AN EXPLICIT FORMULA
FOR THE GREATEST COMMON DIVISOR
OF THREE INTEGERS

Jonas ŠIAULYS, Gediminas STEPANAUSKAS,
Laura VASILIAUSKAITĖ
Vilnius University, Naugarduko 24, LT-03223 Vilnius, Lithuania;
e-mails: jonas siaulys@mif.vu.lt, gediminas.stepanauskas@mif.vu.lt

Dedicated to Professor Antanas Laurinčikas on his 65th birthday

Abstract. One explicit formula for finding the greatest common divisor
of three integers is presented. For the proof, a simple geometrical method
is used.

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In [1], M. Polezzi presented simple geometrical proof of one explicit for-
mula for finding the greatest common divisor of two integers. He proved the
following

Theorem 1. Let $d(m, n)$ be the greatest common divisor of positive integers
$m$ and $n$. Then

$$d(m, n) = 2 \sum_{i=1}^{m-1} \lfloor \frac{in}{m} \rfloor - mn + m + n.$$  \hfill (1)

We present the generalization of formula (1) for finding the greatest com-
mon divisor for three integers. The following assertion is the main in the
paper.
Theorem 2. Let \( d(m,n,s) \) be the greatest common divisor of positive integers \( m, n, \) and \( s. \) Then

\[
d(m,n,s) = 2 \sum_{i=1}^{m-1} \left( \left\lfloor \frac{i \, n}{m} \right\rfloor \left\lfloor \frac{i \, s}{m} \right\rfloor + \left(1 - s\right) \left\lfloor \frac{i \, n}{m} \right\rfloor + (1 - n) \left\lfloor \frac{i \, s}{m} \right\rfloor \right)
+ mns - mn - ms - ns + m + n + s.
\]

(2)

Proof. The set

\[ A = \left\{ (x, y, z) \in \mathbb{Z}^3 : x \geq 0, y \geq 0, z \geq 0, \frac{x}{m} + \frac{y}{n} \leq 1, \frac{z}{s} \leq 1 \right\} \]

is the set of lattice points of the quadrangular pyramid \( OBECA \) (see Figure 1). The number of points of this set when \( x = i, i = 0, 1, \ldots, m, \) is equal to

\[
\left( \left\lfloor n \left(1 - \frac{i}{m}\right) \right\rfloor + 1 \right) \left( \left\lfloor s \left(1 - \frac{i}{m}\right) \right\rfloor + 1 \right).
\]

Therefore, we have

\[
\#\{A\} = \sum_{i=0}^{m} \left( \left\lfloor \frac{m - i}{m} \right\rfloor + 1 \right) \left( \left\lfloor \frac{s - i}{m} \right\rfloor + 1 \right)
\]
\[
= \sum_{i=0}^{m} \left( \left\lfloor \frac{i}{m} \right\rfloor + 1 \right) \left( \left\lfloor \frac{s}{m} \right\rfloor + 1 \right). \tag{3}
\]

The set
\[
\mathbb{B} = \left\{ (x, y, z) \in \mathbb{Z}^3 : y \geq 0, z \leq c, \frac{x}{m} + \frac{y}{n} \leq 1, \frac{x}{m} + \frac{z}{s} \geq 1 \right\}
\]
is the set of lattice points of the triangular pyramid $ADCE$ (see Figure 2). The number of points of the set $\mathbb{B}$ when $x = i$, $i = 0, 1, \ldots, m$, is equal to
\[
\left( \left\lfloor n \left( 1 - \frac{i}{m} \right) \right\rfloor + 1 \right) \sum_{k=[s(1-i/m)]}^{s} 1
\]
\[
= \left( \left\lfloor n \frac{m-i}{m} \right\rfloor + 1 \right) \left( s - \left\lfloor s - \frac{i}{m} \right\rfloor + 1 \right)
\]
\[
= \left( \left\lfloor n \frac{m-i}{m} \right\rfloor + 1 \right) \left( \left\lceil \frac{s}{m} \right\rceil + 1 \right).
\]

Thus,
\[
\#\{\mathbb{B}\} = \sum_{i=0}^{m} \left( \left\lfloor \frac{m-i}{m} \right\rfloor + 1 \right) \left( \left\lfloor \frac{s}{m} \right\rfloor + 1 \right). \tag{4}
\]

![Figure 2](image)

The set
\[
\mathbb{A} \cup \mathbb{B} = \left\{ (x, y, z) \in \mathbb{Z}^3 : x \geq 0, y \geq 0, 0 \leq z \leq s, \frac{x}{m} + \frac{y}{n} \leq 1 \right\}
\]
is the set of lattice points of the triangular prism $OABCDE$ (see Figure 3). Considering the prism as a half of rectangular parallelepiped, we obtain that

$$\#\{A \cup B\} = \frac{(m + 1)(n + 1)(s + 1) + (s + 1)(d(m, n) + 1)}{2},$$

(5)

since the number of lattice points on the hypotenuse $AB$ of the triangle $OAB$ is equal to $d(m, n) + 1$ (for details, see [1]).

![Figure 3](image)

The set

$$\left\{(x, y, z) \in \mathbb{Z}^3 : x \geqslant 0, y \geqslant 0, \frac{x}{m} + \frac{y}{n} = 1, \frac{x}{m} + \frac{z}{s} = 1 \right\}$$

is the set of lattice points of the line segment $AE$ (see Figure 1). The number of lattice points of this segment is equal to $d(m, n, s) + 1$. In fact, the set of integers $i$ between 0 and $m$ such that

$$(i, j, k) = \left(i, n - i \frac{n}{m}, s - i \frac{s}{m}\right) \in \mathbb{Z}^3$$

is

$$\left\{0, \frac{m}{d}, 2\frac{m}{d}, \ldots, (d - 1)\frac{m}{d}, m\right\},$$

where $d = d(m, n, s)$.

The set

$$A \cap B = \left\{(x, y, z) \in \mathbb{Z}^3 : x \geqslant 0, y \geqslant 0, \frac{x}{m} + \frac{y}{n} \leq 1, \frac{x}{m} + \frac{z}{s} = 1 \right\}$$
is the set of lattice points of the triangle $ACE$ (see Figure 3). Considering the triangle as a half of rectangle with the diagonal $AE$, we obtain

$$\# \{A \cap B\} = \frac{(n+1)(d(m,s)+1)+d(m,n,s)+1}{2}.$$  

(6)

On the other hand, the lattice points of the considered sets are connected with the equality

$$\# \{A \cup B\} = \# \{A\} + \# \{B\} - \# \{A \cap B\}.$$  

(7)

Collecting (5), (3), (4), and (6) into (7) and doing some elementary calculations, we get that

$$d(m,n,s) = 2 \sum_{i=0}^{m} \left( \left\lfloor \frac{n}{m} \right\rfloor + 1 \right) \left( \left\lfloor \frac{s}{m} \right\rfloor + 1 \right)$$

$$+ 2 \sum_{i=0}^{m} \left( \left\lfloor \frac{(m-i)n}{m} \right\rfloor + 1 \right) \left( \left\lfloor \frac{i}{m} \right\rfloor + 1 \right)$$

$$- (n+1)d(m,s) - (s+1)d(m,n)$$

$$- mns - mn - ms - ns - m - 2n - 2s - 4.$$  

(8)

Further, writing the expression (1) for $d(m,s)$ and $d(m,n)$ into (8) and performing required operations, we finally obtain that

$$d(m,n,s) = 2 \sum_{i=1}^{m-1} \left( \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{i}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{s}{m} \right\rfloor + \left\lfloor \frac{i}{m} \right\rfloor \left\lfloor \frac{m-i}{m} \right\rfloor \right)$$

$$+ \left( m - i \right) \frac{n}{m} \left( \left\lfloor \frac{i}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor \right)$$

$$- 2(n+1) \sum_{i=1}^{m-1} \left\lfloor \frac{i}{m} \right\rfloor - 2(s+1) \sum_{i=1}^{m-1} \left\lfloor \frac{i}{m} \right\rfloor$$

$$+ mns - mn - ms - ns + m + n + s$$

$$= 2 \sum_{i=1}^{m-1} \left( \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{i}{m} \right\rfloor + \left\lfloor \frac{(m-i)n}{m} \right\rfloor \left\lfloor \frac{i}{m} \right\rfloor \right)$$

$$+ (1-s) \left\lfloor \frac{n}{m} \right\rfloor + (1-n) \left\lfloor \frac{s}{m} \right\rfloor$$

$$+ mns - mn - ms - ns + m + n + s.$$  

References


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