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AN EXPLICIT FORMULA FOR THE GREATEST COMMON DIVISOR OF THREE INTEGERS

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Dedicated to Professor Antanas Laurinčikas on his 65th birthday

Abstract. One explicit formula for finding the greatest common divisor of three integers is presented. For the proof, a simple geometrical method is used.

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In [1], M. Polezzi presented simple geometrical proof of one explicit formula for finding the greatest common divisor of two integers. He proved the following

THEOREM 1. *Let $d(m,n)$ be the greatest common divisor of positive integers m and n. Then*

$$d(m,n) = 2\sum_{i=1}^{m-1} \left\lfloor i\,\frac{n}{m} \right\rfloor - mn + m + n. \tag{1}$$

We present the generalization of formula (1) for finding the greatest common divisor for three integers. The following assertion is the main in the paper.

THEOREM 2. *Let $d(m,n,s)$ be the greatest common divisor of positive integers m,n, and s. Then*

$$\begin{aligned} d(m,n,s) \;=\;& 2\sum_{i=1}^{m-1}\Big(\Big\lfloor i\,\frac{n}{m}\Big\rfloor\Big\lfloor i\,\frac{s}{m}\Big\rfloor+\Big\lfloor (m-i)\frac{n}{m}\Big\rfloor\Big\lfloor i\,\frac{s}{m}\Big\rfloor \\ &+(1-s)\Big\lfloor i\,\frac{n}{m}\Big\rfloor+(1-n)\Big\lfloor i\,\frac{s}{m}\Big\rfloor\Big) \\ &+mns-mn-ms-ns+m+n+s. \end{aligned} \tag{2}$$

Proof. The set

$$\mathbb{A}=\Big\{(x,y,z)\in\mathbb{Z}^3 : x\geqslant 0,\ y\geqslant 0,\ z\geqslant 0,\ \frac{x}{m}+\frac{y}{n}\leqslant 1,\ \frac{x}{m}+\frac{z}{s}\leqslant 1\Big\}$$

is the set of lattice points of the quadrangular pyramid $OBECA$ (see Figure 1). The number of points of this set when $x=i, i=0,1,\ldots,m$, is equal to

$$\Big(\Big\lfloor n\Big(1-\frac{i}{m}\Big)\Big\rfloor+1\Big)\Big(\Big\lfloor s\Big(1-\frac{i}{m}\Big)\Big\rfloor+1\Big).$$

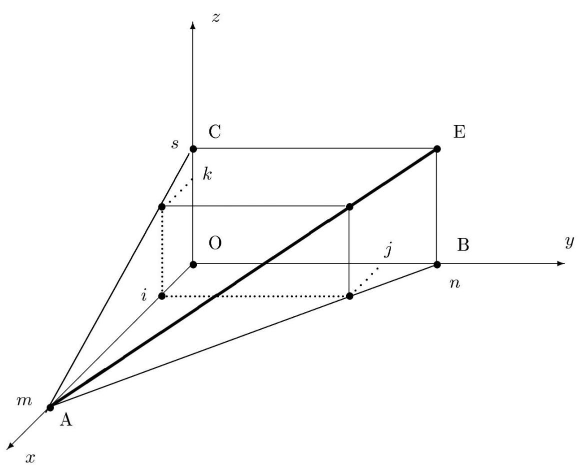

Figure 1

Therefore, we have

$$\#\{\mathbb{A}\} \;=\; \sum_{i=0}^{m}\Big(\Big\lfloor n\frac{m-i}{m}\Big\rfloor+1\Big)\Big(\Big\lfloor s\frac{m-i}{m}\Big\rfloor+1\Big)$$

$$= \sum_{i=0}^{m}\left(\left\lfloor n\frac{i}{m}\right\rfloor+1\right)\left(\left\lfloor s\frac{i}{m}\right\rfloor+1\right). \tag{3}$$

The set

$$\mathbb{B}=\left\{(x,y,z)\in\mathbb{Z}^3 : y\geqslant 0, z\leqslant c, \frac{x}{m}+\frac{y}{n}\leqslant 1, \frac{x}{m}+\frac{z}{s}\geqslant 1\right\}$$

is the set of lattice points of the triangular pyramid $ADCE$ (see Figure 2). The number of points of the set $\mathbb{B}$ when $x=i,\ i=0,1,\ldots,m$, is equal to

$$\begin{aligned}&\left(\left\lfloor n\left(1-\frac{i}{m}\right)\right\rfloor+1\right)\sum_{k=\lceil s(1-i/m)\rceil}^{s}1\\ &=\left(\left\lfloor n\,\frac{m-i}{m}\right\rfloor+1\right)\left(s-\left\lceil s-s\,\frac{i}{m}\right\rceil+1\right)\\ &=\left(\left\lfloor n\,\frac{m-i}{m}\right\rfloor+1\right)\left(\left\lfloor s\,\frac{i}{m}\right\rfloor+1\right).\end{aligned}$$

Thus,

$$\#\{\mathbb{B}\}=\sum_{i=0}^{m}\left(\left\lfloor n\frac{m-i}{m}\right\rfloor+1\right)\left(\left\lfloor s\frac{i}{m}\right\rfloor+1\right). \tag{4}$$

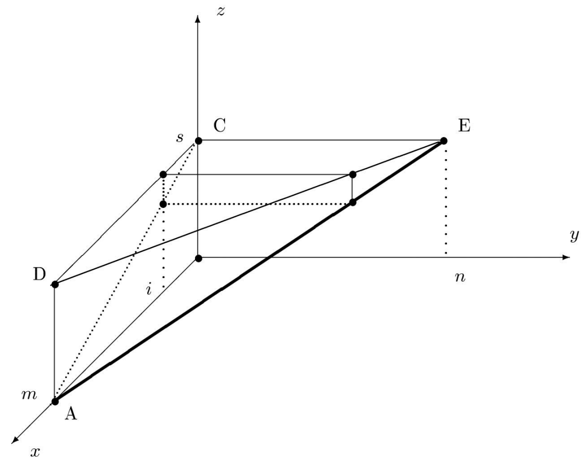

Figure 2

The set

$$\mathbb{A}\cup\mathbb{B}=\left\{(x,y,z)\in\mathbb{Z}^3 : x\geqslant 0, y\geqslant 0, 0\leqslant z\leqslant s, \frac{x}{m}+\frac{y}{n}\leqslant 1\right\}$$

is the set of lattice points of the triangular prism $OABCDE$ (see Figure 3). Considering the prism as a half of rectangular parallelepiped, we obtain that

$$\#\{\mathbb{A} \cup \mathbb{B}\} = \frac{(m+1)(n+1)(s+1) + (s+1)(d(m,n)+1)}{2}, \tag{5}$$

since the number of lattice points on the hypotenuse AB of the triangle OAB is equal to $d(m,n)+1$ (for details, see [1]).

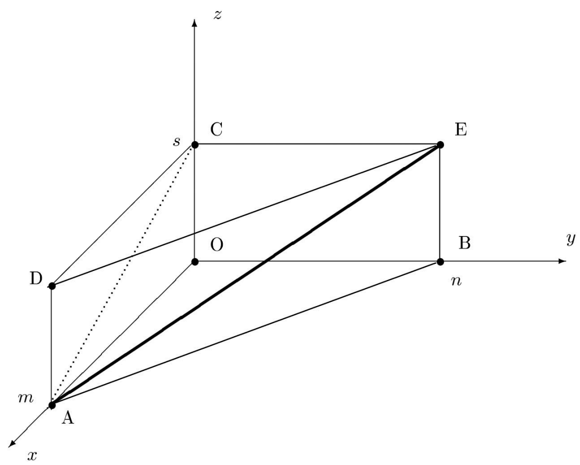

Figure 3

The set

$$\Big\{(x,y,z) \in \mathbb{Z}^3 : x \geqslant 0, y \geqslant 0, \frac{x}{m} + \frac{y}{n} = 1, \frac{x}{m} + \frac{z}{s} = 1\Big\}$$

is the set of lattice points of the line segment AE (see Figure 1). The number of lattice points of this segment is equal to $d(m,n,s)+1$. In fact, the set of integers i between 0 and m such that

$$(i,j,k) = \Big(i, n - i\frac{n}{m}, s - i\frac{s}{m}\Big) \in \mathbb{Z}^3$$

is

$$\Big\{0, \frac{m}{d}, 2\frac{m}{d}, \dots, (d-1)\frac{m}{d}, m\Big\},$$

where $d = d(m,n,s)$.

The set

$$\mathbb{A} \cap \mathbb{B} = \Big\{(x,y,z) \in \mathbb{Z}^3 : x \geqslant 0, y \geqslant 0, \frac{x}{m} + \frac{y}{n} \leqslant 1, \frac{x}{m} + \frac{z}{s} = 1\Big\}$$

is the set of lattice points of the triangle ACE (see Figure 3). Considering the triangle as a half of rectangle with the diagonal AE, we obtain

$$\#\{\mathbb{A}\cap\mathbb{B}\} = \frac{(n+1)(d(m,s)+1)+d(m,n,s)+1}{2}. \tag{6}$$

On the other hand, the lattice points of the considered sets are connected with the equality

$$\#\{\mathbb{A}\cup\mathbb{B}\} = \#\{\mathbb{A}\} + \#\{\mathbb{B}\} - \#\{\mathbb{A}\cap\mathbb{B}\}. \tag{7}$$

Collecting (5), (3), (4), and (6) into (7) and doing some elementary calculations, we get that

$$\begin{aligned} d(m,n,s) \;=\;& 2\sum_{i=0}^{m}\left(\left\lfloor i\frac{n}{m}\right\rfloor+1\right)\left(\left\lfloor i\frac{s}{m}\right\rfloor+1\right) \\ &+2\sum_{i=0}^{m}\left(\left\lfloor (m-i)\frac{n}{m}\right\rfloor+1\right)\left(\left\lfloor i\frac{s}{m}\right\rfloor+1\right) \\ &-(n+1)d(m,s)-(s+1)d(m,n) \\ &-mns-mn-ms-ns-m-2n-2s-4. \end{aligned} \tag{8}$$

Further, writing the expression (1) for $d(m,s)$ and $d(m,n)$ into (8) and performing required operations, we finally obtain that

$$\begin{aligned} d(m,n,s) \;=\;& 2\sum_{i=1}^{m-1}\Bigg(\left\lfloor i\frac{n}{m}\right\rfloor\left\lfloor i\frac{s}{m}\right\rfloor+\left\lfloor i\frac{n}{m}\right\rfloor+\left\lfloor i\frac{s}{m}\right\rfloor \\ &+\left\lfloor (m-i)\frac{n}{m}\right\rfloor\left\lfloor i\frac{s}{m}\right\rfloor+\left\lfloor (m-i)\frac{n}{m}\right\rfloor+\left\lfloor i\,\frac{s}{m}\right\rfloor\Bigg) \\ &-2(n+1)\sum_{i=1}^{m-1}\left\lfloor i\frac{s}{m}\right\rfloor-2(s+1)\sum_{i=1}^{m-1}\left\lfloor i\frac{n}{m}\right\rfloor \\ &+mns-mn-ms-ns+m+n+s \\ =\;& 2\sum_{i=1}^{m-1}\Bigg(\left\lfloor i\frac{n}{m}\right\rfloor\left\lfloor i\frac{s}{m}\right\rfloor+\left\lfloor (m-i)\frac{n}{m}\right\rfloor\left\lfloor i\frac{s}{m}\right\rfloor \\ &+(1-s)\left\lfloor i\frac{n}{m}\right\rfloor+(1-n)\left\lfloor i\,\frac{s}{m}\right\rfloor\Bigg) \\ &+mns-mn-ms-ns+m+n+s. \end{aligned}$$

References

[1] M. Polezzi, A geometrical method for finding an explicit formula for the greatest common divisor, *Am. Math. Mon.*, **104**(5), 445–446 (1997).

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