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ON THE ARITHMETIC TRIANGLES

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Abstract. We reconsider the concept of generalized arithmetic triangles and describe the sum of the elements lying along a finite ray in one type of generalized arithmetic triangles. These sums satisfy linear recurrence relations that can be explicitly determined. A view on the extensions of the regular Pascal triangle and binomial coefficients is also provided.

Key words and phrases: binomial coefficients, combinatorial identities, linear recurrences.

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1. Introduction

The aim of this paper is to reconsider some features connected to Pascal-type triangles concentrating mainly on the sum of elements lying along a finite ray.

The elements $\binom{n}{k}$ of the Pascal triangle represent the coefficients of $x^{n-k}y^k$, $0 \leqslant k \leqslant n$, in the expansion of the polynomial $(x+y)^n$, therefore, they are called binomial coefficients. It is known that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}, \quad 0 \leqslant k \leqslant n, \tag{1}$$

gives the number of ways of choosing k (or $n-k$) objects out of n distinguishable objects. Formula (1) or its combinatorial interpretation provides generalization possibilities of the Pascal triangle. Some of them are listed here.

• AB-based Pascal triangles (see, for instance, [3]). The structure is identical to the regular Pascal triangle. With given real numbers A and B, let $T_{AB}(n,k) = A^{n-k}B^k\binom{n}{k}$ denote the kth entry of the nth row, $0 \leqslant k \leqslant n$. Clearly, the elements $T_{AB}(n,k)$ are the coefficients of $x^{n-k}y^k$ in the expansion of the polynomial $(Ax+$

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$By)^n$. The addition rule has the form

$$T_{AB}(n,k) = B \cdot T_{AB}(n-1,k-1) + A \cdot T_{AB}(n-1,k).$$

• s-Pascal triangles. This type is also planar construction, and the elements $\binom{n}{k}_{\{s\}}$ have the following combinatorial interpretation. The term $\binom{n}{k}_{\{s\}}$ assigns the number of different ways of distributing k uniform objects among n boxes, where each box may contain at most s objects, $0 \leqslant k \leqslant sn$. In other words,

$$\binom{n}{k}_{\{s\}} = \#\left\{ f : \{0,\dots,n-1\} \to \{0,\dots,s\} \text{ such that } \sum_{i=0}^{n-1} f(i) = k \right\}.$$

• Pascal-pyramids. Let $r \geqslant 2$ denote an integer (the dimension), and consider the map $p:\ \mathbb{N}^r \to \mathbb{N}$,

$$(n_1,\dots,n_r) \mapsto \binom{n_1+\ldots+n_r}{n_1,\dots,n_r} = \frac{(n_1+\ldots+n_r)!}{n_1!\ldots n_r!}.$$

The map p provides the number of ways of splitting a set of $n = n_1+\ldots+n_r$ distinguishable objects into pairwise disjoint subsets S_i of cardinality n_i, $i=1,\dots,r$. When $r=2$, the map returns the usual binomial coefficients in the Pascal triangle.
• Arithmetic triangles. It is the term which was used for the original triangle by Pascal himself. Now, let the real sequences $\{a_n\}$ and $\{b_n\}$ be given with $a_0 = b_0$. Ensley [2] defined the object of generalized arithmetic triangle (GAT, in short) for $\{a_n\}$ and $\{b_n\}$ as follows. Let $G(n,0) = a_n$, $G(n,n) = b_n$ and

$$G(n,k) = G(n-1,k-1) + G(n-1,k) \quad \text{if} \quad 1 \leqslant k \leqslant n-1.$$

Ensley discussed some properties of GAT, especially he determined the entry $G(n,k)$ as a sum of certain weighted binomial coefficients. He also described a few particular GATs, for example, when $\{a_n\} = \{b_n\}$ is the Fibonacci sequence, etc. In Section 2 of this paper, we introduce a wider, more general approach of generalized arithmetic triangles.

After reviewing certain generalizations of the regular Pascal triangle, we turn our attention to the extensions of binomial coefficients (1) to the whole lattice of integer vectors. To analyze the different approaches for our purposes we go into details, but it worth noting that the paper of Sprugnoli [6] deals with the possibilities. A conventional extension of binomial coefficients to arbitrary integers n and k takes the following form (see e.g., [3] or [5])

$$\binom{n}{k}_K = \begin{cases} \frac{n(n-1)\ldots(n-k+1)}{k!} & \text{if} \quad k \geqslant 0, \\ 0 & \text{if} \quad k < 0. \end{cases}$$

In this paper, we also use a seemingly simpler extension given by

$$\binom{n}{k}_0 = \begin{cases} \frac{n(n-1)\ldots(n-k+1)}{k!} & \text{if} \quad 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Alternative generalization of binomial coefficients is obtained by the Gamma function as follows. Put

$$\binom{n}{k}_\Gamma = \lim_{\substack{n_1\to n\\ k_1\to k}} \frac{\Gamma(n_1+1)}{\Gamma(k_1+1)\Gamma(n_1-k_1+1)}.$$

Observe that the difference between the three definitions is signed also in the notation $\binom{n}{k}_K$, $\binom{n}{k}_0$ and $\binom{n}{k}_\Gamma$. Sometimes we simply use $\binom{n}{k}$ when we speak obviously about the ordinary binomial coefficients, i.e., $0\leqslant k\leqslant n$.

Note that the three approaches differ only for negative values of n. The first extension is widely used since it conserves Pascal's addition rule $\binom{n-1}{k-1}_K+\binom{n-1}{k}_K=\binom{n}{k}_K$, and therefore most binomial coefficients identities are henceforward true. But we lose the symmetry $\binom{n}{k}_K=\binom{n}{n-k}_K$ for negative integers n. When the upper index n is negative, then by $m=-n$ we get $\binom{-m}{k}_K=(-1)^k\binom{m+k-1}{k}_K$.

Unfortunately, more problems appear in the usage of the other two definitions, because the addition rule is failed at one point:

$$\binom{-1}{-1}_0+\binom{-1}{0}_0=0+0\neq 1=\binom{0}{0}_0,\quad \binom{-1}{-1}_\Gamma+\binom{-1}{0}_\Gamma=1+1\neq 1=\binom{0}{0}_\Gamma, \tag{2}$$

and it causes many collateral inconveniences in the applications of $\binom{n}{k}_0$ and $\binom{n}{k}_\Gamma$. Therefore, one must very carefully proceed if the upper index n is negative.

In this paper, we employ all the three extensions. Theorem 1 and its consequence use $\binom{n}{k}_\Gamma$, Lemma 2 has reference to $\binom{n}{k}_K$. Otherwise, we work with $\binom{n}{k}_0$.

2. GAT

Now we build up a sort of Generalized Arithmetic Triangle which is structurally identical with the regular Pascal triangle. It resembles Ensley's GAT [2], but here we allow $a_0\neq b_0$ in the generator sequences, further we also vary the rule of addition.

Let $\{a_n\}_{n=0}^\infty\in\mathbb{R}^\infty$ and $\{b_n\}_{n=0}^\infty\in\mathbb{R}^\infty$ be two real sequences called generator sequences, further let $A,B\in\mathbb{R}$. The new GAT contains also rows numbered by $0,1,2,\ldots$ such that the nth row possesses the elements $\left\langle {n \atop k}\right\rangle$ in the positions (say columns) $k=0,1,\ldots,n$ as follows.

Let $\left\langle {0 \atop 0}\right\rangle$ be arbitrary denoted by Ω, for positive integer n put $\left\langle {n \atop 0}\right\rangle := A^n a_n$ and $\left\langle {n \atop n}\right\rangle := B^n b_n$, further, for $n\geqslant 2$ and $1\leqslant k\leqslant n-1$, let

$$\left\langle {n \atop k}\right\rangle := B\left\langle {n-1 \atop k-1}\right\rangle + A\left\langle {n-1 \atop k}\right\rangle.$$

Clearly, a GAT depends only on the generator sequences $\{a_n\},\{b_n\}$ and on the real numbers A,B, therefore, sometimes we write $\left\langle {n \atop k}\right\rangle_{\{a_n\},\{b_n\},A,B}$ to emphasize the parameters of the triangle.

Since the definition of GAT is recursive, the aim is arising naturally to give a direct formula for the elements $\left\langle {n \atop k}\right\rangle$ by the terms of the sequences $\{a_n\},\{b_n\}$ and by the values A,B. Theorem 1 admits such a formula.

Theorem 1. *Given* $\{a_n\}, \{b_n\}, A, B$. *Then, for any positive integer* n *and for any non-negative integer* $k \leqslant n$, *we have*

$$\left\langle {n \atop k} \right\rangle = A^{n-k}B^k \left(\sum_{i=0}^{n-k-1} \binom{n-2-i}{k-1}_\Gamma a_{i+1} + \sum_{j=0}^{k-1} \binom{n-2-j}{k-1-j}_\Gamma b_{j+1} \right). \tag{3}$$

Proof. We apply the technique of induction, beginning with the verification of $\left\langle {n \atop 0} \right\rangle$ and $\left\langle {n \atop n} \right\rangle$, $n \geqslant 1$. Clearly, the theorem holds for the sides of the triangle since

$$\left\langle {n \atop 0} \right\rangle = A^n a_n, \quad \left\langle {n \atop n} \right\rangle = B^n b_n.$$

To justify the validity of formula (3) in the interior part of the GAT, suppose that the theorem is true if $n \leqslant N$, $N \geqslant 2$. Hence, for $0 \leqslant k \leqslant N-1$, we find that $\left\langle {N+1 \atop k+1} \right\rangle = B\left\langle {N \atop k} \right\rangle + A\left\langle {N \atop k+1} \right\rangle$ equals

$$\begin{aligned} &A^{N-k}B^{k+1} \\ &\times \left[\sum_{i=0}^{N-k-2} \left(\binom{N-2-i}{k-1}_\Gamma + \binom{N-2-i}{k}_\Gamma \right) a_{i+1} + \binom{k-1}{k-1}_\Gamma a_{N-k} \right. \\ &\quad \left. + \sum_{j=0}^{k-1} \left(\binom{N-2-j}{k-1-j}_\Gamma + \binom{N-2-j}{k-j}_\Gamma \right) b_{j+1} + \binom{N-2-k}{0}_\Gamma b_{k+1} \right], \end{aligned}$$

which coincides with

$$A^{N-k}B^{k+1} \left[\sum_{i=0}^{N-k-1} \binom{N-1-i}{k}_\Gamma a_{i+1} + \sum_{j=0}^{k} \binom{N-1-j}{k-j}_\Gamma b_{j+1} \right]$$

since there is no restriction of the type (2) in using the addition rule in the extension $\binom{\cdot}{\cdot}_\Gamma$.

Example 1. Let $\{a_n\} = \{b_n\} = \{F_n\}$, the Fibonacci sequence, further $A = B = 1$. Figure 1 shows the rows 0 through 4 of the GAT we obtain.

$$\begin{array}{ccccccccc} &&&& \Omega &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 2 && 3 && 3 && 2 & \\ 3 && 5 && 6 && 5 && 3 \\ \vdots && \vdots && \vdots && \vdots && \vdots \end{array}$$

Figure 1. GAT generated by the Fibonacci sequence

The reader can easily show that the sum s_n of the elements in nth row is $s_n = 2^{n+1} - 2F_{n+1}$ (see again [2]). Moreover, the recurrence relations $s_n = s_{n-1} + s_{n-2} + 2^{n-1}$ and $s_n = 3s_{n-1} - s_{n-2} - 2s_{n-3}$ hold for $n \geqslant 2$.

The ascending diagonal sum

$$d_n = \left\langle {n \atop 0} \right\rangle + \left\langle {n-1 \atop 1} \right\rangle + \left\langle {n-2 \atop 2} \right\rangle + \dots$$

satisfies the recursion formula $d_n = d_{n-1} + d_{n-2} + F_{n-2} + \delta_n$, where δ_n equals $F_{n/2-2}$ if n is even and vanishes otherwise.

In the sequel, the case when the sequences $\{a_n\}$ and $\{b_n\}$ both are constant sequences plays an important role. Put $a_n = G_0 \in \mathbb{R}$ and $b_n = G_1 \in \mathbb{R}$ for all $n \in \mathbb{N}$. Then, with $n \geqslant 1$ with the parameters G_0, G_1, A and B, we obtain

$$\begin{aligned}\left\langle {n \atop k} \right\rangle &= A^{n-k}B^k \left(\sum_{i=0}^{n-k-1} \binom{n-2-i}{k-1}_\Gamma G_0 + \sum_{j=0}^{k-1} \binom{n-2-j}{k-1-j}_\Gamma G_1 \right) \\ &= A^{n-k}B^k \left(G_0 \binom{n-1}{k}_\Gamma + G_1 \binom{n-1}{k-1}_\Gamma \right). \qquad (4)\end{aligned}$$

By the properties of the regular Pascal triangle, the latest equality is obviously true if $n \geqslant 1$ and $1 \leqslant k \leqslant n-1$. But it is easy to verify that $\sum_{i=0}^{n-k-1} \binom{n-2-i}{k-1}_\Gamma = \binom{n-1}{k}_\Gamma$ and $\sum_{j=0}^{k-1} \binom{n-2-j}{k-1-j}_\Gamma = \binom{n-1}{k-1}_\Gamma$ hold even if $n \geqslant 1$, $k = 0$ or $n \geqslant 1$, $k = n$. Hence the elements of a GAT, in this specific case, are generated by the linear combinations of two neighbor elements from the previous row of the given extension of Pascal triangle.

Observe that formula (4) contains binomial coefficients of type $\binom{n-1}{t}_\Gamma$, $n \geqslant 1$. Since the upper index $n-1$ is non-negative, therefore independently from the sign of the lower index t, we can switch to the second generalization of binomial coefficients (or equivalently, to the ordinary binomial coefficients). The reason of this turnout will be explained later. Hence, for $n \geqslant 1$, with G_0, G_1, A and B, we obtain

$$\begin{aligned}\left\langle {n \atop k} \right\rangle &= A^{n-k}B^k \left(G_0 \binom{n-1}{k}_0 + G_1 \binom{n-1}{k-1}_0 \right) \\ &= AG_0 \cdot A^{n-k-1}B^k \binom{n-1}{k}_0 + BG_1 \cdot A^{n-k}B^{k-1} \binom{n-1}{k-1}_0 \\ &= AG_0 \cdot A^{n-k-1}B^k \binom{n-1}{k} + BG_1 \cdot A^{n-k}B^{k-1} \binom{n-1}{k-1} \\ &= AG_0 \cdot T_{AB}(n-1,k) + BG_1 \cdot T_{AB}(n-1,k-1).\end{aligned}$$

where T_{AB}'s are from an AB-based Pascal triangle.

The four parameters G_0, G_1, A and B of GAT can allow us to define the real linear recurrence sequence $\{G_n\}_{n=0}^{\infty}$ by the initial values G_0 and G_1, and by the recurrence relation $G_n = AG_{n-1} + BG_{n-2}$, $n \geqslant 2$.

Example 2. Put $G_0 = 0$ and $G_1 = A = B = 1$, i.e., $\{G_n\}$ is the Fibonacci sequence. Thus,

$$\left\langle {n \atop k} \right\rangle_{0,1,1,1} = 0 \cdot \binom{n-1}{k}_0 + 1 \cdot \binom{n-1}{k-1}_0 = \binom{n-1}{k-1}_0 = \binom{n-1}{k-1},$$

i.e., it is a variation of the ordinary Pascal triangle (see Figure 2). Note that by changing $G_0 = 0$ to $G_0 = 1$, the construction returns with the usual Pascal triangle (see again Figure 2). Finally, put $G_0 = G_1 = 1$, and let A, B be arbitrary real numbers. Then $\left\langle {n \atop k} \right\rangle = A^{n-k}B^k\binom{n}{k}$ leads to an AB-based Pascal triangle.

$$
\begin{array}{ccccccccc}
 & & & & \Omega & & & & \\
 & & & 0 & & 1 & & & \\
 & & 0 & & 1 & & 1 & & \\
 & 0 & & 1 & & 2 & & 1 & \\
0 & & 1 & & 3 & & 3 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & \Omega & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Figure 2. $\mathrm{GAT}_{0,1,1,1}$ and $\mathrm{GAT}_{1,1,1,1}$

3. Sum of elements lying along a finite ray

The purpose is to determine the sum of elements lying along a finite ray in a GAT generated by constant sequences. Therefore, anywhere in this section we assume that the generator sequences are given by $a_n = G_0$ and $b_n = G_1$, $n \in \mathbb{N}$. A finite ray of the fixed GAT is given by one of its element $\langle {n \atop k} \rangle$ and by an integer vector (r, q). More precisely, the elements of a finite ray in the GAT form the finite sequence

$$\varrho_k = \left\langle {n-qk \atop m+rk} \right\rangle, \quad k = 0, 1, \ldots, \omega = \left\lfloor \frac{n-m}{r+q} \right\rfloor, \tag{5}$$

where $n, r \in \mathbb{N}^+$, $m \in \mathbb{N}$, $q \in \mathbb{Z}$ and the parameters satisfy the relations

$$0 < r + q \quad, m < r \quad, \quad m \leqslant n. \tag{6}$$

By conditions (6), any ray is uniquely identified, and it has at least one element. Observe that the range $0, 1, \ldots, \omega$ of k in (5) is a consequence of the necessary condition $n - qk \geqslant m + rk$. Obviously, the condition $n - qk \geqslant m + rk$ implies directly $n - qk \geqslant 1$ if $m \geqslant 1$ or $k \geqslant 1$. But in case of $m = 0$ and $k = 0$ we always have $\varrho_0 = \langle {n \atop 0} \rangle$ with $n \geqslant 1$.

Now let the vector (r, q) and the value m be fixed. Clearly, ω depends only on n, so we switch over the notation from ω to ω_n. Varying $n \geqslant m$ the sum of the elements lying along a given finite ray is

$$
\begin{aligned}
S_\varrho(n) &= \sum_{k=0}^{\omega_n} \varrho_k(n) = \sum_{k=0}^{\omega_n} \left\langle {n-qk \atop m+rk} \right\rangle \\
&= AG_0 \sum_{k=0}^{\omega_n} A^{n-m-1-(q+r)k} B^{m+rk} \binom{n-1-qk}{m+rk}_0
\end{aligned}
$$

$$+BG_1\sum_{k=0}^{\omega_n} A^{n-m-(q+r)k}B^{m+rk-1}\binom{n-1-qk}{m-1+rk}_0$$
$$= \sum_{\varepsilon=0}^{1} A^{1-\varepsilon}B^{\varepsilon}G_{\varepsilon}T^{(\varepsilon)}_{q,r,m}(n), \tag{7}$$

where

$$T^{(\varepsilon)}_{q,r,m}(n)=\sum_{k=0}^{\omega_n} A^{n-m-1+\varepsilon-(q+r)k}B^{m+rk-\varepsilon}\binom{n-1-qk}{m-\varepsilon+rk}_0.$$

Thus, to give $S_\varrho(n)$, it is sufficient to determine the sums $T^{(\varepsilon)}_{q,r,m}(n)$.

As a demonstration of (7), one can easily calculate the value $S_\varrho(1)$. First let $m=1$, which entails $\omega_1=0$ and then $k=0$. Thus, $T^{(0)}_{q,r,m}(1)=0$, $T^{(1)}_{q,r,m}(1)=1$, hence, $S_\varrho(1)=G_1B$.

If $m=0$, we distinguish two cases. Suppose first that $r+q>1$. Then $\omega_1=0$, $k=0$ hold again. Thus, $T^{(0)}_{q,r,m}(1)=1$, $T^{(1)}_{q,r,m}(1)=0$, subsequently $S_\varrho(1)=G_0A$. Finally, assume that $r+q=1$. Consequently, $\omega_1=1$, $k=0,1$ and $T^{(0)}_{q,r,m}(1)=1$, $T^{(1)}_{q,r,m}(1)=B^{r-1}$. Hence, $S_\varrho(1)=G_0A+G_1B^r$.

A further scrutiny revealed that if we vary n then both of $T^{(0)}_{q,r,m}(n)$ and $T^{(1)}_{q,r,m}(n)$ satisfy the same recurrence relation of order r if $q\leqslant 0$, and $r+q$ when q is positive (see Theorem 5). Studying these recurrences, first we present three lemmata, the last two are two versions of a useful statement playing a crucial role in determining the recurrence linked to $T^{(\varepsilon)}_{q,r,m}(n)$ and then to $S_\varrho(n)$.

LEMMA 2. *Given non-negative integers x, y and z satisfy $x\geqslant y$. Then*

$$\sum_{k=0}^{x}(-1)^k\binom{y}{k}_0\binom{x-k}{z}_0=\binom{x-y}{x-z}_0$$

holds.

Proof. With $t=0$ and with non-negative integers m and r, we quote the identity (23) of Section 1.2.6 from [5] to have

$$\sum_{k=0}^{r}(-1)^k\binom{s}{k}_K\binom{r-k}{m}_K=\binom{r-s}{r-m}_K.$$

Supposing $s\in\mathbb{N}$, replace r by x, s by y and m by z to get

$$\sum_{k=0}^{x}(-1)^k\binom{y}{k}_K\binom{x-k}{z}_K=\binom{x-y}{x-z}_K. \tag{8}$$

Finally, since the condition $x\geqslant y$ guarantees that the upper index of each binomial coefficient occurring in (8) are non-negative, we can substitute the binomial coefficients $\binom{\cdot}{\cdot}_K$ by $\binom{\cdot}{\cdot}_0$.

Lemma 3. *Let $a, b \in \mathbb{Z}$ and $r \in \mathbb{N}$. Then*

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}_0\binom{a-k}{b}_0 = \begin{cases} (-1)^{a-b}\binom{r-b-1}{a-b}_0 & \text{if} \quad 0 \leqslant b \leqslant a < r, \\ \binom{a-r}{b-r}_0 & \text{otherwise.} \end{cases} \tag{9}$$

Proof. I. First we treat the upper case, when $0 \leqslant b \leqslant a < r$. Let $\mathcal{S}_r$ denote the sum on the left hand side of (9).

If $a = b$ also holds then $a - k \leqslant b$, therefore, all but one summands are zero, i.e.,

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}_0\binom{a-k}{b}_0 = \binom{r}{0}_0\binom{a}{a}_0 = 1.$$

On the other hand, the right hand side of (9) returns with $\binom{r-a-1}{0}_0 = 1$ since here $r - a - 1 \geqslant 0$.

Hence, we may suppose $0 \leqslant b < a < r$, and then use the technique of induction on the positive integer variable r. One can easily see that the statement is true if $r = 2$. Assume now that

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}_0\binom{a-k}{b}_0 = (-1)^{a-b}\binom{r-b-1}{a-b}_0$$

holds for any $r \leqslant R$, where R is a positive integer. Clearly, $\mathcal{S}_{R+1}$ coincides with

$$\sum_{k=0}^{R}(-1)^k\binom{R}{k}_0\binom{a-k}{b}_0 - \sum_{j=0}^{R}(-1)^j\binom{R}{j}_0\binom{a-1-j}{b}_0.$$

Since $R > a > b \geqslant 0$, we can apply now the assumption of the induction or the result on the case $b = a - 1 < R$ to deduce

$$\begin{aligned} \mathcal{S}_{R+1} &= (-1)^{a-b}\binom{R-b-1}{a-b}_0 - (-1)^{a-b-1}\binom{R-b-1}{a-b-1}_0 \\ &= (-1)^{a-b}\left[\binom{R-b-1}{a-b}_0 + \binom{R-b-1}{a-b-1}_0\right] = (-1)^{a-b}\binom{R-b}{a-b}_0. \end{aligned}$$

At the latest equality we used the rule of addition, which is valid because $R-b \geqslant 2$, consequently, $R - b - 1 \neq -1$.

II. This section is split into four principal parts. The union of these parts collects all possible cases with the integers a, b and r, except $r > a \geqslant b \geqslant 0$ (which was previously handled in I).

1. First let $a > 0$, $b > 0$ and $r > 0$ with $a > b$. Note, that $a < r$ would provide the conditions occurred in I, therefore, here we assume $a \geqslant r$. A direct application of Lemma 2 provides

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}_0\binom{a-k}{b}_0 = \binom{a-r}{a-b}_0.$$

Since $a - r \geqslant 0$, by the symmetry property we obtain $\binom{a-r}{a-b}_0 = \binom{a-r}{b-r}_0$. This value is zero when $r > b$, further positive when $r \leqslant b$, especially we get 1 if $r = b$.

2. Both sides of (9) coincide with 0 if

$$a < 0 \text{ or } a = 0,\ r > 0,\ b \neq 0 \text{ or } a > 0,\ r > 0,\ b < 0 \text{ or } b > a > 0,\ r > 0.$$

Observe that the terms $\binom{a-k}{b}_0$ on the left hand side of (9) are trivially zero for all possible values of k.

3. The left hand side of (9) is $\binom{a}{b}_0$ if $a \geqslant r = 0$ or $a = b \geqslant r$. Clearly, the right hand side of (9) is so.

4. Let $a \geqslant r > 0$ and $b = 0$. Then we have

$$\sum_{k=0}^{r}(-1)^k \binom{r}{k}_0 \binom{a-k}{b}_0 = \sum_{k=0}^{r}(-1)^k \binom{r}{k}_0 \binom{a-k}{0}_0 = \sum_{k=0}^{r}(-1)^k \binom{r}{k}_0 = 0.$$

Obviously, $\binom{a-r}{0-r}_0$ is also zero.

A nice combinatorial proof to a particular case of Lemma 3 will be shown now. Since here the upper and lower indexes of the binomial coefficients are non-negative integers, we drop the subscripts of the binomial coefficients to indicate that this lemma is being played in the regular Pascal triangle.

LEMMA 4. *Let $a, b, r \in \mathbb{N}$ satisfy the relations $r \leqslant b \leqslant a$ and $r \leqslant a - b$. Then*

$$\sum_{k=0}^{r}(-1)^k \binom{r}{k} \binom{a-k}{b} = \binom{a-r}{b-r} \tag{10}$$

holds.

Proof. First rewrite (10) by $\alpha = a - r$, $\beta = b - r$, so we equivalently have

$$\sum_{k=0}^{r}(-1)^k \binom{r}{k} \binom{\alpha + r - k}{\beta + r} = \binom{\alpha}{\beta}.$$

Clearly, $\alpha \geqslant \beta$ are non-negative integers. Suppose now that we must choose β objects in all possible ways from a given set H containing α distinguishable objects. Before doing that, r new elements arrive at the set H. Therefore,

- first we choose $\beta + r$ elements from the new set having cardinality $\alpha + r$. It can be done in $\binom{\alpha+r}{\beta+r}$ different ways;
- then, by the sieve theorem, we exclude that cases when not all new elements are among the chosen $\beta + r$.

Let A_k, $k = 1, 2, \ldots, r$, denote the event when the kth new element is not chosen among the $\alpha + r$ objects. Hence, first $|\bar{A}_1 \bar{A}_2 \ldots \bar{A}_r| = \binom{\alpha}{\beta}$. On the other hand, by the sieve, we obtain

$$\begin{aligned}|\bar{A}_1\bar{A}_2\dots\bar{A}_r| &= \binom{\alpha+r}{\beta+r} - \sum_{1\leqslant k\leqslant r}|A_i| + \dots + (-1)^r|A_1A_2\dots A_r| = \\ &= \binom{\alpha+r}{\beta+r} - \binom{r}{1}\binom{\alpha+r-1}{\beta+r} + \dots + (-1)^r\binom{r}{r}\binom{\alpha}{\beta+r}.\end{aligned}$$

Now we turn to the principal theorem of the paper. Essentially, in the proof, we follow the method of Theorem 1 of [1]. But here, by ε, we insert a new term.

Theorem 5. *Let $n > \max\{r, r+q\}$. The terms of the sequence*

$$T^{(\varepsilon)}_{q,r,m}(n) = \sum_{k=0}^{\omega_n} A^{n-m-1+\varepsilon-(q+r)k} B^{m+rk-\varepsilon}\binom{n-1-qk}{m-\varepsilon+rk}_0$$

satisfy the linear recurrence relation

$$\begin{aligned}&T^{(\varepsilon)}_{q,r,m}(n) - A\binom{r}{1}_0 T^{(\varepsilon)}_{q,r,m}(n-1) + \dots + (-1)^r A^r\binom{r}{r}_0 T^{(\varepsilon)}_{q,r,m}(n-r) \\ &\quad = B^r T^{(\varepsilon)}_{q,r,m}(n-q-r). \qquad (11)\end{aligned}$$

Proof. Clearly, $n > \max\{r, r+q\}$ implies $n > m$. First transform the left hand side of (11) as follows. (Recall, that $\omega_{n-i} = \left\lfloor\frac{n-i-m}{r+q}\right\rfloor$.) Put $\mathcal{S} = \sum_{i=0}^{r}(-A)^i\binom{r}{i}_0 T^{(\varepsilon)}_{q,r,m}(n-i)$. Then

$$\begin{aligned}\mathcal{S} &= \sum_{i=0}^{r}(-A)^i\binom{r}{i}_0\sum_{k=0}^{\omega_{n-i}} A^{n-i-m+1-\varepsilon-(q+r)k}B^{m+rk-\varepsilon}\binom{n-i-1-qk}{m-\varepsilon+rk}_0 \\ &= \sum_{i=0}^{r}\sum_{k=0}^{\omega_{n-i}}(-A)^i A^{n-i-m+1-\varepsilon-(q+r)k}B^{m+rk-\varepsilon}\binom{r}{i}_0\binom{n-i-1-qk}{m-\varepsilon+rk}_0 \\ &= \sum_{k=0}^{\omega_n} A^{n-m+1-\varepsilon-(q+r)k}B^{m+rk-\varepsilon}\sum_{i=0}^{r}(-1)^i\binom{r}{i}_0\binom{n-i-1-qk}{m-\varepsilon+rk}_0. \qquad (12)\end{aligned}$$

In the last step, we extended the summation up to ω_n without changing the total sum, since $k > \omega_{n-i}$ implies $m-\varepsilon-rk > n-i-1-qk$, hence, $\binom{n-i-1-qk}{m-\varepsilon+rk}_0 = 0$ independently from the sign of $n-i-1-qk$.

To apply the second branch of Lemma 3 we must exclude the case $r > n-1-qk \geqslant m-\varepsilon+rk \geqslant 0$. If $k=0$, then $r > n-1 \geqslant m-\varepsilon \geqslant 0$ contradict to $n > r$. Similarly, $k=1$ admits $r > n-1-q \geqslant m-\varepsilon+r \geqslant 0$, which are impossible since $n > r+q$. Finally, $k \geqslant 2$ provides the inequalities $r > n-1-qk \geqslant m-\varepsilon+rk \geqslant m-\varepsilon+2r$, and we arrived at a contradiction by comparing the leftmost and rightmost terms. Therefore, we can employ the second branch of Lemma 3 to

(12). Thus,

$$
\begin{aligned}
\mathcal{S} &= \sum_{k=0}^{\omega_n} A^{n-m+1-\varepsilon-(q+r)k} B^{m+rk-\varepsilon} \binom{n-1-qk-r}{m-\varepsilon+rk-r}_0 \\
&= 0 + \sum_{k=1}^{\omega_n} A^{n-m+1-\varepsilon-(q+r)k} B^{m+rk-\varepsilon} \binom{n-1-qk-r}{m-\varepsilon+r(k-1)}_0 \\
&= \sum_{k=0}^{\omega_n-1} A^{n-m+1-\varepsilon-(q+r)(k+1)} B^{m+r(k+1)-\varepsilon} \binom{n-1-q(k+1)-r}{m-\varepsilon+rk}_0 \\
&= B^r \sum_{k=0}^{\omega_n-1} A^{(n-q-r)-m+1-\varepsilon-(q+r)k} B^{m+rk-\varepsilon} \binom{(n-q-r)-1-qk}{m-\varepsilon+rk}_0 .
\end{aligned}
$$

To complete the proof, one must clarify the upper bound of the summation. Clearly,

$$
\omega_{n-q-r} = \left\lfloor \frac{n-q-r-m}{r+q} \right\rfloor = \left\lfloor \frac{n-m}{r+q} - 1 \right\rfloor = \omega_n - 1.
$$

Subsequently,

$$
B^r \sum_{k=0}^{\omega_{n-q-r}} A^{(n-q-r)-m+1-\varepsilon-(q+r)k} B^{m+rk-\varepsilon} \binom{(n-q-r)-1-qk}{m-\varepsilon+rk}_0
$$

coincides with $B^r T^{(\varepsilon)}_{q,r,m}(n-q-r)$.

Corollary 6. *The sequence $S_\varrho(n)$ also satisfies the recurrence relation* (11) *since it is a linear combination of the sequences $T^{(\varepsilon)}_{q,r,m}(n)$, $\varepsilon = 0, 1$. The order of the recurrent sequence $S_\varrho(n)$ is r if q is negative and $r+q$ when q is non-negative.*

Remark. In the proof of Theorem 5, before the step (12) we exploit that the raise of the maximum of k from ω_{n-i} to ω_n has no influence on the total sum since the additional binomial coefficients $\binom{\cdot}{\cdot}_0$ are zero. As we are going to show, it would be false in case of $\binom{\cdot}{\cdot}_\Gamma$ or $\binom{\cdot}{\cdot}_K$. Indeed, by the extension, it can be occurred that the upper index of the binomial coefficients becomes negative meanwhile the lower index is still positive. Consequently, the extension can effect a change in the total sum.

For example, put $r = 1$, $q = 3$ and $m = 0$. Then the order of the linear recurrence is $r + q = 4$. The first element can be computed by (11) is $T^\varepsilon_{3,1,0}(5)$, where $\omega_5 = 1$. To express $T^\varepsilon_{3,1,0}(5)$, we need to determine $T^\varepsilon_{3,1,0}(1), \ldots, T^\varepsilon_{3,1,0}(4)$. Obviously, $\omega_4 = 1$, but $\omega_1 = \omega_2 = \omega_3 = 0$. Hence, the change of, for instance, ω_2 to ω_4 entails that instead of $\sum_{k=0}^{0}$ we have $\sum_{k=0}^{1}$. Put $\varepsilon = 0$. Then

$$
\sum_{k=0}^{1} A^{n-i-m+1-\varepsilon-(q+r)k} B^{m+rk-\varepsilon} \binom{n-i-1-qk}{m-\varepsilon+rk}_0 = A^3 \binom{1}{0}_0 + A^{-1} B \binom{-2}{1}_0
$$

gives A^3. But, $A^3\binom{1}{0}_K + A^{-1}B\binom{-2}{1}_K = A^3\binom{1}{0}_\Gamma + A^{-1}B\binom{-2}{1}_\Gamma = A^3 - 2A^{-1}B$ is different.

This example shows us that we cannot use $\binom{\cdot}{\cdot}_\Gamma$ or $\binom{\cdot}{\cdot}_K$ in Theorem 5 in harmony with the demand $\left\langle {n \atop k} \right\rangle = 0$ for $k < 0$ or $n < k$.

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