeta(s+i\tau;\mathfrak{a}),\zeta(s+i\tau,\alpha;\mathfrak{b})\big)-p_j(s)\big|\\&\quad\leqslant c\max\bigg(\sup_{s\in\hat{K}_1}|\zeta(s+i\tau;\mathfrak{a})-g_1(s)|^{\beta_1},\\&\qquad\qquad\sup_{s\in\hat{K}_2}|\zeta(s+i\tau,\alpha;\mathfrak{b})-g_2(s)|^{\beta_2}\bigg)<\frac{\varepsilon}{2}.\end{aligned}$$

This and (5) show that

$$\begin{aligned}&\liminf_{T\to\infty}\frac{1}{T}\text{meas}\bigg\{\tau\in[0,T]:\\&\quad\sup_{1\leqslant j\leqslant n}\sup_{s\in K_j}|F_j\left(\zeta(s+i\tau;\mathfrak{a}),\zeta(s+i\tau,\alpha;\mathfrak{b})\right)-p_j(s)|<\frac{\varepsilon}{2}\bigg\}>0.\end{aligned}$$

Combining this with (7), we obtain the assertion of the theorem. □

3. Examples

In this section, we present some examples for Theorems 4 and 5.

3.1. Examples for Theorem 4

1. For $(g_1, g_2) \in H^2(D)$, let

$$F(g_1, g_2) = g_1 + g_2.$$

We will prove that $F \in Lip(1,1)$.

Let

$$p(s) = a_n s^n + \cdots + a_1 s + a_0$$

be an arbitrary polynomial, $a_n \neq 0$, and K be an arbitrary subset of D with connected complement. Since K is a bounded set, there exists a number $\hat{a}_0 \in \mathbb{C}$ such that the polynomial

$$\hat{p}(s) = a_n s^n + \cdots + a_1 s + \hat{a}_0 \neq 0$$

on K. Now we choose $b \in \mathbb{C}$ such that $\hat{a}_0 + b = a_0$. Thus, taking $g_1(s) = \hat{p}(s)$ and $g_2(s) = b$, we have that $g_1(s) + g_2(s) = p(s)$. Thus, $(g_1, g_2) \in F^{-1}\{p\}$, and $g_1(s) \neq 0$ on K. Therefore, hypothesis 1° of the class $Lip(\beta_1, \beta_2)$ is satisfied.

Let $K \subset D$ be a compact set with connected complement. Then, for all $(g_{r1}, g_{r2}) \in H^2(D)$, $r = 1, 2$, we have

$$\begin{aligned}
&\sup_{s\in K} |F(g_{11}, g_{12}) - F(g_{21}, g_{22})| \\
&\quad = \sup_{s\in K} |(g_{11}(s) + g_{12}(s)) - (g_{21}(s) + g_{22}(s))| \\
&\quad \leqslant \sup_{s\in K} |g_{11}(s) - g_{21}(s)| + \sup_{s\in K} |g_{12}(s) - g_{22}(s)| \\
&\quad \leqslant 2 \sup_{1\leqslant j\leqslant 2} \sup_{s\in K} |g_{1j}(s) - g_{2j}(s)|.
\end{aligned}$$

Thus, hypothesis 2° of the class $Lip(1, 1)$ is satisfied with $c = 2$ and $K_1 = K_2 = K$, and $F \in Lip(1, 1)$. Hence, the function

$$\zeta(s; \mathfrak{a}) + \zeta(s, \alpha; \mathfrak{b})$$

is universal in the sense of Theorem 4.

2. This example generalizes Example 1 for derivatives of the functions $g_1(s)$ and $g_2(s)$. Denote by $g^{(k)}$ the kth derivative of $g \in H(D)$. For $(g_1, g_2) \in H^2(D)$, let

$$F(g_1, g_2) = c_1 g_1^{(k_1)} + c_2 g_2^{(k_2)},$$

where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $k_1, k_2 \in \mathbb{N}$. We will prove that $F \in Lip(1, 1)$. Let $p(s)$ and K be the same as in Example 1. We take

$$g_1(s) = \frac{a_n s^{n+k_1}}{c_1(n+1)\cdots(n+k_1)} + \cdots + \frac{a_1 s^{1+k_1}}{c_1(k_1+1)!} + \frac{a_0 s^{k_1}}{c_1 k_1!} + c_0,$$

where $c_0 \in \mathbb{C}$ and $|c_0|$ is large enough so that $g_1(s) \neq 0$ for $s \in K$, and $g_2(s) = C$, with arbitrary $C \in \mathbb{C}$. Then, clearly, $(g_1, g_2) \in H^2(D)$, and

$$F(g_1, g_2) = p(s).$$

Therefore, hypothesis 1° of the class $Lip(1, 1)$ is satisfied. It remains to check hypothesis 2° of the class $Lip(1, 1)$.

Let K be an arbitrary compact subset of D with connected complement, and let $K \subset G \subset K_1$, where G is an open set, and $K_1 \subset D$ is a compact subset with connected complement. We take a simple closed contour L lying in $K_1 \setminus G$ and enclosing the the set K. Then, in view of the Cauchy integral formula, we have that, for all $(g_{r1}, g_{r2}) \in H^2(D)$, $r = 1, 2$, and $s \in K$,

$$|F(g_{11}(s), g_{12}(s)) - F(g_{21}(s), g_{22}(s))| = \left|\sum_{j=1}^{2} c_j \frac{k_j!}{2\pi i} \int_L \frac{g_{1j}(z) - g_{2j}(z)}{(z-s)^{k_j+1}} \mathrm{d}z\right|$$
$$\leqslant \sum_{j=1}^{2} |c_j| \hat{c}_j \sup_{s \in L} |g_{1j}(s) - g_{2j}(s)| \leqslant \sum_{j=1}^{2} |c_j| \hat{c}_j \sup_{s \in K_1} |g_{1j}(s) - g_{2j}(s)|$$
$$\leqslant c \sup_{1 \leqslant j \leqslant 2} \sup_{s \in K_1} |g_{1j}(s) - g_{2j}(s)|.$$

Here $c = 2\max_{1\leqslant j \leqslant 2}(|c_j|\hat{c}_j)$, and $\hat{c}_j$ is a positive constant, $j = 1, 2$. This shows that hypothesis 2° of the class $Lip(1, 1)$ is also satisfied. Thus, $F \in Lip(1, 1)$, and for the function

$$c_1\zeta^{(k_1)}(s; \mathfrak{a}) + c_2\zeta^{(k_2)}(s, \alpha; \mathfrak{b})$$

the assertion of Theorem 4 holds.

3.2. Example for Theorem 5

Let, for $r = 1, 2$,

$$F_r(g_1, g_2) = c_r g_r^{(k_r)},$$

where $c_r \in \mathbb{C} \setminus \{0\}$ and $k_r \in \mathbb{N}$. Consider the operator

$$F_{2,2} = (F_1, F_2): \ H^2(D) \to H^2(D).$$

Let $K \subset D$ be an arbitrary set with connected complement, and $p_1 = p_1(s)$, $p_2 = p_2(s)$ be arbitrary polynomials. Similarly to Example 2 for Theorem 4, from the equation

$$F_1(g_1, g_2) = p_1$$

we find $g_1(s) = \hat{g}_1(s) \in H(D)$ such that $\hat{g}_1(s) \neq 0$ on K. The function $g_2(s) \in H(D)$ can be arbitrary. Similarly, solving the equation

$$F_2(\hat{g}_1, g_2) = p_2,$$

we find $g_2(s) = \hat{g}_2(s) \in H(D)$. Thus, we have that

$$F_{2,2}(\hat{g}_1, \hat{g}_2) = (F_1(\hat{g}_1, \hat{g}_2), F_2(\hat{g}_1, \hat{g}_2)) = (p_1, p_2),$$

and $\hat{g}_1(s) \neq 0$ on K. Consequently, hypothesis 1° of the class $Lip_2(1, 1)$ is satisfied.

For a compact subset $K_r \subset D$ with connected complement, let $K_r \subset G_r \subset \widetilde{K}_r$, where G_r is an open set, and $\widetilde{K}_r \subset D$ is a compact subset with connected complement, $r = 1, 2$. Moreover, let L_r be a simple closed contour lying in $\widetilde{K}_r \setminus G_r$ and enclosing the set K_r, $r = 1, 2$. Then, for fixed $r = 1, 2$ and all $(g_{j1}, g_{j2}) \in H^2(D)$, $j = 1, 2$, $s \in K_r$, by the Cauchy integral formula, we have that

$$\begin{aligned}
&|F_r(g_{11}(s), g_{12}(s)) - F_r(g_{21}(s), g_{22}(s))| = \left| c_r \frac{k_r!}{2\pi i} \int_{L_r} \frac{g_{1r}(z) - g_{2r}(z)}{(z-s)^{k_r+1}} \mathrm{d}z \right| \\
&\quad \leqslant |c_r| \hat{c}_r \sup_{s \in L_r} |g_{1r}(s) - g_{2r}(s)| \leqslant |c_r| \hat{c}_r \sup_{s \in \widetilde{K}_r} |g_{1r}(s) - g_{2r}(s)| \\
&\quad \leqslant c \sup_{1 \leqslant j \leqslant 2} \sup_{s \in \widetilde{K}_r} |g_{1j}(s) - g_{2j}(s)|,
\end{aligned}$$

where $c = \max_{1 \leqslant r \leqslant 2}(|c_r| \hat{c}_r)$, and $\hat{c}_r$ is a positive constant, $r = 1, 2$. Hence,

$$\begin{aligned}
&\sup_{1 \leqslant r \leqslant 2} \sup_{s \in K_r} |F_r(g_{11}(s), g_{12}(s)) - F_r(g_{21}(s), g_{22}(s))| \\
&\quad \leqslant c \sup_{1 \leqslant j \leqslant 2} \sup_{s \in \hat{K}} |g_{1j}(s) - g_{2j}(s)|,
\end{aligned}$$

where $\hat{K} \subset D$ is a compact subset with connected complement, such that

$$\widetilde{K}_1 \cup \widetilde{K}_2 \subset \hat{K},$$

and

$$\hat{K}_1 = \hat{K}_2 = \hat{K}.$$

Thus, hypothesis 2° of the class $Lip_2(1,1)$ is satisfied, and we have that $F_{2,2} \in Lip_2(1,1)$. Therefore, by Theorem 5, for every $\varepsilon > 0$ and $k_1, k_2 \in \mathbb{N}$,

$$\begin{aligned}
\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0, T] : &\sup_{s \in K_1} \left| \zeta^{(k_1)}(s + i\tau; \mathfrak{a}) - f_1(s) \right| < \varepsilon, \\
&\sup_{s \in K_2} \left| \zeta^{(k_2)}(s + i\tau, \alpha; \mathfrak{b}) - f_2(s) \right| < \varepsilon \Big\} > 0,
\end{aligned}$$

if $\mathfrak{a}$, α, K_1, K_2 and $f_1(s)$, $f_2(s)$ satisfy the hypotheses of Theorem 5.

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