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ABSOLUTE VORONOI SUMMABILITY OF FOURIER INTEGRALS OF FUNCTIONS OF BOUNDED VARIATION

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Abstract. In the present paper, we obtain sufficient conditions imposed on the function $p(t)$, under which the Fourier integral of the function $f(t) \in L(-\infty, \infty)$ is absolutely summable by Voronoi method if the function $\Phi(t) = f(x+t) + f(x-t) - 2f(x)$ is of bounded variation on $(0; +\infty)$.

Key words and phrases: Cesaro method, Fourier integral, summability, function of bounded variation, Voronoi method.

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The functional summability of integrals has been considered by G.F. Voronoi in his work [5]. In the foreign mathematical literature, this method is known as Nerlund method. The history of the summation theory by Voronoi functional method and its applications to trigonometric Fourier integrals, fundamental objects of this theory and review of the achieved results are contained in [4].

Let $p(t)$ be an integrable function on the real semiaxis. If $f(u)$ is integrable in $(0, \infty)$, $S(u) = \int_0^u f(t)\mathrm{d}t$, $P(y) = \int_0^y p(t)\mathrm{d}t \neq 0$, and

$$\tau(y) = \frac{1}{P(y)} \int_0^y P(y-u) f(u)\mathrm{d}u = \frac{1}{P(y)} \int_0^y p(y-u) S(u)\mathrm{d}u,$$

then the integral $\int_0^\infty f(u)\mathrm{d}u$ is said to be absolutely summable by Voronoi method, or $|W, p(y)|$-summable, if $\int_0^\infty |\tau'(y)|\mathrm{d}y < \infty$.

For a function $f(t) \in L(-\infty, \infty)$, its Fourier integral is defined (see [6]) as

$$\frac{1}{\pi} \int_0^\infty \mathrm{d}u \int_{-\infty}^\infty f(t) \cos u(x-t)\mathrm{d}t = \int_0^\infty A(u,x)\mathrm{d}u.$$

We put

$$\phi(t) = \phi(x,t) = f(x+t) + f(x-t) - 2f(x).$$

The following theorem holds.

THEOREM 1. *Let $f(t) \in L(-\infty,\infty)$. If the function $\phi(t)$ is of bounded variation on $(0,\infty)$, the function $p(y)$ is positive non-increasing continuously differentiable such that $P(y) \to \infty$ as $y \to \infty$, and*

$$\int_y^\infty \frac{\mathrm{d}u}{uP(u)} \leqslant \frac{K}{P(y)} \quad \textit{for all} \quad y > 0, \tag{1}$$

then the Fourier integral of the function $f(t)$ is $|W,p(y)|$-summable.

To prove the theorem, we use the following lemma from [2].

LEMMA 1. *Let $p(u)$ be a non-negative, decreasing function defined on $[0,\infty)$, continuously differetiable in $[a,b]$, $0 \leqslant a < b \leqslant \infty$. Then, for any positive t and y,*

$$\left|\int\limits_0^y p(u)e^{i(y-u)t}du\right| \leqslant K \cdot P\left(\frac{1}{t}\right).$$

Proof of Theorem 1. We have

$$\begin{aligned}
\tau(y) &= \frac{1}{P(y)}\int_0^y P(y-u)A(u,x)\mathrm{d}u \\
&= \frac{1}{\pi P(y)}\int_0^y P(y-u)\mathrm{d}u\int_{-\infty}^{\infty} f(t)\cos u(x-t)\mathrm{d}t \\
&= \frac{1}{\pi P(y)}\int_0^y P(y-u)\mathrm{d}u\int_0^{\infty} [f(x+t)+f(x-t)]\cos ut\mathrm{d}t.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
&\int_0^\infty [f(x+t)+f(x-t)]\cos ut\mathrm{d}t \\
&= [f(x+t)+f(x-t)]\frac{\sin ut}{u}\Big|_0^\infty - \int_0^\infty \mathrm{d}\phi(t)\frac{\sin ut}{u} \\
&= -\int_0^\infty \mathrm{d}\phi(t)\frac{\sin ut}{u},
\end{aligned}$$

since $\lim_{t\to\pm\infty} f(t) = 0$ [3]. Then

$$\tau(y) = -\frac{1}{\pi P(y)}\int_0^y P(y-u)\mathrm{d}u\int_0^\infty \mathrm{d}\phi(t)\frac{\sin ut}{u},$$

and

$$\tau'(y) = \frac{p(y)}{\pi P^2(y)}\int_0^y P(y-u)\mathrm{d}u\int_0^\infty \mathrm{d}\phi(t)\frac{\sin ut}{u}$$

$$\begin{aligned}
& -\frac{1}{\pi P(y)}\int_0^y p(y-u)\mathrm{d}u\int_0^\infty \mathrm{d}\phi(t)\frac{\sin ut}{u}\\
= & \ \frac{1}{\pi}\int_0^y \frac{p(y)P(y-u)-p(y-u)P(y)}{P^2(y)}\mathrm{d}u\int_0^\infty \mathrm{d}\phi(t)\frac{\sin ut}{u}\\
= & \ \frac{1}{\pi}\int_0^\infty \mathrm{d}\phi(t)\int_0^y \frac{p(y)P(y-u)-p(y-u)P(y)}{P^2(y)}\mathrm{d}u.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \pi\int_0^\infty |\tau'(y)|\mathrm{d}y\\
& \leqslant \int_0^\infty |\mathrm{d}\phi(t)|\int_0^\infty \mathrm{d}y\left|\int_0^y \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\frac{\sin ut}{u}\mathrm{d}u\right|.
\end{aligned}$$

It is enough to prove that

$$I=\int_0^\infty \mathrm{d}y\left|\int_0^y \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\,\frac{\sin ut}{u}\mathrm{d}u\right|$$

is uniformly bounded in t.

Let's write I in the form

$$I=\int_0^{1/t}+\int_{1/t}^\infty=U+V.$$

Since $\frac{P(y-u)}{P(y)}\uparrow 1$ as $y\to\infty$ for each u, we have

$$\begin{aligned}
U & \leqslant t\int_0^{1/t}\mathrm{d}y\int_0^y \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\mathrm{d}u\\
& = t\int_0^{1/t}\mathrm{d}u\int_u^{1/t}\left(\frac{P(y-u)}{P(y)}\right)_y'\mathrm{d}y\\
& = t\int_0^{1/t}\left(\frac{P(1/t-u)}{P(1/t)}-\frac{P(0)}{P(1/t)}\right)\mathrm{d}u\\
& = t\int_0^{1/t}\frac{P(1/t-u)}{P(1/t)}\mathrm{d}u\leqslant K.
\end{aligned}$$

Next,

$$\begin{aligned}
V & = \int_{1/t}^\infty \mathrm{d}y\left|\int_0^y \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\,\frac{\sin ut}{u}\mathrm{d}u\right|\\
& \leqslant \int_{1/t}^\infty \mathrm{d}y\left|\int_0^{1/t} \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\,\frac{\sin ut}{u}\mathrm{d}u\right|\\
& \quad +\int_{1/t}^\infty \mathrm{d}y\left|\int_{1/t}^y \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\,\frac{\sin ut}{u}\mathrm{d}u\right|=W+X,
\end{aligned}$$

where

$$\begin{aligned} W &\leqslant t\int_{1/t}^{\infty} \mathrm{d}y \int_0^{1/t} \left(\frac{P(y-u)}{P(y)}\right)'_y \mathrm{d}u = t\int_0^{1/t} \mathrm{d}u \int_{1/t}^{\infty} \left(\frac{P(y-u)}{P(y)}\right)'_y \mathrm{d}y \\ &= t\int_0^{1/t} \left(1-\frac{P(1/t-u)}{P(1/t)}\right) \mathrm{d}u \leqslant K \end{aligned}$$

and

$$X = \int_{1/t}^{2/t} \mathrm{d}y \left|\int_{1/t}^{y}\right| + \int_{2/t}^{\infty} \mathrm{d}y \left|\int_{1/t}^{y/2}\right| + \int_{2/t}^{\infty} \mathrm{d}y \left|\int_{y/2}^{y}\right| = X' + Y + Z.$$

The boundedness of X' can be shown in the same way as that for W, i.e.,

$$\begin{aligned} X' &\leqslant t\int_{1/t}^{2/t} \mathrm{d}u \int_u^{2/t} \left(\frac{P(y-u)}{P(y)}\right)'_y \mathrm{d}y = t\int_{1/t}^{2/t} \frac{P(2/t-u)}{P(2/t)} \mathrm{d}u \\ &\leqslant Kt\left(\frac{2}{t}-\frac{1}{t}\right) = K. \end{aligned}$$

Integrating again by parts, we obtain

$$\begin{aligned} &\int_{1/t}^{y/2} \frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)} \frac{\sin ut}{u} \mathrm{d}u \\ &= -\frac{p(y-u)P(y)-p(y)P(y-u)}{uP^2(y)} \frac{\cos ut}{t}\bigg|_{1/t}^{y/2} \\ &\quad + \int_{1/t}^{y/2} \left(\frac{p(y-u)P(y)-p(y)P(y-u)}{uP^2(y)}\right)'_u \frac{\cos ut}{t} \mathrm{d}u \\ &= \frac{p(y-1/t)P(y)-p(y)P(y-1/t)}{P^2(y)} \cos 1 \\ &\quad -2\frac{p(y/2)P(y)-p(y)P(y/2)}{yP^2(y)} \frac{\cos(yt/2)}{t} \\ &\quad + \int_{1/t}^{y/2} \left(\frac{p(y-u)P(y)-p(y)P(y-u)}{uP^2(y)}\right)'_u \frac{\cos ut}{t} \mathrm{d}u, \end{aligned}$$

since

$$\begin{aligned} Y &\leqslant K\int_{2/t}^{\infty} \frac{p(y-1/t)P(y)-p(y)P(y-1/t)}{P^2(y)} \mathrm{d}y \\ &\quad + \int_{2/t}^{\infty} \mathrm{d}y \left|\int_{1/t}^{y/2} \left(\frac{p(y-u)P(y)-p(y)P(y-u)}{uP^2(y)}\right)'_u \frac{\cos ut}{t} \mathrm{d}u\right| \\ &\quad + \frac{K}{t}\int_{2/t}^{\infty} \left|\frac{p(y/2)P(y)-p(y)P(y/2)}{yP^2(y)} \cos(yt/2)\right| \mathrm{d}y = Y_1 + Y_2 + Y_3, \end{aligned}$$

where

$$Y_1 \leqslant K\int_{2/t}^{\infty} \left(\frac{P(y-1/t)}{P(y)}\right)'_y \mathrm{d}y = K\frac{P(y-1/t)}{P(y)}\bigg|_{2/t}^{\infty}$$

$$
\begin{aligned}
&\leqslant K + K\frac{P(1/t)}{P(2/t)} \leqslant K,\\
Y_3 &\leqslant \frac{K}{t}\left(\int_{2/t}^{\infty}\frac{P(y/2)}{yP(y)}\mathrm{d}y + \int_{2/t}^{\infty}\frac{P(y)}{yP(y)}\mathrm{d}y\right)\\
&\leqslant \frac{K}{t}\int_{2/t}^{\infty}\frac{\mathrm{d}y}{y^2} = -\frac{K}{t}\frac{1}{y}\bigg|_{2/t}^{\infty} \leqslant K,\\
Y_2 &\leqslant \int_{2/t}^{\infty}\mathrm{d}y\\
&\quad\times\left|\int_{1/t}^{y/2}\frac{(p(y)p(y-u)-p'(y-u)P(y))u-p(y-u)P(y)+p(y)P(y-u)}{u^2P^2(y)}\right.\\
&\quad\left.\times\frac{\cos ut}{t}\mathrm{d}u\right|\\
&\leqslant \frac{K}{t}\int_{2/t}^{\infty}\mathrm{d}y\int_{1/t}^{y/2}\frac{1}{u}\frac{p(y)p(y-u)-p'(y-u)P(y)}{P^2(y)}\mathrm{d}u\\
&\quad+\frac{K}{t}\int_{2/t}^{\infty}\mathrm{d}y\int_{1/t}^{\infty}\frac{1}{u^2}\frac{p(y-u)P(y)-p(y)P(y-u)}{P^2(y)}\mathrm{d}u\\
&= -\frac{K}{t}\int_{2/t}^{\infty}\frac{\mathrm{d}u}{u}\int_{2u}^{\infty}\left(\frac{p(y-u)}{P(y)}\right)_y'\mathrm{d}y\\
&\quad+\frac{K}{t}\int_{2/t}^{\infty}\frac{\mathrm{d}u}{u^2}\int_{2u}^{\infty}\left(\frac{P(y-u)}{P(y)}\right)_y'\mathrm{d}y\\
&= -\frac{K}{t}\int_{2/t}^{\infty}\frac{\mathrm{d}u}{u}\frac{p(y-u)}{P(y)}\bigg|_{2u}^{\infty} + \frac{K}{t}\int_{2/t}^{\infty}\frac{\mathrm{d}u}{u^2}\frac{P(y-u)}{P(y)}\bigg|_{2u}^{\infty}\\
&= \frac{K}{t}\int_{2/t}^{\infty}\frac{p(u)}{uP(2u)}\mathrm{d}u + \frac{K}{t}\int_{2/t}^{\infty}\left(1-\frac{P(u)}{P(2u)}\right)\frac{\mathrm{d}u}{u^2}\\
&\leqslant \frac{K}{t}\int_{2/t}^{\infty}\frac{1}{u^2}\mathrm{d}u \leqslant K.
\end{aligned}
$$

At last,

$$
\begin{aligned}
Z &= \int_{2/y}^{\infty}\mathrm{d}y\left|\int_{y/2}^{y}\frac{P(y)p(y-u)-p(y)P(y-u)}{P^2(y)}\cdot\frac{\sin ut}{u}\mathrm{d}u\right|\\
&\leqslant \int_{2/y}^{\infty}\frac{dy}{P(y)}\left|\int_{y/2}^{y}\frac{p(y-u)\sin ut}{u}\mathrm{d}u\right|\\
&\quad+\int_{2/t}^{\infty}\frac{p(y)\mathrm{d}y}{P^2(y)}\left|\int_{y/2}^{y}\frac{P(y-u)\sin ut}{u}\mathrm{d}u\right| = Z_1+Z_2.
\end{aligned}
$$

In view of Lemma 1,

$$
\left|\int_{y/2}^{y}\frac{p(y-u)\sin ut}{u}\mathrm{d}u\right|
$$

$$
\begin{aligned}
&\leqslant \left|\frac{1}{y}\int_0^y p(y-z)\sin zt\mathrm{d}z\right| + \frac{2}{y}\left|\int_0^{y/2} p(y-z)\sin zt\mathrm{d}z\right| \\
&\quad + \left|\int_{y/2}^y \frac{\mathrm{d}u}{u^2}\int_0^u p(y-z)\sin zt\mathrm{d}z\right| \\
&\leqslant \frac{1}{y}KP\left(\frac{1}{t}\right) + \int_{y/2}^y \frac{\mathrm{d}u}{u^2}P\left(\frac{1}{t}\right) \leqslant K\frac{P(1/t)}{y},
\end{aligned}
$$

whence, due to (1), we arrive at

$$
Z_1 \leqslant KP\Big(\frac{1}{t}\Big)\int_{2/t}^\infty \frac{\mathrm{d}y}{yP(y)} \leqslant K.
$$

Now, the integration by parts gives

$$
\begin{aligned}
&\left|\int_{y/2}^y \frac{P(y-u)\sin ut}{u}\mathrm{d}u\right| \\
&= \left|\frac{2P(y/2)\cos(yt/2)}{ty} + \int_{y/2}^y \frac{p(y-u)\cos ut}{ut}\mathrm{d}u + \int_{y/2}^y \frac{P(y-u)\cos ut}{u^2 t}\mathrm{d}u\right| \\
&\leqslant \frac{2P(y/2)}{ty} + \frac{2}{ty}\int_{y/2}^y p(y-u)\mathrm{d}u + \frac{P(y/2)}{ty} \leqslant K\frac{P(y/2)}{ty},
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
Z_2 &= \int_{2/t}^\infty \frac{p(y)\mathrm{d}y}{P^2(y)}\left|\int_0^y \frac{P(y-u)\sin ut}{u}\mathrm{d}u\right| \\
&\leqslant \frac{K}{t}\int_{2/t}^\infty \frac{p(y)P(y/2)}{yP^2(y)}\mathrm{d}y \leqslant \frac{K}{t}\int_{2/t}^\infty \frac{\mathrm{d}y}{y^2} \leqslant K.
\end{aligned}
$$

This completes the proof of the theorem.

The next theorem is contained in [1].

THEOREM 2. *Let the function $\phi(t)$ be of bounded variation on $(0,\infty)$, the function $p(y)$ be positive, differentiable and such that $P(y) \to \infty$ as $y \to \infty$, and the following conditions are fulfilled:*

- $\frac{yp(y)}{P(y)}$ *is of bounded variation on $(0,\infty)$;*
- *for all $y \geqslant 0$,*

$$
P(y)\int_y^\infty \frac{\mathrm{d}t}{tP(t)} \leqslant K. \tag{2}
$$

Then the Fourier integral of the function $f(t)$ is $|W, p(y)|$-summable.

As it is seen from the proof of Theorem 1, in the case where the function $p(t)$ is non-increasing, the condition (2) may be omitted.

In the particular situation, when $p(t) = \alpha t^{\alpha-1}$, $\alpha > 0$, the Voronoi method turns into the well-known Cesaro method (C,α), $\alpha > 0$.

The next assertion follows from Theorem 1.

Corollary 1. *Let $f(t) \in L(-\infty,\infty)$. If the function $\phi(t)$ is of bounded variation on $(0,\infty)$, then the Fourier integral of the function $f(t)$ is (C,α)-summable, $0 < \alpha < 1$.*

It is known that the absolute summability of an integral implies its ordinary one. This allows to apply the Voronoi means and Cesaro means of the Fourier integral in the approximation theory.

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