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ON ZEROS AND c-VALUES OF EPSTEIN ZETA-FUNCTIONS

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Dedicated to Professor Antanas Laurinčikas
on the occasion of his 65th birthday

Abstract. Let $\mathcal{Q}$ be a positive definite $n \times n$ matrix and $\zeta(s;\mathcal{Q})$ be the Epstein zeta-function associated with $\mathcal{Q}$. In the present paper, we prove that, for arbitrary given complex number c, the equation $\zeta(s;\mathcal{Q}) = c$ has at least CT, for some positive constant C, solutions in the region $\operatorname{Re} s > \frac{n-1}{2}$ when $n \geqslant 4$ is even and $\mathcal{Q}$ satisfies certain conditions. As a corollary, we show that $\zeta(s;I_{2k})$, where $\mathbb{N} \ni k \neq 1,2,4$ and I_n is the n-dimensional unit matrix, have complex zeros in the strip $k - \frac{1}{2} < \operatorname{Re} s < k$.

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1. Introduction

1.1. Epstein zeta-functions

In the beginning of the twentieth century, Epstein [12] introduced zeta-functions associated with quadratic forms. As mentioned in [32, Introduction], these zeta-functions are interesting analytical objects which play an important role in algebraic number theory, the theory of modular forms (see, for example, Siegel [30]), and, recently, in chemistry and physics (see, for instance, Buhler and Crandall [6], Elizalde and Romeo [10], and Elizalde [11]).

Let $\mathcal{Q}$ be a positive definite $n \times n$ matrix, and write $\mathcal{Q}[\mathbf{x}] := \mathbf{x}^t \mathcal{Q} \mathbf{x}$ for $\mathbf{x} \in \mathbb{Z}^n$. Then the Epstein zeta-function associated with $\mathcal{Q}$ is given by

$$\zeta(s;\mathcal{Q}) := \sum_{\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} \mathcal{Q}[\mathbf{x}]^{-s}, \quad \operatorname{Re} s > \frac{n}{2}.$$

As some examples of Epstein zeta-functions, we have the following (see also Section 3.1). Denote by I_n the n-dimensional unit matrix. Let $\zeta(s)$ be the Riemann zeta-function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, where $\operatorname{Re} s > 1$, $L(s,\chi)$ be the Dirichlet L-function $L(s,\chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s}$, where $\operatorname{Re} s > 1$, and χ_{-4} be the non-principal Dirichlet character of mod 4. Then we have

$$\zeta(s;I_1) = 2\zeta(2s), \quad \zeta(s;I_2) = 4\zeta(s)L(s,\chi_{-4}),$$
$$\zeta(s;I_4) = 8(1-2^{2-2s})\zeta(s)\zeta(s-1).$$

For any positive definite $n \times n$ matrix $\mathcal{Q}$, the Epstein zeta-function $\zeta(s;\mathcal{Q})$ is continued analytically elsewhere, except for a simple pole at $s = \frac{n}{2}$. Moreover, the Epstein zeta-function satisfies a functional equation of the Riemann-type

$$\pi^{-s}\Gamma(s)\zeta(s;\mathcal{Q}) = (\det \mathcal{Q})^{-1/2} \pi^{s-n/2} \Gamma\left(\frac{n}{2} - s\right) \zeta\left(\frac{n}{2} - s; \mathcal{Q}^{-1}\right). \quad (1)$$

This functional equation implies that $\zeta(s;\mathcal{Q})$ vanishes at the so-called trivial zeros $s = -m$, $m \in \mathbb{N}$. All other zeros are said to be nontrivial and are denoted by $\rho = \beta + i\gamma$.

If the Riemann hypothesis, i.e., all nontrivial zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$, is true, then all nontrivial zeros of the function $\zeta(s;I_1)$ lie on the critical line $\sigma = \frac{1}{4}$. Additionally, if the analogue of the Riemann hypothesis for the Dirichlet L-function $L(s,\chi_{-4})$ holds, then all nontrivial zeros of the function $\zeta(s;I_2)$ lie on the critical line $\sigma = \frac{1}{2}$ or $\sigma = \frac{1}{4}$. The function $\zeta(s;I_4)$ is expected to have most of its zeros on the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{3}{2}$, but infinitely many zeros lie on $\sigma = 1$.

The zero-distribution of these Epstein zeta-functions with $n = 2$ was firstly investigated by Potter and Titchmarsh [27]. They proved that infinitely many zeros lie on the critical line $\sigma = \frac{1}{2}$. Bateman and Grosswald [2] showed that Epstein zeta-functions attached to positive definite quadratic forms $ax^2+bxy+cy^2$ with discriminant $\Delta := b^2-4ac$ have a real zero between

$\frac{1}{2}$ and 1 if $k := (2a)^{-1}\sqrt{\Delta} > 7.00556$. Note that this result was announced by Chowla and Selberg [7] but they have never published a proof. Deuring [9] and Stark [31] showed that all zeros of these Epstein zeta-functions in the rectangle $-1 < \mathrm{Re}\, s < 2$, $|\mathrm{Im}\, s| \leqslant 2k$ lie on the critical line $\sigma = \frac{1}{2}$ and are simple with the exception of two real zeros between 0 and 1, provided k is sufficiently large.

It is known that if the binary quadratic form $\mathcal{Q}[\mathbf{x}]$ has class number one, then the Epstein zeta-function is up to a constant factor equal to the Dedekind zeta-function of the related quadratic number field; in this case, $\zeta(s;\mathcal{Q})$ has an Euler product and is expected to satisfy the analogue of the Riemann hypothesis. If the class number is larger than one, Davenport and Heilbronn [8] proved an infinitude of zeros in the half-plane of absolute convergence $\sigma > 1$. Hejhal [16] and Bombieri and Hejhal [4] proved that almost all zeros of Epstein zeta-functions associated with binary quadratic forms lie on the critical line subject to the truth of the generalized Riemann hypothesis in combination with an unproved but widely believed conjecture on the spacing of zeros of L-functions for ideal class characters. Recently, Bombieri and Mueller [5] obtained upper and lower bounds for the rate of approach of zeros to the boundary of the zero-free half-plane for certain Epstein zeta functions, associated to positive definite binary quadratic forms with class number 2. Moreover, Lee [22] showed an asymptotic formula for the number of zeros in any strip $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$.

It turns out that the zero-distribution of Epstein zeta-functions attached to quadratic forms in more than two variables has a rather different nature. For Epstein zeta-functions attached to certain quadratic forms of rank $n = 4$, Fujii [13] investigated the real zeros of Epstein zeta functions with $\mathcal{Q}[\mathbf{x}] = x_1^2 + x_2^2 + d(x_3^2 + x_4^3)$. Terras [35] gave examples which have real zeros of the critical line $\sigma = \frac{n}{4}$ for arbitrary n. Steuding [32] proved that the mean value of the real parts of the nontrivial zeros of the Epstein zeta-function is equal to the abscissa of the critical line $\sigma = \frac{n}{4}$. Let $N(T;\mathcal{Q})$ count the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s;\mathcal{Q})$ with $|\gamma| \leqslant T$. Denote by $\mathrm{m}(\mathcal{Q})$ the minimum of the values of the quadratic form $\mathcal{Q}[\mathbf{x}]$ for $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and, finally, let $\mathrm{N}(\mathcal{Q})$ count the number of $\mathbf{x}$ for which $\mathcal{Q}[\mathbf{x}] = \mathrm{m}(\mathcal{Q})$. Then Steuding showed the following result (see [32], Theorem 1).

Theorem A. *As T tends to infinity,*

$$N(T;\mathcal{Q}) \quad = \quad \frac{2T}{\pi} \log \frac{T}{\pi \mathrm{e}\sqrt{\mathrm{m}(\mathcal{Q})\mathrm{m}(\mathcal{Q}^{-1})}} + \mathrm{O}(\log T),$$

$$\sum_{|\gamma|<T}\left(\beta-\frac{n}{4}\right) = -\frac{T}{4\pi}\log\bigl(\Sigma(\mathcal{Q})\bigr)+\mathrm{O}(\log T),$$

where $\Sigma(\mathcal{Q})$ *is defined by*

$$\Sigma(\mathcal{Q}) := (\det\mathcal{Q})^2\left(\frac{\mathrm{N}(\mathcal{Q})}{\mathrm{N}(\mathcal{Q}^{-1})}\right)^4\left(\frac{\mathrm{m}(\mathcal{Q}^{-1})}{\mathrm{m}(\mathcal{Q})}\right)^n.$$

The first asymptotic formula is a Riemann-von Mangoldt formula for the Epstein zeta-function. It should be noted that if $\Sigma(\mathcal{Q})$ is different from 1, $\zeta(s;\mathcal{Q})$ has infinitely many zeros off the critical line $\sigma=\frac{n}{4}$ (see Remark 3.1). The combination of both asymptotic formulae of the above theorem leads to

$$\frac{1}{N(T;\mathcal{Q})}\sum_{|\gamma|<T}\beta=\frac{n}{4}+\mathrm{O}\left(\frac{1}{\log T}\right),$$

and thus the mean-value of the real parts of the nontrivial zeros of the Epstein zeta-function $\zeta(s;\mathcal{Q})$ exists and is equal to $\frac{n}{4}$.

It turns out that the distribution of the c-values, i.e., the roots of the equation $\zeta(s;\mathcal{Q})=c$, behaves similarly to the zero-distribution showed in Theorem A. Namely, Steuding (see [33], Theorem 1) proved the analogue of a Riemann-von Mangoldt formula for the counting function $N_c(\sigma,\infty,T;\mathcal{Q})$ and asymmetry in the c-values distribution, where $N_c(\sigma,\infty,T;\mathcal{Q})$ counts the number of c-values $\rho_c=\beta_c+i\gamma_c$ with $\sigma<\beta_c<\infty$ and $0<\gamma_c<T$. Moreover, one can conclude from Theorem 2 in [33] the following fact.

Proposition 1.1. *Let c be a complex number and $\sigma_1>\max\left\{\frac{1}{4},\frac{n-1}{2}\right\}$ fixed. Then, for sufficiently large T, we have*

$$\sum_{\substack{\beta_c>\sigma_1\\ 0<\gamma_c<T}}(\beta_c-\sigma_1)\ll T.$$

In particular,

$$N_c(\sigma_1,\infty,T;\mathcal{Q})\ll T \quad \textit{as} \quad T\to\infty.$$

It should be noted that Theorem 2 from [33] gives the same estimation but for c-values with the imaginary part satisfying $T<\gamma_c<2T$. However, the first part of the proposition can be easily showed by applying this theorem with $2^{-n}T$ instead of T and summing up all resulting estimations.

The second part of Proposition 1.1 can be easily verified as well, since

$$\sum_{\substack{\beta_c>\sigma_1\\0<\gamma_c<T}} 1 \leqslant \frac{1}{\sigma_1-\sigma_1'} \sum_{\substack{\beta_c>\sigma_1\\0<\gamma_c<T}} (\beta_c-\sigma_1') \leqslant \frac{1}{\sigma_1-\sigma_1'} \sum_{\substack{\beta_c>\sigma_1'\\0<\gamma_c<T}} (\beta_c-\sigma_1') \ll T,$$

where $\sigma_1 > \sigma_1' > \max\left\{\frac{1}{4}, \frac{n-1}{2}\right\}$.

Our main propose of this paper is to give a lower bound for the counting function $N_c(\sigma_1, \sigma_2, T; \mathcal{Q})$, provided that $\frac{n-1}{2} < \sigma_1 < \sigma_2 < \frac{n}{2}$ and the order n of $\mathcal{Q}$ is even and greater that or equal to 4. For a precise formulation, we refer to Theorem 3.1. As a corollary, we obtain new examples of Epstein zeta-functions which have complex zeros off the critical line $\operatorname{Re} s = \frac{n}{4}$ (see Remark 3.1).

2. Auxiliary results

2.1. Hybrid universality

In 1975, Voronin [36] showed the universality theorem for the Riemann zeta-function $\zeta(s)$. To state it, let $D := \left\{s \in \mathbb{C} : \frac{1}{2} < \operatorname{Re} s < 1\right\}$ and $K \subset D$ be a compact set with connected complement. Denote by $\mu(A)$ the Lebesgue measure of the set A, and, for $T > 0$, write $\nu_T\{\ldots\} := T^{-1}\mu\{\tau \in [0,T] : \ldots\}$ where the dots stands for a condition satisfied by τ. Let $H(K)$ denote the space of non-vanishing continuous functions on K, which are analytic in the interior, equipped with the supremum norm $\|\cdot\|_K$. The modern version of the universality for $\zeta(s)$ is as follows.

Theorem B. *For any $f \in H(K)$ and any $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T\left\{\|\zeta(s+i\tau) - f(s)\|_K < \varepsilon\right\} > 0.$$

Roughly speaking, this theorem implies that any non-vanishing analytic function can be uniformly approximated by the Riemann zeta-function. Subsequently many mathematicians have considered generalizations of universality (see, for instance, [34]). For example, Voronin also proved the joint universality theorem, which implies that a collection of Dirichlet L-functions with non-equivalent characters uniformly and simultaneously approximates non-vanishing analytic functions. Note that two characters are non-equivalent if they are not induced by the same character. In slightly different form this was also established by Gonek [14] and Bagchi [1], independently (both of these papers are unpublished Doctoral Theses). Let

$K_1, \ldots, K_m \subset D$ be compact sets with connected complement. The modern version of the joint universality for Dirichlet L-functions $L(s,\chi)$ states:

THEOREM C. *Let $\chi_1 \bmod q_1, \ldots, \chi_m \bmod q_m$ be pairwise non-equivalent Dirichlet characters, and $f_l(s)$ be a non-vanishing continuous function on K_l which is analytic in the interior of K_l for $1 \leqslant l \leqslant m$. Then, for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \nu_T\Big\{\max_{1\leqslant l\leqslant m} \|L(s+i\tau,\chi_l) - f_l(s)\|_{K_l} < \varepsilon\Big\} > 0.$$

In the present paper, we use so-called hybrid universality, which is a connection between the Voronin theorem and the classical Kronecker approximation theorem. The authors used this property to obtain universality theorems for certain combinations of L-functions with general Dirichlet series as coefficients in [23] and [24]. Moreover, the authors gave a lower and upper bound for the number of zeros of certain polynomials of L-functions in [25]. We adopt the approach from [25] to prove the main theorem on c-values of Epstein zeta-functions (see Theorem 3.1 below).

We denote the distance to the nearest integer by $\|\cdot\|$. The precise definition of the hybrid universality is as follows.

DEFINITION 2.1. *Hybrid joint universality for the set of functions $\{L_l(s)\}_{1\leqslant l\leqslant m}$ is the following property: Let $K_l \subset D$, $f_l \in H(K_l)$ and $\{\alpha_j\}_{1\leqslant j\leqslant k}$ be real numbers linearly independent over $\mathbb{Q}$. Then, for any $\varepsilon > 0$ and any real numbers $\{\theta_j\}_{1\leqslant j\leqslant k}$, we have*

$$\liminf_{T\to\infty} \nu_T\Big\{\max_{1\leqslant l\leqslant m} \|L_l(s+i\tau) - f_l(s)\|_{K_l} < \varepsilon,\ \max_{1\leqslant j\leqslant k} \|\tau\alpha_j - \theta_j\| < \varepsilon\Big\} > 0. \quad (2)$$

The first result on hybrid universality was proved in weaker form by Gonek [14] and slightly improved using different method by Kaczorowski and Kulas [19]. They showed that Dirichlet L-functions satisfy the inequality in above definition for $\alpha_n = \log p_n$, where p_n denotes the nth prime number. The second author [26] proved the hybrid universality in the most general form for an axiomatically defined wide class of L-functions having Euler product which contains, for instance, Dirichlet L-functions.

It is worth to note that the Diophantine inequality appeared in (2) plays a crucial role in estimating the number of c-values from below. Using this restriction on t, one can prove almost periodicity of an absolutely convergent general Dirichlet series defined by

$$\sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n s}, \quad \text{where} \quad a_n \in \mathbb{C}, \quad \lambda_n \in \mathbb{R}\setminus\{0\}.$$

More precisely, the following lemma based on the approach of Bohr [3], or Sander and Steuding [28, Section 2], holds.

LEMMA 2.1. *Let* $P_k(s) = \sum_{n=1}^{\infty} a_{kn} \mathrm{e}^{-\lambda_{kn} s}$ *be a general Dirichlet series. Then, for every* $\varepsilon > 0$ *and every compact set* K *lying in the half-plane of absolute convergence, there exist* $\delta > 0$, $N \in \mathbb{Z}_+$ *and a finite set* $J \in \{1, 2, \ldots, N\} \times \mathbb{Z}_+$ *such that the numbers* λ_{jm}, $(j,m) \in J$, *are linearly independent over* $\mathbb{Q}$ *and, moreover, if*

$$\max_{(k,n)\in J} \left\| \frac{\tau \lambda_{kn}}{2\pi N} \right\| < \delta,$$

then it holds that

$$\max_{1\leqslant k\leqslant n} \left\| P_k(s+i\tau) - P_k(s) \right\|_K < \varepsilon.$$

Proof. Let us take $\varepsilon > 0$, and a compact set K is fixed. Then there exists a positive integer M such that

$$\max_{1\leqslant k\leqslant n} \max_{s\in K} \sum_{n>M} |a_{kn}| \mathrm{e}^{-\lambda_{kn} \mathrm{Re}\, s} < \frac{\varepsilon}{3}.$$

Now let us take the set J of indices such that $\{\lambda_{jm} : (j,m) \in J\}$ is the basis of a vector space over $\mathbb{Q}$ generated by all numbers λ_{kn} with $n \leqslant M$. Moreover, let N be a positive integer such that all these numbers can be expressed as a linear combination of elements $\frac{\lambda_{jm}}{N}$, $(j,m) \in J$, with integer coefficients. Then, for $s \in K$ and $1 \leqslant k \leqslant n$, we have

$$\begin{aligned} |P_k(s+i\tau) - P_k(s)| &\leqslant \sum_{n\leqslant M} |a_{kn}| \left| (\mathrm{e}^{-\lambda_{kn}(s+i\tau)} - \mathrm{e}^{-\lambda_{kn} s}) \right| \\ &\quad +2 \sum_{n>M} |a_{kn}|\, \mathrm{e}^{-\lambda_{kn} \mathrm{Re}\, s} \\ &\ll \max_{n\leqslant M} \left| \mathrm{e}^{-i\tau\lambda_{kn}} - 1 \right| + \frac{2\varepsilon}{3} \ll \max_{(j,m)\in J} \left\| \frac{\tau\lambda_{jm}}{N} \right\| + \frac{2\varepsilon}{3}. \end{aligned}$$

Finally, taking suitable $\delta > 0$ completes the proof. □

Thus, since hybrid universality combines Diophantine approximations and universality, one can use this property to show that any linear combination of hybrid universal L-functions with an absolutely convergent Dirichlet series as coefficients approximates (in the sense of Voronin's theorem) an analytic function having at least one c-values in certain disk. More details will be given in Section 3.2. Then the classical Rouché theorem gives the lower bound for the number of c-values for such a linear combination.

2.2. Epstein zeta-functions and L-functions

It is well known that if $\Delta < 0$ is a fundamental discriminant, then equivalence classes of binary quadratic forms of discriminant Δ are in one-to-one correspondence with ideal classes of the field $\mathbb{Q}(\sqrt{\Delta})$. The number of representations of a number k by a quadratic form is the number of integer ideals of norm k in the corresponding ideal class, times the number w of roots of unity in $\mathbb{Q}(\sqrt{\Delta})$. Let $\chi'_1, \dots, \chi'_m$ be characters of the class field $\mathbb{Q}(\sqrt{\Delta})$ and $\mathcal{Q}_\Delta[\mathbf{x}]$ be a quadratic form with integer coefficients whose discriminant is equal to the discriminant of $\mathbb{Q}(\sqrt{\Delta})$. Then it follows that

$$\zeta(s; \mathcal{Q}_\Delta) = \sum_{k=1}^{m} \alpha_k L(s, \chi'_k),$$

where $\alpha_k \in \mathbb{C}$, $\chi'_k \neq \chi'_j$ and $\chi'_k \neq \overline{\chi}'_j$ for $k \neq j$. Note that the number m in the equation above is greater than 1 under the condition $\Delta < -1$ (see, for example, [20, p. 283]). Similarly to Theorem 7.4.3 form [20], we obtain the following theorem by the equation above and joint universality for Dirichlet L-functions (see Theorem C).

Theorem D. *Suppose that the class number of the field $\mathbb{Q}(\sqrt{\Delta})$, where Δ is a negative integer, is greater than 1. Then, for any $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$ and for T sufficiently large, the region $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ contains at least CT c-values of $\zeta(s; \mathcal{Q}_\Delta)$, where $C = C(\sigma_1, \sigma_2, \mathcal{Q}_\Delta) > 0$ does not depend on T, and c is an arbitrary given complex number.*

Hereafter, assume that $\mathcal{Q}[\mathbf{x}] \in \mathbb{Z}$ for any $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. For $k \in \mathbb{Z}_{\geqslant 0}$, we define $r_\mathcal{Q}(k)$ by the number of $\mathbf{x} \in \mathbb{Z}^n$ which satisfies $\mathcal{Q}[\mathbf{x}] = k$. Then the Epstein zeta-function $\zeta(s; \mathcal{Q})$ is also expressed by

$$\zeta(s; \mathcal{Q}) = \sum_{k=1}^{\infty} \frac{r_\mathcal{Q}(k)}{k^s}, \quad \operatorname{Re} s > \frac{n}{2}.$$

It is well-known (see, for example, [15]) that the corresponding theta series

$$\theta(z; \mathcal{Q}) := \sum_{k=0}^{\infty} r_\mathcal{Q}(k) \mathrm{e}^{2\pi i k z}$$

becomes a modular form of weight $\frac{n}{2}$, which decomposes into the summation of an Eisenstein series and a cusp form. More precisely, we have

$$\theta(z; \mathcal{Q}) = E_\mathcal{Q}(z) + S_\mathcal{Q}(z),$$

where $E_{\mathcal{Q}}(z) := \sum_{k=0}^{\infty} \mathrm{e}_{\mathcal{Q}}(k)\mathrm{e}^{2\pi ikz}$ is an Eisenstein series and $S_{\mathcal{Q}}(z) := \sum_{k=1}^{\infty} s_{\mathcal{Q}}(k)\mathrm{e}^{2\pi ikz}$ is a cusp form. Moreover, it is known that the coefficient $s_{\mathcal{Q}}(k)$ is evaluated by

$$s_{\mathcal{Q}}(k) \ll \begin{cases} k^{n/4-1/2+\varepsilon} & \text{if} \quad n \quad \text{is even}, \\ k^{n/4-1/4+\varepsilon} & \text{if} \quad n \quad \text{is odd}. \end{cases} \tag{3}$$

Therefore, the Epstein zeta-function $\zeta(s;\mathcal{Q})$ is expressed by

$$\zeta(s;\mathcal{Q}) = \widehat{E}_{\mathcal{Q}}(s) + \widehat{S}_{\mathcal{Q}}(s), \tag{4}$$

where $\widehat{E}_{\mathcal{Q}}(s)$ and $\widehat{S}_{\mathcal{Q}}(s)$ are defined by

$$\widehat{E}_{\mathcal{Q}}(s) := \sum_{k=1}^{\infty} \frac{e_{\mathcal{Q}}(k)}{k^s}, \quad \widehat{S}_{\mathcal{Q}}(s) := \sum_{k=1}^{\infty} \frac{s_{\mathcal{Q}}(k)}{k^s}, \qquad \operatorname{Re} s > \frac{n}{2}.$$

Hence, $\zeta(s;\mathcal{Q})$ is decomposed into the summation of the L-function associated to the Eisenstein series and the L-function associated to the cusp form.

Now we suppose that n is even and $n \geqslant 4$. Then the Eisenstein series $E_{\mathcal{Q}}(z)$ is a modular form of weight $\frac{n}{2}$ and level N, where N is a positive integer such that NA^{-1} becomes an integral matrix for $A = 2\mathcal{Q}$ (see, for example, [18, p. 185]). In Theorem 44 from [17], Hecke showed the following.

THEOREM E. *Let n be even and $n \geqslant 4$. Then the series $\widehat{E}_{\mathcal{Q}}(s)$ is expressed by some linear combination of the form $(t_1t_2)^{-s}L(s,\chi_1)L\big(s-\frac{n}{2}+1,\chi_2\big)$, where t_1, t_2 are positive divisors of level N and χ_1, χ_2 are Dirichlet characters modulo $\frac{N}{t_1}$, $\frac{N}{t_2}$, respectively.*

3. Main result

3.1. Main Theorem

Before we formulate the main result of this paper, we shall express a given Epstein zeta-function as a linear combination of Dirichlet L-functions. Recall that a Dirichlet polynomial is a finite Dirichlet series $\sum_{n=1}^{m} a_n n^{-s}$ with complex coefficients a_n. Now let n be even and $n \geqslant 4$. It should be noted that the series expression of $\widehat{S}_{\mathcal{Q}}(s)$ converges absolutely in the region $\operatorname{Re} s > \frac{n-1}{2}$ because of (3). Therefore, by (4) and Theorem E, we have

$$\zeta(s;\mathcal{Q}) = \sum_{l=1}^{L}\sum_{j=1}^{J} P_{jl}(s)L(s,\varphi_j)L\left(s-\frac{n}{2}+1,\psi_l\right) + \sum_{k=1}^{\infty}\frac{s_{\mathcal{Q}}(k)}{k^s}, \tag{5}$$

where $\operatorname{Re} s > \frac{n-1}{2}$, Dirichlet polynomials $P_{jl}(s)$ are not identically zero, the Dirichlet characters φ_j, $1 \leqslant j \leqslant J$, are non-equivalent as well as ψ_l, $1 \leqslant l \leqslant L$.

Now we state the main theorem on c-values of $\zeta(s;\mathcal{Q})$ with $2\mathbb{N} \ni n \geqslant 4$. Note that Theorem D is c-values of $\zeta(s;\mathcal{Q})$ with $n = 2$.

Theorem 3.1. *Let $c \in \mathbb{C}$ and $n \geqslant 4$ be even. Suppose $L > 1$ or $\widehat{S}_{\mathcal{Q}}(s) \not\equiv c$, namely, $\widehat{S}_{\mathcal{Q}}(s)$ is not constantly equal to c. Then, for any $\frac{n-1}{2} < \sigma_1 < \sigma_2 < \frac{n}{2}$, and, for sufficiently large T, we have*

$$N_c(\sigma_1, \sigma_2, T; \mathcal{Q}) \geqslant CT,$$

where $C = C(\sigma_1, \sigma_2, \mathcal{Q}) > 0$ does not dependent on T.

Remark 3.2. Obviously, the uniqueness theorem for Dirichlet series implies immediately that $\widehat{S}_{\mathcal{Q}}(s) \equiv c$ is equivalent to the condition $s_{\mathcal{Q}}(1) = c$ and $s_{\mathcal{Q}}(k) = 0$ for $k \geqslant 2$.

Before we prove the above theorem, let us give some important examples of Epstein zeta-functions for which Theorem 3.1 holds.

Let $\Delta(q)$ be Ramanujan's τ-function defined by

$$\Delta(q) := \sum_{m=1}^{\infty} \tau(m) q^m = q \sum_{m=1}^{\infty} (1 - q^m)^{24}$$

with $q := \exp(2\pi i \tau)$ and τ from the upper half-plane. We define the L-function $L(s;\Delta)$ associated to Δ by $L(s;\Delta) := \sum_{m=1}^{\infty} \tau(m) m^{-s}$. By Theorem 3.1, the following Epstein zeta-functions have infinitely many c-values, particularly zeros, off the critical line $\operatorname{Re} s = \frac{n}{4}$:

$$\begin{aligned}
\zeta(s; I_6) &= -4\big(\zeta(s)L(s-2,\chi_{-4}) - 4\zeta(s-2)L(s,\chi_{-4})\big), \\
\zeta(s; I_{10}) &= \frac{4}{5}\big(\zeta(s)L(s-2,\chi_{-4}) + 4^2\zeta(s-4)L(s,\chi_{-4})\big) \\
&\quad -2 \sum_{0 \neq \lambda \in \mathbb{Z}[i]} \frac{\lambda^4}{(\lambda\bar{\lambda})^s}, \\
\zeta(s; I_{12}) &= c_1(2^6 - 2^{6-s})\zeta(s)\zeta(s-4) + c_2 L(s;\sqrt{\Delta}),
\end{aligned}$$

where c_1 and c_2 are some constants and $L(s;\sqrt{\Delta})$ the L-function (Dirichlet series) associated to $\sqrt{\Delta}$. It is well-known that if $2\mathbb{N} \ni n \geqslant 10$, the cuspital part of $\theta(z; I_n)$ is non-trivial (see, for example, [18, p. 187]). Therefore, $s_{I_n}(k) \neq 0$ for some $k \in \mathbb{N}$ when $n \geqslant 10$ is even.

Let $\mathfrak{L}_{24}$ be the matrix related to the Leech lattice. A different example is given by

$$\zeta(s;\mathfrak{L}_{24})=\frac{65520}{691}\big(\zeta(s)\zeta(s-11)-L(s;\Delta)\big).$$

Remark 3.3. One has $\Sigma(\mathcal{Q})=1$ for $\mathcal{Q}=I_6,\ I_n$, where $2\mathbb{N}\ni n\geqslant 10$, and $\mathfrak{L}_{24}$ in Theorem A. It should be mentioned that Siegel [29] proved that the number of zeros of $\zeta(s;\mathcal{Q})$ in $2\leqslant \operatorname{Re} s\leqslant \frac{n}{2}-2$, $0<\operatorname{Im} s\leqslant T$ is $\frac{T}{\pi}\log 2+\mathrm{O}(1)$ when $\mathcal{Q}$ belongs to the genus of I_n, $n\geqslant 12$.

3.2. Proof of Theorem 3.1

First, suppose that $L=1$ and $\widehat{S}_{\mathcal{Q}}(s)\not\equiv c$. Let $s_0\in\mathbb{C}$ be such that $\frac{n-1}{2}<\sigma_1<\operatorname{Re} s_0<\sigma_2<\frac{n}{2}$ and $\sum_{j=1}^{J}P_j(s_0)L(s_0,\varphi_j)\neq 0$, and take $0\neq z_0\in\mathbb{C}$ such that

$$z_0\sum_{j=1}^{J}P_j(s_0)L(s_0,\varphi_j)+\sum_{k=1}^{\infty}\frac{s_{\mathcal{Q}}(k)}{k^{s_0}}=c.$$

Let us notice that $\sum_{k=1}^{\infty}s_{\mathcal{Q}}(k)k^{-s}$ and the series expression of $L(s,\varphi_j)$ converge absolutely when $\operatorname{Re} s>\frac{n-1}{2}$.

Hybrid universality (see Definition 2.1) gives that $L\big(s-\frac{n}{2}+1;\psi_1\big)$ can approximate (in the sense of Voronin) the function $f(s)=z_0\exp(s-s_0)$ uniformly on the closed disk $\mathcal{K}$ with center s_0 and radius $r<\min\big\{\sigma_0-\frac{n-1}{2},\frac{n}{2}-\sigma_1\big\}$. Next, assume that

$$g(s):=z_0\exp(s-s_0)\sum_{j=1}^{J}P_j(s)L(s,\varphi_j)+\sum_{k=1}^{\infty}\frac{s_{\mathcal{Q}}(k)}{k^{s}}-c$$

does not have zeros on the boundary of $\mathcal{K}$. Obviously, $g(s_0)=0$. On the other hand, by Lemma 2.1 and the fact that $L\big(s-\frac{n}{2}+1;\psi_1\big)$ is hybridly universal, one has

$$\max_{|s-s_0|=r}\big|\big(\zeta(s+i\tau;\mathcal{Q})-c\big)-g(s)\big|<\min_{|s-s_0|=r}\big|g(s)\big| \tag{6}$$

An application of Rouché's theorem shows that whenever the inequality (6) holds, the equation $\zeta(s+i\tau;\mathcal{Q})=c$ has a root in the interior of $\mathcal{K}$ (see, for example, Section 8.1 in [34]). From the view point of the hybrid universality, the measure of such $\tau\in[0,T]$ is $\gg T$ (see also proof of Theorem 8.4.7 from [21]) which completes the proof.

In order to consider the case $L \geqslant 2$, let us assume (see (5)) that

$$\zeta(s;\mathcal{Q}) = \sum_{l=1}^{L} Q_l(s) L\Big(s - \frac{n}{2} + 1, \psi_l\Big) + Q_0(s), \quad \sigma > \frac{n-1}{2},$$

where Q_l, $0 \leqslant l \leqslant L$, are absolutely convergent Dirichlet series and $Q_l(s)$, $1 \leqslant l \leqslant L$, are not identically zero.

Let σ_0 be a real number satisfying $\frac{n-1}{2} < \sigma_1 < \sigma_0 < \sigma_2 < \frac{n}{2}$, and t_0 and $\delta > 0$ be such that $Q_l(s) \neq 0$ for $|s - s_0| \leqslant \delta$, where $s_0 = \sigma_0 + it_0$. The choice of t_0 and δ is possible, since Dirichlet polynomials Q_l are not identically zero. Moreover, we require that δ is such that $h(s) := s - s_0 + Q_0(s) - Q_0(s_0) \neq 0$ for $|s - s_0| = \delta$, and the disk with center s_0 and radius δ lies in the strip of complex numbers with the real part between $\frac{n-1}{2}$ and $\frac{n}{2}$.

Now, applying hybrid joint universality of $L(s,\psi_1), \ldots, L(s,\psi_L)$ yields that, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left\{ \begin{array}{l} \max\limits_{|s-s_0|\leqslant\delta} \left| L\Big(s+i\tau-\frac{n}{2}+1,\psi_1\Big) - \frac{s}{Q_1(s)} \right| < \varepsilon, \\ \max\limits_{|s-s_0|\leqslant\delta} \left| L\Big(s+i\tau-\frac{n}{2}+1,\psi_2\Big) - \frac{(c-Q_0(s_0)-s_0)}{Q_2(s)} \right| < \varepsilon, \\ \max\limits_{3\leqslant l\leqslant L} \max\limits_{|s-s_0|\leqslant\delta} \left| L\Big(s+i\tau-\frac{n}{2}+1,\psi_l\Big) - \frac{\varepsilon}{(L-2)Q_l(s)} \right| < \varepsilon, \\ \max\limits_{0\leqslant l\leqslant L} \max\limits_{|s-s_0|\leqslant\delta} |Q_l(s+i\tau) - Q_l(s)| < \varepsilon \end{array} \right\}$$

is positive. It should be noted here that, for s_0 satisfying $c - Q_0(s_0) - s_0 = 0$, we need to replace $\frac{c-Q_0(s_0)-s_0}{Q_1(s)}$ by $\frac{c-Q_0(s_0)-s_0}{Q_1(s)} + \varepsilon$.

Next easy calculations show that

$$\liminf_{T\to\infty} \nu_T \Big\{ \max_{|s-s_0|<\delta} \big| (\zeta(s+i\tau;\mathcal{Q}) - c) - \big(s + (c - Q_0(s_0) - s_0) + \varepsilon + Q_0(s) - c\big) \big| \ll \varepsilon \Big\} > 0.$$

Hence, by taking sufficiently small $\varepsilon > 0$, we obtain

$$\liminf_{T\to\infty} \nu_T \left\{ \max_{|s-s_0|=\delta} |(\zeta(s+i\tau;\mathcal{Q}) - c) - h(s)| \leqslant \min_{|s-s_0|=\delta} |h(s)| \right\} > 0$$

Therefore, using Rouché's theorem and the fact that $h(s_0) = 0$ completes the proof.

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References

[1] B. Bagchi, *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series*, PhD. Thesis, Calcutta, Indian Statistical Institute, 1981.

[2] P.T. Bateman, E. Grosswald, On Epstein's zeta function, *Acta Arith.*, **9**, 365–373 (1964).

[3] H. Bohr, Über eine quasi-periodische eigenschaft Dirichletscher reihen mit anwendung auf die Diriehletschen L-Funktionen, *Math. Ann.*, **85**(1), 115–122 (1922).

[4] E. Bombieri, D.A. Hejhal, On the zeros of Epstein zeta functions, *C. R. Acad. Sci. Paris Sér. I Math.* **304**(9), 213–217 (1987).

[5] E. Bombieri, J. Mueller, On the zeros of certain Epstein zeta functions, *Forum Math.*, **20**(2), 359–385 (2008).

[6] J.P. Buhler, R.E. Crandall, Elementary function expansions for Madelung constants, *J. Phys. Series A.*, **20**, 5497–5510 (1987).

[7] S. Chowla, A. Selberg, On Epstein's zeta function. I, *Proc. Nat. Acad. Sci. USA*, **35**, 371–374 (1949).

[8] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series. I, II, *J. London Math. Soc.*, **11**, 181–185, 307–313 (1936).

[9] M. Deuring, Zetafunktionen quadratischer Formen, *J. Reine Angew. Math.*, **172**, 226–252 (1935).

[10] E. Elizalde, A. Romeo, Regularization of general multidimensional Epstein zeta-functions, *Rev. Math. Phys.*, **1**, 113–128 (1989).

[11] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, 2nd ed., Lecture Notes in Physics, **855**, Springer, Heidelberg, 2012.

[12] P. Epstein, Zur Theorie allgemeiner Zetafunctionen, *Math. Ann.*, **56**, 615–644 (1903).

[13] A. Fujii, On the zeros of the Epstein zeta functions, *J. Math. Kyoto Univ.*, **36**(4), 697–770 (1996).

[14] S.M. Gonek, *Analytic Properties of Zeta and L-Functions*, Ph.D. Thesis, Universality of Michigan, 1979.

[15] O.M. Fomenko, Order of the Epstein zeta-function in the critical strip, *J. Math. Sci.*, **110**(6), 3150–3163 (2002).

[16] D.A. Hejhal, Zeros of Epstein zeta functions and supercomputers, in: *Proc. of the Intern. Congr. of Math.*, Vol. **1**, **2**, Berkeley, Calif., 1986, Amer. Math. Soc., Providence, RI, 1362–1384, 1987.

[17] E. Hecke, Über Modulfunktionen und Dirichletschen Reihen mit Eulerscher Productentwicklung. I, II, *Math. Ann.*, **114**, 1–28, 316–351 (1937).

[18] H. Iwaniec, *Topics in Classical Automorphic Forms*, Amer. Math. Soc. Graduate Studies in Mathematics, **17**, 1997.

[19] J. Kaczorowski, M. Kulas, On the non-trivial zeros off line for L-functions from extended Selberg class, *Monatsh. Math.*, **150**(3), 217–232 (2007).

[20] A.A. Karatsuba, S.M. Voronin, *The Riemann Zeta-Function*, Walter de Gruyter & Co., Berlin, 1992.

[21] A. Laurinčikas, R. Garunkšitis, *The Lerch Zeta-Function,* Kluwer Academic Publishers, Dordrecht, 2002.

[22] Y. Lee, On the Zeros of Epstein zeta functions, *Forum Math*, to appear.

[23] T. Nakamura, Ł. Pańkowski, Applications of hybrid universality to multivariable zeta-functions, *J. Number Theory*, **131**(11), 2151–2161 (2011).

[24] T. Nakamura, Ł. Pańkowski, On universality of linear combinations of L-functions, *Monatsh. Math.*, **165**(3), 433–446 (2012).

[25] T. Nakamura, Ł. Pańkowski, Any non-monomial polynomial of the Riemann zeta-function has complex zeros off the critical line, submitted, arXiv:1212.5890.

[26] Ł. Pańkowski, Hybrid joint universality theorem for Dirichlet L-functions, *Acta Arith.*, **141**(1), 59–72 (2010).

[27] H.S.A. Potter, E.C. Titchmarsh, The zeros of Epstein's zeta-functions, *Proc. London Math. Soc.*, **39**, 372–384 (1935).

[28] J. Sander, J. Steuding, Joint universality for sums and products of Dirichlet L-functions, *Analysis*, **26**, 295–312 (2006).

[29] C.L. Siegel, Contributions to the theory of the Dirichlet L-series and the Epstein zeta-functions, *Ann. of Math.*, **44**(2), 143–172 (1943).

[30] C.L. Siegel, *Advanced Analytic Number Theory*, 2nd ed., Tata Institute of Fundamental Research, Bombay, 1980.

[31] H.M. Stark, On the zeros of Epstein's zeta function, *Mathematika*, **14**, 47–55 (1967).

[32] J. Steuding, On the zero-distribution of Epstein zeta-functions, *Math. Ann.*, **333**(3), 689–697 (2005).

[33] J. Steuding, Value-distribution of Epstein zeta-functions, *Publ. Mat.*, Proc. of the Primeras Jornadas de Teoria de Numeros, 221–244 (2007).

[34] J. Steuding, *Value-Distribution of L-functions*, Lecture Notes in Math., **1877**, Springer, Berlin, 2007.

[35] A. Terras, The minima of quadratic forms and the behavior of Epstein and Dedekind zeta functions, *J. Number Theory.*, **12**, 258–272 (1980).

[36] S.M. Voronin, Theorem on the universality of the Riemann zeta function, *Izv. Akad. Nauk SSSR Ser. Mat.*, **39**, 475–486 (1975) (in Russian) = *Math. USSR Izv.*, **9**, 443–453 (1975).

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