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APPLICATIONS OF THE BETA-TRANSFORM

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To Professor Hidenori Fujiwara on the occasion of his 65th birthday, with friendship and respect

Abstract. The importance of the Hecke gamma-transform in number theory cannot be overstated. Along with this, there has been equally effective applications of the method of beta-transform, or the binomial expansion, leading to the K-Bessel series. The purpose here is to elucidate the explicit and implicit use of the beta-transform in the transformation of the perturbed Dirichlet series and reveal the hidden structure as the Fourier-Bessel expansion. In the first instance, we shall deal with Stark's method along with a recent result of Murty-Sinha as a manifestation of the beta-transform. Later and in the main part, we shall resurrect the long-forgotten important work of Koshlyakov [10, 11, 12, 13] showing that Koshlyakov's formula for the perturbed Dedekind zeta-functions for a real quadratic field is in fact Lipschitz summation formula, and that for an imaginary quadratic field the K-Bessel function is intrinsic to the Hecke functional equation. Similarly, we shall elucidate Koshlyalov's σ-series in terms of Kelvin functions.

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1. Introduction

We begin by elucidating Stark's method of deriving the Dirichlet class number formula and the special values at negative integers of the Hurwitz zeta-function as a manifestation of the beta-transform. A remarkable feature is that there is no need to appeal to the functional equation. However, to derive Lerch's formula (2.9), we need to make a recourse to a formula of Ramanujan and Yoshimoto.

We shall also materialize the Murty-Sinha theorem (Theorem 2.1) when the original Dirichlet series are assumed to satisfy the functional equation, as their Fourier-Bessel expansion [8], which is equivalent to the functional equation.

Then we will discuss why the work in [10, 11, 12, 13] is dominated by the modified K-Bessel function. In particular, we shall elucidate [11, pp. 243–247] in this vein and show that Koshlyakov's formula for the perturbed Dedekind zeta-functions for a real quadratic field is in fact Lipschitz summation formula [15] (cf. also [14]) on one hand. On the other, in the imaginary quadratic case, Proposition 4.1 explains the situation where the K-Bessel function is intrinsic with the Hecke type functional equation (i.e., with the gamma factor $\Gamma(s)$).

At the end, we shall elucidate the nature of Koshlyakov's K-function in terms of the modified Kelvin functions.

2. Stark's method

2.1. Murty-Sinha theorem

One can find in the literature many works devoted to the "analytic continuation by Taylor expansion" of a class of zeta-functions. One direction is to appeal to the binomial expansion due to Wilton [25], e.g. [16, (4.6), p. 152], [17, 21], etc., and another is to apply the Euler transformation, e.g., [5, 6, 19] with differencing, which has been recently elucidated in [23]. The paper of Murty and Sinha [17] contains a general theorem which gives an analytic continuation for the perturbed Dirichlet series once an analytic continuation is known for the original Dirichlet series. We state their theorem and realize it in view of the binomial expansion, or the beta-transform.

THEOREM 2.1 ([17], Theorem 3.1). *If the Dirichlet series*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^s}$$

with abscissa σ_φ of absolute convergence admits a meromorphic continuation to the whole plane, then its perturbation

$$\varphi(s,a) = \sum_{n=0}^{\infty} \frac{\alpha_n}{(\lambda_n + a)^s}$$

also extends meromorphically to the whole plane, and the poles of $\varphi(s,a)$ are contained in the positive integral translates of poles of $\varphi(s)$.

The proof of the theorem relies on the "binomial principle of analytic continuation" stated as

$$\varphi(s,a) = \sum_{r=0}^{\infty} \binom{-s}{r} \varphi(s+r) a^r \tag{2.1}$$

for $0 < |a| < 1$.

We may interpret (2.1) as the beta-transform (cf. [3, p. 34–35]) (which turns out to be an analytic expression for the binomial expansion),

$$(1+x)^{-z} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z+s)\,\Gamma(-s)}{\Gamma(z)}\, x^s \mathrm{d}s = \frac{1}{\Gamma(z)} G^{1,1}_{1,1}\left(x^{-1} \,\middle|\, \begin{matrix} 1 \\ z \end{matrix}\right) \tag{2.2}$$

for $x > 0$, $-\xi < c < 0$. Here (c) denotes the vertical Bromwich path $s = c + it$ with $-\infty < t < \infty$, and $G^{1,1}_{1,1}$ is the Meijer G-function.

This beta-transform has been extensively used in the literature by many authors including Mellin, Barnes, Hardy, Berndt, Koshlyakov, Katsurada, Matsumoto et al., who referred to it as the Mellin-Barnes integral, which turns out to be the beta-function integral $H^{1,1}_{1,1}$, where H indicates the Fox H-function, to be introduced below.

For $\xi = \operatorname{Re} z > 1$, the Lipschitz-Lerch transcendent is defined by

$$L(\lambda, a, z) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2\pi i n\lambda}}{(n+a)^z}, \tag{2.3}$$

where $\lambda \in \mathbb{R}$ and $a \in \mathbb{C} \setminus (\mathbb{Z} \cup \{0\})$, but we usually restrict to $\operatorname{Re} a > 0$ (cf. e.g. [20, p. 122]). The Hurwitz zeta-function is a special case of (2.3) with $\lambda \in \mathbb{Z}$:

$$\zeta(z, a) = L(\lambda, a, z) = \sum_{m=0}^{\infty} (m+a)^{-z}$$

for $a > 0$.

We illustrate Theorem 2.1 by the following example.

Example 1. We apply (2.2) to the Lipschitz-Lerch transcendent and obtain the integral representation

$$L(\lambda, a+x, s) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} \phi(\lambda, a, s+w) x^w \mathrm{d}w. \tag{2.4}$$

This is valid for $1 - \sigma < c < 0$.

Moving the line of integration to the right (thereby using Stirling's formula to make sure that the horizontal integrals vanish) up to $\operatorname{Re} w = a_K$, where

$$K < a_K < K+1, \quad K \in \mathbb{N},$$

as a result we get

$$\begin{aligned} L(\lambda, a+x, s) &= \frac{1}{2\pi i}\int_{(a_K)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} L(\lambda, a, s+w) x^w \mathrm{d}w \\ &\quad + \frac{1}{\Gamma(s)} \sum_{k=0}^{K} \Gamma(s+k) \frac{(-1)^k}{k!} L(\lambda, a, s+k) x^k. \end{aligned} \tag{2.5}$$

The relation (2.5) holds true for $\sigma > -K$, and the integral is absolutely convergent. Thus, it gives an analytic continuation for $\phi(\lambda, a+x, s)$ in that domain. The sum in (2.5) is

$$\sum_{k=0}^{K} \binom{-s}{k} \phi(\lambda, a, s+k) x^k.$$

This is the Kth partial sum of the Taylor series around a.

Remark 1. If the original Dirichlet series satisfies the functional equation, then the Murty-Sinha theorem amounts to the correspondence $H_{1,1}^{1,1}$ and its counterpart. Our Theorems 3.2, 4.2, and 5.1 illustrate this principle.

The Taylor expansion [20, p. 125] (for $|a| > 0$) of L which is due to Klusch [9] is given by

$$L(x, s, a+\xi) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(s)_\nu}{\nu!} L(x, s+\nu, a) \xi^{\nu} \tag{2.6}$$

in $|\xi| < |a|$, where $(s)_\nu = s(s+1)\dots(s+\nu-1)$ denotes the Pochhammber symbol. Equation (2.6) with $x \in \mathbb{Z}$ reduces to Wilton's formula [25]

$$\zeta(s, a+\xi) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(s)_\nu}{\nu!} \zeta(s+\nu, a)(\xi)^{\nu}. \tag{2.7}$$

2.2. Stark's method

We consider the special case $\zeta(s,x) = L(1,x,s)$ of the Hurwitz zeta-function. To apply (2.2), we need to separate the term with $n = 0$ and apply it with $\frac{x}{n}$. Then, correspondingly to (2.4), we get

$$\zeta(s,x) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} \zeta(s+w) x^w \mathrm{d}w$$

valid for $1 - c < \sigma$ with $c < 0$. Correspondingly to (2.5), one gets

$$\begin{aligned} \zeta(s,x) &= x^{-s} + \frac{1}{2\pi i} \int_{(a_K)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} \zeta(s+w) x^w \mathrm{d}w \\ &\quad + \sum_{k=0}^{K} \binom{-s}{k} \zeta(s+k) x^k \end{aligned} \tag{2.8}$$

which is valid for $\sigma > -K$ and the integral is absolutely convergent, and thus, gives an analytic continuation for $\zeta(s,x)$ in that domain. If we move the line of integration far to the right as in the proof of Example 2.1, we obtain the Taylor expansion (2.7) with $a = 1$.

Stark [21] uses only first a few terms of the expansion to deduce the Dirichlet class number formula and the special values of the relevant zeta and L-functions.

The Dirichlet L-function can be written in terms of Hurwitz zeta-function

$$L(s,\chi) = \frac{1}{q^s}\sum_{a=1}^{q-1}\chi(a)\zeta\left(s,\frac{a}{q}\right),$$

where χ is a Dirichlet character mod q. Thus, it suffices to give closed formulas for $\zeta(s,x)$. We now have (similar to [21, (13)]), correspondingly to (2.5),

$$\sum_{n=1}^{\infty}\left((n+x)^{-s} - \sum_{j=0}^{K}\binom{-s}{k} n^{-n-j}x^j\right) = \frac{1}{2\pi i}\int_{(a_K)}\frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)}\zeta(s+w)x^w\mathrm{d}w.$$

If one follows Stark's method more closely, write $K+1$ for K in (2.5) and set $s=-K$ with $0 \leqslant K \in \mathbb{Z}$ in (2.5), then one can deduce

$$\zeta(-K,x) = x^K + \sum_{k=0}^{K+1}\binom{K}{k}\zeta(-K+k)x^k$$

with the understanding that the term with $k = K+1$ is

$$\binom{-s}{K+1}\zeta(-s+K+1)\Big|_{s=-K}x^{K+1} = \frac{(-s)\dots(-s-K)\zeta(-s+K+1)\Big|_{s=-K}}{(K+1)!}x^{K+1} = \frac{x^{K+1}}{K+1}$$

on using the fact that $\operatorname{Res}\zeta(s) = 1$, and the term with $k=K$ is $\zeta(0)x^K$.

PROPOSITION 2.2. *We have, for any* $0 \leqslant K \in \mathbb{Z}$,

$$\zeta(-K,x) = \frac{x^{K+1}}{K+1} + \Big(1+\zeta(0)\Big)x^K + \sum_{k=0}^{K-1}\binom{K}{k}\zeta(-K+k)x^k.$$

In particular, (putting $\zeta(0) = -\frac{1}{2}$*)*

$$\zeta(0,x) = \frac{1}{2} - x = -B_1(x),$$

where $B_1(x)$ *is the first Bernoulli polynomial.*

Proof. Putting $s=-K$ in (2.8) and noting that $\frac{1}{\Gamma(-K)} = 0$ for $0 \leqslant K \in \mathbb{Z}$, the proof follows.

If we use $\zeta(-n) = \frac{B_{n+1}}{n+1}$ for $0 \leqslant n \in \mathbb{Z}$, we rediscover the well-known formula

$$\zeta(-K, x) = -\frac{B_{K+1}(x)}{K+1}.$$

Now we give a proof of Lerch's formula starting from the beta-transform.

Proposition 2.3. (Lerch's formula)

$$\zeta'(0, x) = \log \frac{\Gamma(x)}{\sqrt{2\pi}}. \tag{2.9}$$

Proof. We use the integral representation (2.8) with $K = 1$ which reads

$$\begin{aligned} \zeta(s, x) &= x^{-s} + \frac{1}{2\pi i} \int_{(a_K)} \frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)} \zeta(s+w) x^w \mathrm{d}w \\ &\quad + \zeta(s) - s\zeta(s+1)x. \end{aligned} \tag{2.10}$$

Differentiating and putting $s = 0$ in (2.10), we find, similarly as in Stark's method, that those terms, except for the integral, say I, becomes

$$\zeta'(0, x) = \log x + \zeta'(0) + \gamma x + \frac{\mathrm{d}}{\mathrm{d}t} I\big|_{s=0}, \tag{2.11}$$

where γ indicates the Euler constant. Clearly, only the term with $-\frac{\Gamma'(s)}{\Gamma^2(s)} = -\frac{s\Gamma'(s) - \Gamma(s+1)}{\Gamma^2(s+1)}$ counts, other terms vanishing in view of $\frac{1}{\Gamma(0)} = 0$. Therefore, we only need to evaluate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I\big|_{s=0} &= \frac{1}{2\pi i} \int_{(a_K)} \Gamma(-w)\Gamma(w)\zeta(w) x^w \mathrm{d}w \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{(a_K)} \frac{\pi}{w \sin \pi w} \left(\frac{x}{n}\right)^w \mathrm{d}w \end{aligned}$$

with $1 < a_K < 2$. Moving the line of integration far to the left, counting the residues and noting that the resulting integral tends to 0, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} I\big|_{s=0} = \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k} \left(-\frac{x}{n}\right)^k = \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} (-x)^k. \tag{2.12}$$

Now we recall the formula [7, Theorem 3.2, p. 62] (also [20, (2), p. 159]) of Ramanujan and Yoshimoto with $\alpha = 1$, which reads

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k} (-x)^k = \log \Gamma(x+1) + \gamma x, \quad |x| < 1.$$

Thus, with this, the equation (2.12) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} I\big|_{s=0} = \log \Gamma(x+1) - \gamma x.$$

Substituting this in (2.11) and appealing to the difference equation $\log \Gamma(x+1) = \log \Gamma(x) + \log x$ for $x > 0$, we conclude (2.9).

3. The main formula for modular relations

In this section, we state a special case of the main formula in [8] and some of applications. This will be of use in the forthcoming sections.

Consider two Dirichlet series $\{\phi_h(s)\}$, $1 \leqslant h \leqslant H$, and $\{\psi_i(s)\}$, $1 \leqslant i \leqslant I$, that satisfy the generalized functional equation (3.2) in the following sense.

Let $\{\lambda_k^{(h)}\}_{k=1}^{\infty}$, $\{\mu_k^{(i)}\}_{k=1}^{\infty}$ denote increasing sequences of positive terms, and let $\{\alpha_k^{(h)}\}_{k=1}^{\infty}$, $\{\beta_k^{(i)}\}_{k=1}^{\infty}$ be complex sequences. Consider

$$\varphi_h(s) = \sum_{k=1}^{\infty} \frac{\alpha_k^{(h)}}{\lambda_k^{(h)s}} \quad \text{and} \quad \psi_i(s) = \sum_{k=1}^{\infty} \frac{\beta_k^{(i)}}{\mu_k^{(i)s}},$$

with finite abscissas of absolute convergence σ_{φ_h} and σ_{ψ_i}, respectively.

For the set of coefficients

$$\Delta = \begin{pmatrix} \{(a_j, A_j)\}_{j=1}^{n}; \{(1-a_j, A_j)\}_{j=n+1}^{p} \\ \{(b_j, B_j)\}_{j=1}^{m}; \{(1-b_j, B_j)\}_{j=m+1}^{q} \end{pmatrix},$$

we introduce the (processing) gamma factor

$$\Gamma(w \mid \Delta) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j w) \prod_{j=1}^{n} \Gamma(a_j - A_j w)}{\prod_{j=n+1}^{p} \Gamma(a_j + A_j w) \prod_{j=m+1}^{q} \Gamma(b_j - B_j w)}, \quad A_j,\ B_j > 0.$$

The H-function is defined by

$$\begin{aligned} &H_{p,q}^{m,n}\left(z \middle| \Delta\right) \\ &= H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (1-a_1, A_1), \ldots, (1-a_n, A_n), (a_{n+1}, A_{n+1}), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_m, B_m), (1-b_{m+1}, B_{m+1}), \ldots, (1-b_q, B_q) \end{matrix}\right) \\ &= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(a_j - A_j s)}{\prod_{j=n+1}^{p} \Gamma(a_j + A_j s) \prod_{m+1}^{q} \Gamma(b_j - B_j s)} z^{-s} \mathrm{d}s, \end{aligned} \tag{3.1}$$

where the path L is subject to the poles separation conditions similar to the ones given below, and $0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant q$, $A_j, B_j > 0$.

Let there exist a meromorphic function $\chi(s)$, which satisfies (for a real number r) the functional equation

$$\chi(s) = \begin{cases} \displaystyle\sum_{h=1}^{H} \frac{\prod_{j=1}^{M^{(h)}} \Gamma\big(d_j^{(h)} + D_j^{(h)} s\big) \prod_{j=1}^{N^{(h)}} \Gamma\big(c_j^{(h)} - C_j^{(h)} s\big)}{\prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\big(c_j^{(h)} + C_j^{(h)} s\big) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\big(d_j^{(h)} - D_j^{(h)} s\big)} \varphi_h(s), \\ \qquad\qquad \mathrm{Re}(s) > \max_{1 \leqslant h \leqslant H} \big(\sigma_{\varphi_h}\big) \\ \displaystyle\sum_{i=1}^{I} \frac{\prod_{j=1}^{\tilde{N}^{(i)}} \Gamma\big(e_j^{(i)} + E_j^{(i)} (r-s)\big) \prod_{j=1}^{\tilde{M}^{(i)}} \Gamma\big(f_j^{(i)} - F_j^{(i)} (r-s)\big)}{\prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}} \Gamma\big(f_j^{(i)} + F_j^{(i)} (r-s)\big) \prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}} \Gamma\big(e_j^{(i)} - E_j^{(i)} (r-s)\big)} \psi_i(r-s), \\ \qquad\qquad \mathrm{Re}(s) < \min_{1 \leqslant i \leqslant I} \big(r - \sigma_{\psi_i}\big), \end{cases} \tag{3.2}$$

with $C_j^{(h)}, D_j^{(h)}, E_j^{(i)}, F_j^{(i)} > 0$.

We further assume that only finitely many of the poles s_k, $1 \leqslant k \leqslant L$, of $\chi(s)$ are neither a pole of

$$\frac{\prod_{j=1}^{N^{(h)}} \Gamma\Big(c_j^{(h)} - C_j^{(h)} s\Big)}{\prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\Big(c_j^{(h)} + C_j^{(h)} s\Big) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\Big(d_j^{(h)} - D_j^{(h)} s\Big)},$$

nor a pole of

$$\frac{\prod_{j=1}^{\tilde{M}^{(i)}} \Gamma\Big(f_j^{(i)} - F_j^{(i)} r + F_j^{(i)} s\Big)}{\prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}} \Gamma\Big(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} s\Big) \prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}} \Gamma\Big(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} s\Big)}.$$

We also assume the required growth condition (for any real numbers u_1, u_2, $u_1 < u_2$)

$$\lim_{|v| \to \infty} \Gamma(u + iv - s \,|\, \Delta)\, \chi(u + iv) = 0$$

uniformly in $u_1 \leqslant u \leqslant u_2$.

We choose $L_1(s)$ so that the poles of

$$\frac{\prod_{j=1}^{n} \Gamma(a_j + A_j s - A_j w) \prod_{j=1}^{N^{(h)}} \Gamma\Big(c_j^{(h)} - C_j^{(h)} w\Big)}{\prod_{j=n+1}^{p} \Gamma(a_j - A_j s + A_j w) \prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\Big(c_j^{(h)} + C_j^{(h)} w\Big)} \times \frac{1}{\prod_{j=m+1}^{q} \Gamma(b_j + B_j s - B_j w) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\Big(d_j^{(h)} - D_j^{(h)} w\Big)}$$

lie on the right of $L_1(s)$, and those of

$$\frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s + B_j w) \prod_{j=1}^{M^{(h)}} \Gamma\Big(d_j^{(h)} + D_j^{(h)} w\Big)}{\prod_{j=m+1}^{q} \Gamma(b_j + B_j s - B_j w) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\Big(d_j^{(h)} - D_j^{(h)} w\Big)} \times \frac{1}{\prod_{j=n+1}^{p} \Gamma(a_j - A_j s + A_j w) \prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\Big(c_j^{(h)} + C_j^{(h)} w\Big)}$$

lie on the left of $L_1(s)$, and choose $L_2(s)$ so that the poles of

$$\frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s + B_j w) \prod_{j=1}^{\tilde{M}^{(i)}} \Gamma\Big(f_j^{(i)} - F_j^{(i)} r + F_j^{(i)} w\Big)}{\prod_{j=m+1}^{q} \Gamma(b_j + B_j s - B_j w) \prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}} \Gamma\Big(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} w\Big)} \times \frac{1}{\prod_{j=n+1}^{p} \Gamma(a_j - A_j s + A_j w) \prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}} \Gamma\Big(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} w\Big)}$$

lie on the left of $L_2(s)$, and those of

$$\frac{\prod_{j=1}^{n}\Gamma(a_j+A_js-A_jw)\prod_{j=1}^{\tilde{N}^{(i)}}\Gamma\Big(e_j^{(i)}+E_j^{(i)}r-E_j^{(i)}w\Big)}{\prod_{j=n+1}^{p}\Gamma(a_j-A_js+A_jw)\prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}}\Gamma\Big(e_j^{(i)}-E_j^{(i)}r+E_j^{(i)}w\Big)} \times \frac{1}{\prod\limits_{j=m+1}^{q}\Gamma(b_j+B_js-B_jw)\prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}}\Gamma\Big(f_j^{(i)}+F_j^{(i)}r-F_j^{(i)}w\Big)}$$

lie on the right of $L_2(s)$. Further, they squeeze a compact set $\mathcal{S}$ such that $s_k \in \mathcal{S}$, $1 \leqslant k \leqslant L$.

Under these conditions the χ-function, *key-function*, $\mathrm{X}(z,s\,|\,\Delta)$ is defined as

$$\mathrm{X}(z,s\,|\,\Delta)=\frac{1}{2\pi i}\int_{L_1(s)}\Gamma(w-s\,|\,\Delta)\chi(w)\,z^{-(w-s)}\mathrm{d}w.$$

Then we have the following modular relation (due to the third author [22]).

Theorem 3.1.

$$\mathrm{X}(z,s\,|\,\Delta) = \begin{cases} \displaystyle\sum_{h=1}^{H}\sum_{k=1}^{\infty}\frac{\alpha_k^{(h)}}{\lambda_k^{(h)s}}H^{m+M^{(h)},n+N^{(h)}}_{p+P^{(h)},q+Q^{(h)}}\left(z\lambda_k^{(h)}\,\middle|\,\begin{matrix} a_{11}, & a_{12}\\ a_{21}, & a_{22}\end{matrix}\right), \\ \qquad \textit{if } L_1(s) \textit{ can be taken to the right of } \max\limits_{1\leqslant h\leqslant H}(\sigma_{\varphi_h}) \\ \displaystyle\sum_{i=1}^{I}\sum_{k=1}^{\infty}\frac{\beta_k^{(i)}}{\mu_k^{(i)r-s}}H^{n+\tilde{N}^{(i)},m+\tilde{M}^{(i)}}_{q+\tilde{Q}^{(i)},p+\tilde{P}^{(i)}}\left(\frac{\mu_k^{(i)}}{z}\,\middle|\,\begin{matrix} b_{11}, & b_{12}\\ b_{21}, & b_{22}\end{matrix}\right) \\ \qquad +\displaystyle\sum_{k=1}^{L}\mathrm{Re}\,s\Big(\Gamma(w-s\,|\,\Delta)\chi(w)\,z^{s-w}, w=s_k\Big) \\ \qquad \textit{if } L_2(s) \textit{ can be taken to the left of } \min\limits_{1\leqslant i\leqslant I}(r-\sigma_{\psi_i}), \end{cases} \tag{3.3}$$

is equivalent to the functional equation (3.2), *where*

$$\begin{aligned}
a_{11} &= \{(1-a_j,A_j)\}_{j=1}^{n}, \{(1-c_j^{(h)}+C_j^{(h)}s, C_j^{(h)})\}_{j=1}^{N^{(h)}},\\
a_{12} &= \{(a_j,A_j)\}_{j=n+1}^{p}, \{(c_j^{(h)}+C_j^{(h)}s, C_j^{(h)})\}_{j=N^{(h)}+1}^{P^{(h)}},\\
a_{21} &= \{(b_j,B_j)\}_{j=1}^{m}, \{(d_j^{(h)}+D_j^{(h)}s, D_j^{(h)})\}_{j=1}^{M^{(h)}},\\
a_{22} &= \{(1-b_j,B_j)\}_{j=m+1}^{q}, \{(1-d_j^{(h)}+D_j^{(h)}s, D_j^{(h)})\}_{j=M^{(h)}+1}^{Q^{(h)}},\\
b_{11} &= \{(1-b_j,B_j)\}_{j=1}^{m}, \{(1-f_j^{(i)}+F_j^{(i)}(r-s), F_j^{(i)})\}_{j=1}^{\tilde{M}^{(i)}},\\
b_{12} &= \{(b_j,B_j)\}_{j=m+1}^{q}, \{(f_j^{(i)}+F_j^{(i)}(r-s), F_j^{(i)})\}_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}},\\
b_{21} &= \{(a_j,A_j)\}_{j=1}^{n}, \{(e_j^{(i)}+E_j^{(i)}(r-s), E_j^{(i)})\}_{j=1}^{\tilde{N}^{(i)}},\\
b_{22} &= \{(1-a_j,A_j)\}_{j=n+1}^{p}, \{(1-e_j^{(i)}+E_j^{(i)}(r-s), E_j^{(i)})\}_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}}.
\end{aligned}$$

We now apply the modular relation (3.3) in the following special case.

values	proc. gamma	LHSFE	RHSFE	values
$m+M^{(h)}=1$	$m=0$	$M^{(h)}=1$	$M^{(i)}=0$	$m+\tilde{M}^{(i)}=0$
$n+N^{(h)}=1$	$n=1$	$N^{(h)}=0$	$N^{(i)}=1$	$n+\tilde{N}^{(i)}=2$
$p+P^{(h)}=1$	$p=1$	$P^{(h)}=0$	$P^{(i)}=1$	$p+\tilde{P}^{(i)}=2$
$q+Q^{(h)}=1$	$q=0$	$Q^{(h)}=1$	$Q^{(i)}=0$	$q+\tilde{Q}^{(i)}=0$

Table 0. Choice of the parameters

THEOREM 3.2. *The functional equation*

$$\Gamma(d_1+D_1 s)\varphi(s)=\Gamma(e_1+E_1(r-s))\psi(r-s),$$

with $\Gamma(s|\Delta)=\Gamma(a_1-A_1 s)$, *is equivalent to*

$$z^s\,\mathrm{X}(z,s\,|\,\Delta) = \begin{cases} \displaystyle\sum_{k=1}^{\infty}\frac{\alpha_k}{\lambda_k^s}H_{1,1}^{1,1}\left(z\lambda_k \,\middle|\, \begin{matrix}(1-a_1,A_1)\\ (d_1+D_1 s,D_1)\end{matrix}\right) \\ \qquad \textit{if } L_1(s) \textit{ can be taken to the right of } \max\limits_{1\leqslant h\leqslant H}(\sigma_{\varphi_h}), \\ \displaystyle\sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^{r-s}}H_{0,2}^{2,0}\left(\frac{\mu_k}{z} \,\middle|\, \begin{matrix}-\\ (a_1,A_1),(e_1+E_1(r-s),E_1)\end{matrix}\right) \\ \displaystyle+\sum_{k=1}^{L}\mathrm{Res}\Big(\Gamma(a_1-A_1(w-s))\chi(w)\,z^{s-w},w=s_k\Big) \\ \qquad \textit{if } L_2(s) \textit{ can be taken to the left of } \min\limits_{1\leqslant i\leqslant I}(r-\sigma_{\psi_i}). \end{cases} \tag{3.4}$$

3.1. Intermission

We now specify the parameters to get concrete formulas. To this end, we appeal to the following closed formulas for G-functions. We start from the modified Bessel function $K_\nu(z)$ of the 3rd kind which is referred to as the K-Bessel function and defined by

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^{\nu}\int_0^{\infty}\mathrm{e}^{-t-z^2/(4t)}t^{-\nu-1}\mathrm{d}t,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\quad |\arg z|<\frac{\pi}{4}.$$

The K-Bessel function reduces ([4, (40), p. 10]) to

$$K_{\nu+\frac{1}{2}}(x)=\left(\frac{\pi}{2x}\right)^{1/2}\mathrm{e}^{-x}\sum_{m=0}^{\nu}(2x)^{-m}\frac{\Gamma(\nu+m+1)}{m!\,\Gamma(\nu+1+m)}.$$

We also note the following limit relation and symmetry property of the Bessel function which will turn out to be relevant to the functional equation [3, (6.22), p. 110]

$$\lim_{z\to 0} z^{\nu} K_{\nu}(2az) = \frac{1}{2} a^{-\nu}\Gamma(\nu), \quad \operatorname{Re}\nu \geq 0, \quad \nu \neq 0. \tag{3.5}$$

$$K_{-\nu}(z) = K_{\nu}(z). \tag{3.6}$$

We shall frequently use the following basic properties and special cases of the H- and G-functions in what follows.

The reduction-augmentation formula reads

$$H\left(z \,\middle|\, \Delta \oplus \begin{pmatrix} (c,C); & - \\ - & ;(c,C) \end{pmatrix}\right) = H\left(z \,\middle|\, \Delta \oplus \begin{pmatrix} - & ;(c,C) \\ (c,C); & - \end{pmatrix}\right) = H(z|\,\Delta). \tag{3.7}$$

The reciprocity formula reads

$$H\left(z \,\middle|\, \Delta \oplus \begin{pmatrix} -;(c,C) \\ -;(c,C) \end{pmatrix}\right) = \frac{1}{2\pi i}\Big\{ \mathrm{e}^{c\pi i}\, H\big(\mathrm{e}^{-C\pi i} z|\,\Delta\big) - \mathrm{e}^{-c\pi i}\, H\big(\mathrm{e}^{C\pi i} z|\,\Delta\big)\Big\}. \tag{3.8}$$

The duplication formula reads

$$\begin{aligned} &H^{2m+m',2n+n'}_{2p+p',2q+q'}\left(z \,\middle|\, \begin{matrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{matrix}\right) \\ &= (2\pi)^{m+n-(p+q)/2}\, 2^{\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j - (p-q)/2} \\ &\quad \times H^{m+m',n+n'}_{p+p',q+q'}\left(4^{-\sum_{j=1}^{p} A_j + \sum_{j=1}^{q} B_j} z \,\middle|\, \begin{matrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{matrix}\right), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} a_{11} &= \{(\tfrac{a_j}{2}, A_j), (\tfrac{a_j+1}{2}, A_j)\}_{j=1}^{n}, \{(a_j', A_j')\}_{j=1}^{n'}, \\ a_{21} &= \{(\tfrac{b_j}{2}, B_j), (\tfrac{b_j+1}{2}, B_j)\}_{j=1}^{m}, \{(b_j', B_j')\}_{j=1}^{m'}, \\ a_{12} &= \{(\tfrac{a_j}{2}, A_j), (\tfrac{a_j+1}{2}, A_j)\}_{j=n+1}^{p}, \{(a_j', A_j')\}_{j=n'+1}^{p'}, \\ a_{22} &= \{(\tfrac{b_j}{2}, B_j), (\tfrac{b_j+1}{2}, B_j)\}_{j=m+1}^{q}, \{(b_j', B_j')\}_{j=m'+1}^{q'}, \\ b_{11} &= \{(a_j, 2A_j)\}_{j=1}^{n}, \{(a_j', A_j')\}_{j=1}^{n'}, \\ b_{21} &= \{(b_j, 2B_j)\}_{j=1}^{m}, \{(b_j', B_j')\}_{j=1}^{m'}, \\ b_{12} &= \{(a_j, 2A_j)\}_{j=n+1}^{p}, \{(a_j', A_j')\}_{j=n'+1}^{p'}, \\ b_{22} &= \{(b_j, 2B_j)\}_{j=m+1}^{q}, \{(b_j', B_j')\}_{j=m'+1}^{q'}. \end{aligned}$$

e.g., the elements $\left(\frac{b_j}{2}, B_j, \frac{b_j+1}{2}, B_j\right)$ contract to $(b_j, 2B_j)$ with the new factor $2^{1-b_j}\pi^{1/2}$

in the integrand in (3.1). The $H \to G$ formula reads

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (a_1, \frac{1}{C}), \dots, (a_n, \frac{1}{C}), (a_{n+1}, \frac{1}{C}), \dots, (a_p, \frac{1}{C}) \\ (b_1, \frac{1}{C}), \dots, (b_m, \frac{1}{C}), (b_{m+1}, \frac{1}{C}), \dots, (b_q, \frac{1}{C}) \end{matrix}\right) = C\, G_{p,q}^{m,n}\left(z^C \,\middle|\, \begin{matrix} a_1, \dots, a_n, a_{n+1}, \dots, a_p \\ b_1, \dots, b_m, b_{m+1}, \dots, b_q \end{matrix}\right), \quad C > 0. \tag{3.10}$$

The following special cases of the G-functions are frequently used, sometimes without notice:

$$\begin{aligned} \Gamma(a+s) : G_{0,1}^{1,0}\left(z \,\middle|\, \begin{matrix} - \\ a \end{matrix}\right) &= z^a \mathrm{e}^{-z}, \\ \Gamma(a-s)\Gamma(b+s) : G_{1,1}^{1,1}\left(z \,\middle|\, \begin{matrix} a \\ b \end{matrix}\right) &= \Gamma(1-a+b)\, z^b\, (z+1)^{a-b-1}, \\ \Gamma(a+s)\Gamma(b+s) : G_{0,2}^{2,0}\left(z \,\middle|\, \begin{matrix} - \\ a, b \end{matrix}\right) &= 2\, z^{\frac{1}{2}(a+b)} K_{a-b}\big(2\sqrt{z}\,\big). \end{aligned} \tag{3.11}$$

3.2. Specialization of Theorem 3.2

We are ready to specify (3.4) with $D_1 = E_1 = A_1$. The right-hand side H-function of the modular relation (3.4) becomes

$$\begin{aligned} H_{1,1}^{1,1}\left(z\lambda_k \,\middle|\, \begin{matrix} (1-a_1, A_1) \\ (d_1 + A_1 s, A_1) \end{matrix}\right) &= A_1^{-1} G_{1,1}^{1,1}\left((z\lambda_k)^{\frac{1}{A_1}} \,\middle|\, \begin{matrix} 1-a_1 \\ d_1 + A_1 s \end{matrix}\right) \\ &= A_1^{-1}\Gamma(1-a_1+d_1+A_1 s)\lambda_k^s (z\lambda_k)^{\frac{d_1}{A_1}} \left((z\lambda_k)^{\frac{1}{A_1}} + 1\right)^{-a_1-d_1} \\ &\quad \times \frac{1}{\left(\lambda_k^{1/A_1} + (z^{-1})^{1/A_1}\right)^s}. \end{aligned} \tag{3.12}$$

Changing z for z^{-1} and choosing (for simplicity) $a_1 = d_1 = 0$ reduces (3.12) into

$$H_{1,1}^{1,1}\left(z^{-1}\lambda_k \,\middle|\, \begin{matrix} (1, A_1) \\ (A_1 s, A_1) \end{matrix}\right) = A_1^{-1}\Gamma(1 + A_1 s)\lambda_k^s \frac{1}{\left(\lambda_k^{1/A_1} + z^{1/A_1}\right)^s}.$$

Similarly, the series on right-hand side of (3.12) becomes for $z \to z^{-1}$ and $E_1 = A_1$:

$$\begin{aligned} &H_{0,2}^{2,0}\left(\mu_k z \,\middle|\, \begin{matrix} - \\ (a_1, A_1), (e_1 + A_1(r-s), A_1) \end{matrix}\right) \\ &= A_1^{-1} G_{0,2}^{2,0}\left((\mu_k z)^{\frac{1}{A_1}} \,\middle|\, \begin{matrix} - \\ a_1, e_1 + A_1(r-s) \end{matrix}\right) \\ &= 2A_1^{-1}(\mu_k z)^{1/2(a_1+e_1+A_1(r-s))} K_{a_1-e_1-A_1(r-s)}\left(2(\mu_k z)^{1/(2A_1)}\right). \end{aligned}$$

Further, specifying $a_1 = d_1 = e_1 = 0$, we deduce the following corollary.

COROLLARY 3.3. *The functional equation*

$$\Gamma(A_1 s)\varphi(s) = \Gamma(A_1(r-s))\psi(r-s)$$

is equivalent to (with $\Gamma(w|\Delta)) = \Gamma(-A_1 w)$*)*

$$\begin{aligned} \mathrm{X}(s, z^{-1}|\Delta) &= \frac{1}{A_1}\Gamma(A_1 s)\sum_{k=1}^{\infty}\frac{\alpha_k}{\left(\lambda_k^{1/A_1} + z^{1/A_1}\right)^{A_1 s}} \\ &= 2{A_1}^{-1} z^{1/2A_1(r-s)}\sum_{k=1}^{\infty}\frac{\beta_k}{{\mu_k}^{1/2A_1(r-s)}}K_{A_1(r-s)}\left(2(\mu_k z)^{1/(2A_1)}\right) \\ &\quad + \sum_{k=1}^{L}\mathrm{Res}\Big(\Gamma(A_1(s-w))\Gamma(A_1 s)\varphi(s)\, z^{w-s}, w = s_k\Big). \end{aligned} \tag{3.13}$$

EXAMPLE 2. A special case of Corollary 3.3 is the Dirichlet series satisfying the Riemann type functional equation with $A_1 = \frac{1}{2}$, i.e.,

$$\Gamma\left(\frac{1}{2}s\right)\varphi(s) = \Gamma\left(\frac{1}{2}(r-s)\right)\psi(r-s) \tag{3.14}$$

with a simple pole at $s = r > 0$ with residue ρ. Thus, from (3.13) we have

$$\begin{aligned} 2\Gamma\left(\frac{1}{2}s\right)\sum_{k=1}^{\infty}\frac{\alpha_k}{(\lambda_k^2 + z^2)^{1/2s}} &= 4z^{1/2(r-s)}\sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^{1/2(r-s)}}K_{1/2(r-s)}(2\mu_k z) \\ &+ \sum_{k=1}^{L}\mathrm{Res}\Big(\Gamma\left(\frac{1}{2}(s-w)\right)\Gamma\left(\frac{1}{2}w\right)\varphi(w)\, z^{w-s}, w = 0, r\Big), \end{aligned}$$

which gives

$$\begin{aligned} &2\Gamma\left(\frac{1}{2}s\right)\sum_{k=1}^{\infty}\frac{\alpha_k}{(\lambda_k^2 + z^2)^{1/2s}} \\ &= 4z^{1/2(r-s)}\sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^{1/2(r-s)}}K_{1/2(r-s)}(2\mu_k z) + 2\Gamma\left(\frac{1}{2}s\right)\varphi(0)\, z^{-s} \\ &\quad + \rho\Gamma\left(\frac{1}{2}(s-r)\right)\Gamma\left(\frac{1}{2}s\right)\, z^{r-s}. \end{aligned} \tag{3.15}$$

In the case of the Riemann zeta-function, $\alpha_k = \beta_k = 1$, $\lambda_k = \sqrt{\pi}k$, $\mu_k = \sqrt{\pi}k$, $r = 1$ and $\rho = 1$, and it satisfies (3.14) with $\varphi(s) = \psi(s) = \pi^{-s/2}\zeta(s)$. Hence, expressing the left-hand side of (3.15) as the sum over all integers, we deduce Watson's formula [24]

$$\begin{aligned}\Gamma\left(\frac{1}{2}s\right) &\sum_{0\neq k\in\mathbb{Z}} \frac{1}{(\pi k^2+z^2)^{1/2s}} \\ &= 4z^{1/4(1-s)}\sum_{k=1}^{\infty}\frac{1}{k^{1/2(1-s)}}K_{1/2(1-s)}(2\sqrt{\pi}kz) \\ &\quad + \Gamma\left(\frac{1}{2}(s-1)\right)\Gamma\left(\frac{1}{2}s\right)\,z^{1-s} + \Gamma\left(\frac{1}{2}s\right)\zeta(0)\,z^{-s}. \end{aligned} \tag{3.16}$$

That is,

$$\pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{1}{(n^2+\alpha^2)^s} = 2\alpha^{1/2-s}\sum_{n=-\infty}^{\infty}|n|^{s-1/2}K_{s-1/2}\left(2\pi\alpha|n|\right),$$

where the term corresponding to $n=0$ on the right-hand side is to be understood to mean

$$\lim_{u\to 0} u^{s-1/2}K_{s-1/2}(2\pi\alpha u) = \frac{\sqrt{\pi}}{2}\alpha^{1/2-s}\Gamma\left(s-\frac{1}{2}\right),$$

which is (3.5). The special case $s=2$ of (3.16) gives the partial fraction expansion for the hyperbolic cotangent function

$$\sum_{k\in\mathbb{Z}}\frac{1}{k^2+w^2} = \frac{\pi}{w}\Big(\frac{2}{e^{2\pi w}-1}+1\Big) = \frac{\pi}{w}\coth \pi w.$$

Example 3. Another more special case of Corollary 3.3 is the Dirichlet series satisfying the Hecke type functional equation with $A_1 = 1$, i.e.,

$$\Gamma\left(s\right)\varphi(s) = \Gamma\left(r-s\right)\psi(r-s) \tag{3.17}$$

with a simple pole at $s=r$ with residue ρ. In this case, (3.13) gives

$$\begin{aligned}\Gamma\left(s\right)\sum_{k=1}^{\infty}\frac{\alpha_k}{\left(\lambda_k+z\right)^s} &= 2z^{1/2(r-s)}\sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^{1/2(r-s)}}K_{r-s}\left(2\sqrt{\mu_k z}\right) \\ &\quad + \mathrm{Res}\Big(\Gamma\left(s-w\right)\chi(w)\,z^{w-s}, w=r\Big),\end{aligned}$$

which becomes

$$\begin{aligned}\Gamma\left(s\right)&\sum_{k=1}^{\infty}\frac{\alpha_k}{\left(\lambda_k+z\right)^s} \\ &= 2z^{1/2(r-s)}\sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^{1/2(r-s)}}K_{r-s}\left(2\sqrt{\mu_k}\right) + \rho\Gamma\left(s-r\right)\,z^{r-s}.\end{aligned}$$

4. Dedekind zeta-function

We consider Dedekind zeta-function of a rational or a quadratic field Ω with degree $\varkappa = r_1 + 2r_2 \leqslant 2$ and discriminant Δ. Let

$$A = \frac{2^{r_2}\pi^{\varkappa/2}}{\sqrt{|\Delta|}}$$

and let $r = r_1 + r_2 - 1$ denote the rank of the unit group. In the case of the Dedekind zeta-function $\zeta_\Omega(s)$ of a quadratic field, $\alpha_k = \beta_k = \alpha(k)$, with $\alpha(k)$ denoting the number of ideals of norm k. Also $\lambda_k = Ak$, $\mu_k = Ak$, $r = 1$ and

$$\rho = \frac{2^{r+1}\pi^{r_2}Rh}{w\sqrt{|\Delta|}} = -\frac{2^{r+1}\pi^{r_2}\zeta_\Omega^{(r)}(s)}{\sqrt{|\Delta|}}. \tag{4.1}$$

Here w, h and R are the number of roots of unity in Ω, the class number and the regulator, respectively. In this case, the functional equation becomes

$$\Gamma^{r_1}\left(\frac{1}{2}s\right)\Gamma^{r_2}(s)\varphi(s) = \Gamma^{r_1}\left(\frac{1}{2} - \frac{1}{2}s\right)\Gamma^{r_2}(1-s)\varphi(1-s), \tag{4.2}$$

where

$$\varphi(s) = A^{-s}\zeta_\Omega(s) = \sum_{k=1}^{\infty}\frac{\alpha(k)}{(Ak)^s}. \tag{4.3}$$

Corollary 3.3 with $A_1 = 1$ gives

$$\begin{aligned}
&\Gamma(s)\sum_{k=1}^{\infty}\frac{\alpha(k)}{(Ak+z)^s}\\
&= 2z^{1/2(1-s)}\sum_{k=1}^{\infty}\frac{\alpha(k)}{(Ak)^{1/2(1-s)}}K_{1-s}\left(2\sqrt{Akz}\right)\\
&+\Gamma(s)\varphi(0)\,z^{-s} + \rho\Gamma(s-1)\,z^{1-s}.
\end{aligned}$$

We replace $\frac{z}{A} = \omega$ with $\operatorname{Re}\omega > 0$ to get

$$\begin{aligned}
&A^{-s}\sum_{k=1}^{\infty}\frac{\alpha(k)}{(k+\omega)^s}\\
&= \frac{2}{\Gamma(s)}\omega^{1/2(1-s)}\sum_{k=1}^{\infty}\frac{\alpha(k)}{k^{1/2(1-s)}}K_{1-s}\left(2A\sqrt{k\omega}\right) + A^{-s}\varphi(0)\,\omega^{-s}\\
&+A^{1-s}\rho\frac{\Gamma(s-1)}{\Gamma(s)}\,\omega^{1-s}.
\end{aligned}$$

Therefore, we have

Proposition 4.1. *In case of imaginary quadratic fields with $r=1$, the functional equation* (3.17) *is equivalent to the Fourier-Bessel series*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\alpha(k)}{(k+\omega)^s} &= A^s \frac{2}{\Gamma(s)} \omega^{1/2(1-s)} \sum_{k=1}^{\infty} \frac{\alpha(k)}{k^{1/2(1-s)}} K_{1-s}\left(2A\sqrt{k\omega}\right) \\ &\quad + \zeta_\Omega(0)\,\omega^{-s} + A\rho\,\omega^{1-s}\frac{1}{s-1}.\end{aligned}$$

We shall now show that Proposition 4.1 gives the Lipschitz summation formula as in Koshlyakov [11, p. 245–246], where he takes a weighted sum of $\zeta_\Omega(s)$ (one reason may be to cancel the pole at $s=1$). The essential reason will be discussed in more detail in §5.. For a character analogue of the Lipschitz summation formula, (cf. Berndt [2]), as in [11, p. 243], we consider for $0 < \mathrm{Re}\,\omega < 1$ (in [11, p. 243], the limiting case $\mathrm{Re}\,\omega = 0$ is considered)

$$\begin{aligned}&\frac{1}{2}e^{\pi i s/2} \sum_{k=1}^{\infty} \frac{\alpha(k)}{(k+iw)^s} + \frac{1}{2}e^{-\pi i s/2} \sum_{k=1}^{\infty} \frac{\alpha(k)}{(k-iw)^s} \\ &= \frac{1}{2}A^s \frac{2}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{\alpha(k)}{k^{1/2(1-s)}} \\ &\quad \times \Big(e^{\pi i s/2}(iw)^{1/2(1-s)} K_{1-s}\left(2A\sqrt{ikw}\right) \\ &\quad + e^{-\pi i s/2}(-iw)^{1/2(1-s)} K_{1-s}\left(2A\sqrt{-ikw}\right) \Big) + \zeta_\Omega(0)\, w^{-s}. \qquad (4.4)\end{aligned}$$

If one writes $\varepsilon = e^{\pi i/4}$, then the right-hand side of (4.4) becomes

$$\begin{aligned}&= \frac{1}{2}A^s \frac{2w^{1/2(1-s)}}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{\alpha(k)}{k^{1/2(1-s)}} \\ &\quad \times \left(\varepsilon^{s+1} K_{1-s}\left(2A\varepsilon\sqrt{kw}\right) + \bar\varepsilon^{s+1} K_{1-s}\left(2A\bar\varepsilon\sqrt{kw}\right) \right) + \frac{\zeta_\Omega(0)}{w^s}. \qquad (4.5)\end{aligned}$$

At this point, in conformity with Koshlyakov [11, p. 241] (cf. (4.3)), we consider the perturbed Dedekind zeta-function

$$\zeta_\Omega(s,\omega) = -\frac{2\zeta_\Omega(0)}{w} + \sum_{k=1}^{\infty} \frac{\alpha(k)}{(k+\omega)^s}.$$

Then (4.4) and (4.5) give the Lipschitz summation formula

$$\begin{aligned}\sum_{n=-\infty}^{\infty}{}' \frac{\alpha(n)}{(w+in)^s} &= \mathrm{e}^{\pi i/2s}\zeta_\Omega(s,iw) + \mathrm{e}^{-\pi i/2s}\zeta_\Omega(s,-iw) \\ &= A^s \frac{w^{1/2(1-s)}}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{\alpha(k)}{k^{1/2(1-s)}} \\ &\quad \times \left(\varepsilon^{s+1}K_{1-s}\left(2A\varepsilon\sqrt{kw}\right) + \bar{\varepsilon}^{s+1}K_{1-s}\left(2A\bar{\varepsilon}\sqrt{kw}\right)\right) - \frac{\zeta_\Omega(0)}{w^s}. \end{aligned} \tag{4.6}$$

This is the corrected form of [11, (23.15)] and in the summation the term corresponding to $n = 0$ is excluded.

The Dedekind zeta-function of a real quadratic field satisfies the functional equation (4.2) with $r_1 = 2, r_2 = 0$. Hence, it take the form

$$\Gamma^2\Big(\frac{1}{2}s\Big)\varphi(s) = \Gamma^2\Big(\frac{1}{2} - \frac{1}{2}s\Big)\varphi(1-s), \tag{4.7}$$

where $\varphi(s)$ is defined by (4.3).

In this form, there would be no proper modular relation with $H_{1,1}^{1,1}$ unless we combine the perturbed Dirichlet series, which we shall state in Section 5.

Here we view (4.7) to mean

$$\frac{\Gamma\big(1/2s\big)}{\Gamma\big((1-s)/2s\big)}\varphi(s) = \frac{\Gamma\big((1-s)/2s\big)}{\Gamma\big((1-(1-s))/2s\big)}\varphi(1-s).$$

As before, we choose the values of the parameters as in the following

values	proc. gamma	LHSFE	RHSFE	values
$m + M^{(h)} = 1$	$m = 0$	$M^{(h)} = 1$	$M^{(i)} = 0$	$m + \tilde{M}^{(i)} = 0$
$n + N^{(h)} = 1$	$n = 1$	$N^{(h)} = 0$	$N^{(i)} = 1$	$n + \tilde{N}^{(i)} = 2$
$p + P^{(h)} = 1$	$p = 1$	$P^{(h)} = 0$	$P^{(i)} = 2$	$p + \tilde{P}^{(i)} = 3$
$q + Q^{(h)} = 1$	$q = 0$	$Q^{(h)} = 1$	$Q^{(i)} = 0$	$q + \tilde{Q}^{(i)} = 0$

Table 1. Choice of parameters

In this set-up, the modular relation gives the following result.

THEOREM 4.2. *The functional equation*

$$\frac{\Gamma(d_1 + D_1 s)}{\Gamma(d_2 - D_2 s)}\varphi(s) = \frac{\Gamma(e_1 + E_1(r-s))}{\Gamma(e_2 - E_2(r-s))}\psi(r-s), \tag{4.8}$$

with $\Gamma(s|\Delta) = \Gamma(a_1 - A_1 s)$, *is equivalent to*

$$z^s\,\mathrm{X}(z,s\,|\,\Delta)$$

$$= \begin{cases} \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^s} H_{1,1}^{1,1}\left(z\lambda_k \,\middle|\, \begin{matrix} (1-a_1, A_1) \\ (d_1 + D_1 s, D_1) \end{matrix} \right) \\ \qquad \textit{if } L_1(s) \textit{ can be taken to the right of } \sigma_\varphi, \\ \sum_{k=1}^{\infty} \frac{\beta_k}{\mu_k^{r-s}} H_{0,3}^{2,0}\left(\frac{\mu_k}{z} \,\middle|\, \begin{matrix} - \\ (a_1, A_1), (e_1 + E_1(r-s), E_1), (1-e_2+E_2(r-s), E_2) \end{matrix} \right) \\ \quad + \sum_{k=1}^{L} \mathrm{Res}\Big(\Gamma(a_1 - A_1(w-s))\chi(w)\, z^{s-w}, w = s_k \Big) \\ \qquad \textit{if } L_2(s) \textit{ can be taken to the left of } r - \sigma_\psi. \end{cases}$$

The right-hand side of the above contains the summand

$$H_{0,3}^{2,0}\left(\mu_k z \,\middle|\, \begin{matrix} - \\ c_{11} \end{matrix} \right) = H_{0,3}^{2,0}\left(\mu_k z \,\middle|\, \begin{matrix} - \\ (0,1), (\frac12(1-s), \frac12), (\frac12 + \frac12(1-s), \frac12) \end{matrix} \right)$$

with

$$c_{11} = (a_1, A_1), \left(e_1 + \frac12 A_1(r-s), \frac12 A_1 \right), \left(1 - e_2 + \frac12 A_1(r-s), (\frac12)A_1 \right).$$

We may work with a slightly more general form of the right-hand side of the modular relation

$$\begin{aligned} & H_{0,3}^{2,0}\left(z \,\middle|\, \begin{matrix} - \\ (s, \frac12 A_1), (a_1, A_1), (s+\frac12, \frac12 A_1) \end{matrix} \right) \\ &= H_{1,4}^{3,0}\left(z \,\middle|\, \begin{matrix} (s+\frac12, \frac12 A_1) \\ (s, \frac12 A_1), (s+\frac12, \frac12 A_1), (a_1, A_1), (s+\frac12, \frac12 A_1) \end{matrix} \right) \\ &= 2\sqrt{\pi}\, 4^{-s} H_{1,3}^{2,0}\left(2^{A_1} z \,\middle|\, \begin{matrix} (s+\frac12, \frac12 A_1) \\ (2s, A_1), (a_1, A_1), (s+\frac12, \frac12 A_1) \end{matrix} \right) \\ &= \frac{4^{-s}}{\sqrt{\pi}\, i} \Big\{ \mathrm{e}^{(s+1/2)\pi i} H_{0,2}^{2,0}\left(2^{A_1} \mathrm{e}^{-\pi/2 A_1 i} z \,\middle|\, \begin{matrix} - \\ (2s, A_1), (a_1, A_1) \end{matrix} \right) \\ &\quad - \mathrm{e}^{-(s+1/2)\pi i} H_{0,2}^{2,0}\left(2^{A_1} \mathrm{e}^{\pi/2 A_1 i} z \,\middle|\, \begin{matrix} - \\ (2s, A_1), (a_1, A_1) \end{matrix} \right) \Big\} \\ &= \frac{4^{-s}}{\sqrt{\pi}\, i A_1} \Big\{ \mathrm{e}^{(s+1/2)\pi i} G_{0,2}^{2,0}\left(2\mathrm{e}^{-\pi/2 i} z^{1/A_1} \,\middle|\, \begin{matrix} - \\ 2s, a_1 \end{matrix} \right) \\ &\quad - \mathrm{e}^{-(s+1/2)\pi i} G_{0,2}^{2,0}\left(2\, e^{\frac{\pi}{2} i} z^{1/A_1} \,\middle|\, \begin{matrix} - \\ 2s, a_1 \end{matrix} \right) \Big\} \\ &= \frac{2}{\sqrt{\pi} A_1} \Big\{ (2\mathrm{e}^{-\pi/2 i} z^{1/A_1})^{a_1} \left(\frac{\mathrm{e}^{-\pi/2 i} z^{1/A_1}}{2} \right)^s K_{2s-a_1}\left(2\sqrt{2} \mathrm{e}^{-\pi i/4} z^{1/2A_1} \right) \\ &\quad + (2\mathrm{e}^{\pi/2 i} z^{1/A_1})^{a_1} \left(\frac{\mathrm{e}^{\pi/2 i} z^{1/A_1}}{2} \right)^s K_{2s-a_1}\left(2\sqrt{2} \mathrm{e}^{\pi i/4} z^{1/(2A_1)} \right) \Big\} \end{aligned} \tag{4.9}$$

by (3.7), (3.9), (3.8) and (3.11). Now, equation (4.9) with $s \leftrightarrow \frac{1-s}{2}$ and $A_1 = 1$

takes the form

$$H_{0,3}^{2,0}\left(z \,\middle|\, \begin{matrix} - \\ \left(\frac{1-s}{2}, \frac{1}{2}\right), (a_1, 1), \left(\frac{1-s}{2} + \frac{1}{2}, \frac{1}{2}\right) \end{matrix}\right)$$
$$= \frac{2}{\sqrt{\pi}}\Big\{(2\mathrm{e}^{-\pi/2i}z)^{a_1}\left(\frac{\mathrm{e}^{-\pi/2i}z}{2}\right)^{(1-s)/2} K_{1-s-a_1}\left(2\mathrm{e}^{-\pi i/4}\sqrt{2z}\right)$$
$$+ (2\mathrm{e}^{\pi/2i}z)^{a_1}\left(\frac{\mathrm{e}^{\pi/2i}z}{2}\right)^{(1-s)/2} K_{1-s-a_1}\left(2\mathrm{e}^{\pi i/4}\sqrt{2z}\right)\Big\}. \tag{4.10}$$

Further, using $a_1 = 0$, the equation (4.10) becomes

$$H_{0,3}^{2,0}\left(z \,\middle|\, \begin{matrix} - \\ \left(\frac{1-s}{2}, \frac{1}{2}\right), (0, 1), \left(\frac{1-s}{2} + \frac{1}{2}, \frac{1}{2}\right) \end{matrix}\right)$$
$$= \frac{2}{\sqrt{\pi}}\Big\{\left(\frac{\mathrm{e}^{-\pi/2i}z}{2}\right)^{(1-s)/2} K_{1-s}\left(2\mathrm{e}^{-\pi i/4}\sqrt{2z}\right)$$
$$+ \left(\frac{\mathrm{e}^{\pi/2i}z}{2}\right)^{(1-s)/2} K_{1-s}\left(2\mathrm{e}^{\pi i/4}\sqrt{2z}\right)\Big\}.$$

This is an analogue of Koshlyakov's result, however, with the basic sequences being squared (since we must choose $E_1 = 2$ in (4.8) and thus we are forced to choose $A_1 = \frac{1}{2}$).

5. Elucidation of Koshlyakov's result in the real quadratic case

We elucidate the reason behind the appearance of the K-Bessel function for a real quadratic field Ω. Koshlyakov considers the integral with the integrand being of the form

$$\Gamma(z)\varphi(z) = \Gamma(z)\frac{\Gamma^2\big((1-z)/2\big)}{\Gamma^2\big(z/2\big)}\varphi(1-z),$$

which is in a form similar to that of (4.7). He applies the duplication formula to the first factor and cancels $\Gamma\left(\frac{z}{2}\right)$ in the denominator. Then he writes $\frac{1}{\Gamma(z/2)}$ as $\Gamma\left(1-\frac{z}{2}\right)\frac{\sin \pi/2z}{\pi}$ and also expresses the product $\Gamma\left(\frac{z}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1-z}{2}\right)$ as $\frac{\pi}{\sin \pi(1-z)/2}$. Then finally, he combines $\Gamma\left(\frac{1-z}{2}\right)$ and $\Gamma\left(1-\frac{z}{2}\right)$ to give $2^{-(1-z)}\pi^{1/2}\Gamma(1-z)$. It follows that

$$\Gamma(z)\zeta_\Omega(z) = \tan\frac{\pi}{2}z\Gamma(1-z)\zeta_\Omega(1-z)$$

which is a reminiscent of the Wigert-Bellman divisor problem [1]. Thus,

THEOREM 5.1. *The generating Dirichlet series for the Wigert-Bellman divisor problem*

$$\zeta_\Omega(s,\omega) = \frac{\omega^{-s}}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\alpha(k)}{k}\frac{1}{2\pi i}\int_{(c)}\tan\frac{\pi}{2}z\Gamma(1-z)\Gamma(s-z)\left(\frac{1}{k\omega}\right)^{-z}\mathrm{d}z$$

amounts to the Lipschitz summation formula

$$\begin{aligned}\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{\alpha(n)}{(w+in)^s} &= \frac{1}{2}\mathrm{e}^{\pi i/2s}\zeta_\Omega(s,iw)+\frac{1}{2}\mathrm{e}^{-\pi i/2s}\zeta_\Omega(s,-iw)\\ &= A^s\frac{w^{1/2(1-s)}}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\alpha(k)}{k^{1/2(1-s)}}\\ &\quad\times\left(\frac{\varepsilon^{s+1}}{i}K_{1-s}\left(2A\varepsilon\sqrt{kw}\right)-\frac{\bar\varepsilon^{s+1}}{i}K_{1-s}\left(2A\bar\varepsilon\sqrt{kw}\right)\right).\end{aligned} \tag{5.1}$$

This in fact is the corrected form of [11, (23.16)]. Proving the above result amounts to showing that, in contrast to the case of imaginary quadratic fields in Proposition 4.1, the left-hand side cancels the denominator $\cos\frac{\pi}{2}z$, which is given in

$$\mathrm{e}^{\pi i/2s}\zeta_\Omega(s,iw)+\mathrm{e}^{-\pi i/2s}\zeta_\Omega(s,-iw). \tag{5.2}$$

The integral expression for (5.2) contains the integrand

$$\frac{w^z}{\pi}\sin\frac{\pi}{2}z\Gamma(1-z)\zeta_\Omega(1-z),$$

and so applying the inverse Heaviside integral (3.11) one can deduce (5.1).

6. Koshlyakov's K-series

Let $G(s)$ denote the quotient of gamma factors in (4.2),

$$G(1-s)=\frac{\Gamma^{r_1}(s/2)\,\Gamma^{r_2}(s)}{\Gamma^{r_1}((1-s)/2)\,\Gamma^{r_2}(1-s)},$$

and consider the kernel (which we call Koshlyakov's K-function)

$$K(x)=K_{r_1,r_2}(x)=\frac{1}{2\pi i}\int_{(c)}\frac{\pi}{2\cos(\pi/2s)}\frac{G(1-s)}{x^s}\mathrm{d}s$$

for $c>0$ and $x>0$ ($\mathrm{Re}\,x>0$ is allowed).

Then Koshlyakov considers the basic series

$$\sigma(x)=\sigma_{r_1,r_2}(x)=\frac{A}{\pi}\sum_{n=1}^{\infty}F(n)K_{r_1,r_2}\big(A^2xn\big). \tag{6.1}$$

He transforms the series (6.1) by Cauchy's residue theorem and applies a form of the beta transform (2.2) ([10, (3.3)]), at the last stage,

$$\frac{1}{2\pi i}\int_{(c)}\frac{\pi}{2\sin(\pi/2w)}\frac{\mathrm{d}w}{\alpha^w}=\frac{1}{1+\alpha^s},\quad \alpha>0,\quad 0<c<2.$$

He proves the following theorem.

THEOREM 1 (Koshlyakov). *One has*

$$\sigma(x) = -\frac{1}{2}\rho - \frac{\zeta_\Omega(0)}{\pi}\frac{1}{x} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^2+x^2} \tag{6.2}$$

for $\operatorname{Re} x > 0$*, where* ρ *is defined in* (4.1)*. Thus, one gets the closed forms for the* K*-function:*

$$\begin{aligned} K_{1,0}(x) &= \sqrt{\pi}\mathrm{e}^{-2x}, & (6.3)\\ K_{0,1}(x) &= -2\,\mathrm{kei}_0\left(4\sqrt{\frac{x}{4}}\right) = \frac{1}{i}\left(K_0\left(2\varepsilon\sqrt{x}\,\right) - K_0\left(2\bar{\varepsilon}\sqrt{x}\,\right)\right), & (6.4)\\ K_{2,0}(x) &= 4\,\mathrm{ker}_0(4\sqrt{x}\,) = 2\left(K_0\left(\varepsilon\sqrt{x}\,\right) + K_0\left(\bar{\varepsilon}\sqrt{x}\,\right)\right), \end{aligned}$$

where kei_0 *and* ker_0 *are modified Kelvin functions* ([4, p. 6], [18]) *and*

$$\varepsilon = \mathrm{e}^{\pi/4i}, \quad \bar{\varepsilon} = \mathrm{e}^{-\pi/4i}.$$

The proof of (6.2) follows by taking the limit as $s \to 0$ in formulas (4.6). The main difference is that one expresses the resulting functions as Kelvin functions.

We turn to the elucidation of Koshlyakov's K-function (6.). We distinguish three cases. In the case of $\mathbb{Q}$, one has $r_1 = 1$, $r_2 = 0$, and this leads to Example 3.2. and (6.3) has been deduced already above.

In the imaginary quadratic case, $r_1 = 0$, $r_2 = 1$, we need to locate Koshlyakov's formula [10, (2.5)] (except for the middle member in (6.4))

$$K_{0,1}(x) = \frac{1}{i}\Big(K_0\big(2\bar{\varepsilon}\sqrt{x}\,\big) - K_0\big(2\varepsilon\sqrt{x}\,\big)\Big). \tag{6.5}$$

Since

$$K_{0,1}(x) = \frac{1}{2}\,H^{2,1}_{1,3}\left(x \,\middle|\, \begin{matrix} (\frac{1}{2},\frac{1}{2}) \\ (\frac{1}{2},\frac{1}{2}),(0,1),(0,1) \end{matrix}\right),$$

by applying (3.7), we see that

$$\begin{aligned} K_{0,1}(x) &= \frac{1}{4}\,H^{3,1}_{1,5}\left(\frac{x}{4} \,\middle|\, \begin{matrix} (\frac{1}{2},\frac{1}{2}) \\ (\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}) \end{matrix}\right)\\ &= \frac{1}{4}\,H^{3,0}_{0,4}\left(\frac{x}{4} \,\middle|\, \begin{matrix} - \\ (0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}) \end{matrix}\right)\\ &= \frac{1}{4}\,H^{4,0}_{1,5}\left(\frac{x}{4} \,\middle|\, \begin{matrix} (0,\frac{1}{2}) \\ (0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}) \end{matrix}\right). \end{aligned} \tag{6.6}$$

Hence, by (3.8), we conclude that

$$\begin{aligned} K_{0,1}(x) = \frac{1}{8\pi i}&\left\{H^{4,0}_{0,4}\left(\frac{\mathrm{e}^{-\pi/2i}x}{4} \,\middle|\, \begin{matrix} - \\ (0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}) \end{matrix}\right)\right.\\ &\left. - H^{4,0}_{0,4}\left(\frac{\mathrm{e}^{\pi/2i}x}{4} \,\middle|\, \begin{matrix} - \\ (0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2}) \end{matrix}\right)\right\}\\ = \frac{1}{2i}&\left\{G^{2,0}_{0,2}\left(\mathrm{e}^{-\pi/2i}x \,\middle|\, \begin{matrix} - \\ 0,0 \end{matrix}\right) - G^{2,0}_{0,2}\left(\mathrm{e}^{\pi/2i}x \,\middle|\, \begin{matrix} - \\ 0,0 \end{matrix}\right)\right\}. \end{aligned} \tag{6.7}$$

In the last step, we applied (3.7) and then (3.10). The equation (6.7) amounts to (6.5) by (3.6).

We observe that we have to transit from (6.6) to

$$K_{0,1}(x) = \frac{1}{2}\, G^{3,0}_{0,4}\left(\left(\frac{x}{4}\right)^2 \,\middle|\, \begin{matrix} - \\ 0, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix}\right).$$

We notice that the right-hand side is one of the modified Kelvin's functions

$$\frac{1}{2}\cdot(-4)\,\mathrm{kei}_0\left(4\sqrt{\frac{x}{4}}\right)$$

[18, 13, p. 675].

The modified Kelvin functions satisfy

$$\mathrm{ker}_\nu(z) \pm i\mathrm{kei}_\nu(z) = \mathrm{e}^{\mp i\nu/2\pi} K_\nu\left(z\mathrm{e}^{\pm\pi i/4}\right), \tag{6.8}$$

and so,

$$2i\mathrm{kei}_\nu(z) = \mathrm{e}^{-\nu\pi/2i} K_\nu\left(z\mathrm{e}^{\pi i/4}\right) - \mathrm{e}^{\nu\pi/2i} K_\nu\left(z\mathrm{e}^{-\pi i/4}\right). \tag{6.9}$$

Thus, one has

$$K_{0,1}(x) = -2\,\mathrm{kei}_0\left(4\sqrt{\frac{x}{4}}\right) = \frac{1}{i}\left(K_0\big(2\varepsilon\sqrt{x}\,\big) - K_0\big(2\bar{\varepsilon}\sqrt{x}\,\big)\right). \tag{6.10}$$

Note that while deriving (6.10), we have also given a proof of (6.9). We are now left to deal with the real quadratic case, where

$$K_{2,0}(x) = \frac{1}{2\pi i}\int_{(c)} \frac{1}{2}\frac{\Gamma^2(w/2)\,\Gamma((1+w)/2)}{\Gamma((1-w)/2)}\, x^{-w}\mathrm{d}w.$$

The above expression is already in the desired shape

$$\begin{aligned} K_{2,0}(x) &= \frac{1}{2} H^{3,0}_{0,4}\left(x \,\middle|\, \begin{matrix} - \\ (0,\frac{1}{2}), (0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2}), (\frac{1}{2},\frac{1}{2}) \end{matrix}\right) \\ &= G^{3,0}_{0,4}\left(x^2 \,\middle|\, \begin{matrix} - \\ 0, 0, \frac{1}{2}, \frac{1}{2} \end{matrix}\right) \end{aligned}$$

by (3.10), which is $4\,\mathrm{ker}_0(4\sqrt{x}\,)$ ([18, 13, p.675]). Further from (6.8), we have

$$2\,\mathrm{ker}_\nu(z) = \mathrm{e}^{-\nu\pi i/2} K_\nu\left(z\mathrm{e}^{\pi i/4}\right) + \mathrm{e}^{\nu\pi i/2} K_\nu\left(z\mathrm{e}^{-\pi i/4}\right).$$

Hence, it follows that

$$K_{2,0}(x) = G^{3,0}_{0,4}\left(x^2 \,\middle|\, \begin{matrix} - \\ 0, 0, \frac{1}{2}, \frac{1}{2} \end{matrix}\right) = 4\,\mathrm{ker}_0(4\sqrt{x}\,) = 2\left(K_0\big(\varepsilon\sqrt{x}\,\big) + K_0\big(\bar{\varepsilon}\sqrt{x}\,\big)\right).$$

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