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A GENERALIZATION OF GALLAGHER'S LEMMA FOR EXPONENTIAL SUMS

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Abstract. First we generalize a famous lemma of Gallagher on the mean square estimate for exponential sums by plugging a weight in the right-hand side of Gallagher's original inequality. Then we apply it in the special case of the Cesàro weight in order to establish some results mainly concerning the classical Dirichlet polynomials and the Selberg integrals of an arithmetic function f, that are tools for studying the distribution of f in short intervals. Furthermore, we describe the smoothing process via self-convolutions of a weight that is involved into our Gallagher type inequalities, and compare it with the analogous process via the so-called correlations. Finally, we discuss a comparison argument in view of refinements on the Gallagher weighted inequalities according to different instances of the weight.

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1. Introduction and statement of the main results

In 1970, Gallagher ([8], Lemma 1) gave a general mean value estimate for series of the type

$$S(t) \stackrel{\text{def}}{=} \sum_{\nu} s(\nu)e(\nu t),$$

where $e(x) \stackrel{\text{def}}{=} \mathrm{e}^{2\pi i x}$ as usual, the frequencies ν run over a (strictly increasing) sequence of real numbers and the coefficients $s(\nu)$ are complex numbers. More precisely, if $S(t)$ converges absolutely and $\delta, T > 0$ are real numbers such that

$\theta \stackrel{\text{def}}{=} \delta T \in (0,1)$, then

$$\|S\|_{2,T}^2 \ll_\theta \delta^{-2} \int_{\mathbb{R}} \Big| \sum_{x<\nu\le x+\delta} s(\nu) \Big|^2 \mathrm{d}x, \tag{1.1}$$

where, for brevity, we have written

$$\|S\|_{2,T} \stackrel{\text{def}}{=} \|S\|_{L^2(-T,T)} = \left(\int_{-T}^{T} |S(t)|^2 \mathrm{d}t \right)^{1/2}.$$

Hereafter, $A \ll_\theta B$ stands for $|A| \leqslant CB$, where $C > 0$ is an unspecified constant that depends on θ. Typically, in the present context the bounds hold for $T \to 0$ or $T \to \infty$ (more precisely, for $|T| \leqslant T_0$ with a sufficiently small $T_0 > 0$ or for $T > T_0$ with a sufficiently large $T_0 > 1$).

We refer the reader to [9] for a first introduction to the large-sieve results, which constitute the main context where the inequality (1.1) has been widely applied (beware that in [9], Part III, (1.1) is referred to as Gallagher's second Lemma). In particular, an immediate and renowned consequence is the so-called Gallagher's Lemma for the Dirichlet series (see [8], Theorem 1). Indeed, since an absolutely convergent Dirichlet series can be written as

$$\sum_n a_n n^{it} = \sum_\nu s(\nu) e(\nu t)$$

by taking $\nu = (2\pi)^{-1} \log n$ and $s(\nu) = a_{e(\nu/i)}$, then, making the substitution $x = \theta \log y$ in (1.1) with $\theta = (2\pi)^{-1}$ and recalling that $T = \theta\delta^{-1}$, one immediately has

$$\Big\| \sum_n a_n n^{it} \Big\|_{2,T}^2 \ll T^2 \int_0^{+\infty} \Big| \sum_{y<n\leqslant ye^{1/T}} a_n \Big|^2 \frac{\mathrm{d}y}{y}. \tag{1.2}$$

Inequalities of this type are important tools for studying large values of the so-called Dirichlet polynomials, i.e., finite Dirichlet series. In fact, they have large values frequencies strictly related to the behavior of their moments: see Chapter 9 of [11] and Chapter 7 of [13] for basic knowledge on this topic.

Here we give a more general version of (1.1) by plugging a weight into the right-hand side of it. To this end, for any weight $w : \mathbb{R} \to \mathbb{C}$, we set $w_\delta(x) = w(x)$ or 0 according as $|x| \leqslant \delta$ or not. The Fourier transform of w_δ is denoted by

$$\widehat{w_\delta}(y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} w_\delta(t) e(-ty) \mathrm{d}t = \int_{-\delta}^{\delta} w(t) e(-ty) \mathrm{d}t.$$

Moreover, we recall that $L^1_{\text{loc}}(\mathbb{R})$ denotes the space of all the locally summable functions on $\mathbb{R}$.

LEMMA. *Let the real numbers $\delta, T > 0$ be fixed. Assume that, given a (strictly increasing) sequence of real numbers ν and complex coefficients $s(\nu)$, the series $S(t) = \sum_\nu s(\nu)e(\nu t)$ is absolutely convergent. Then, for every weight $w \in L^1_{\mathrm{loc}}(\mathbb{R})$, one has*

$$\|S\|_{2,T}^2 \min_{|t|\leqslant T} |\widehat{w_\delta}(t)|^2 \leqslant \int_{\mathbb{R}} \Big|\sum_\nu s(\nu)w_\delta(x-\nu)\Big|^2 \mathrm{d}x. \tag{1.3}$$

More general, if $w \in L^1(\mathbb{R})$, then

$$\|S\|_{2,T}^2 \min_{|t|\leqslant T} |\widehat{w}(t)|^2 \leqslant \int_{\mathbb{R}} \Big|\sum_\nu s(\nu)w(x-\nu)\Big|^2 \mathrm{d}x. \tag{1.4}$$

Though the proof of (1.3) closely parallels Gallagher's original one, it is fully provided in Section 2 together with the proof of (1.4), while some further aspects of the lemma are discussed in Section 4.

REMARKS. 1. If $\min_{|t|\leqslant T}|\widehat{w_\delta}(t)|^2 = 0$, as when w_δ is odd, then (1.3) is trivial. Now, let w_δ be an even weight whose self-convolution

$$(w_\delta * w_\delta)(x) \overset{\mathrm{def}}{=} \int_{\mathbb{R}} w_\delta(t)w_\delta(x-t)\mathrm{d}t$$

is nonnegative. If $|s(\nu)| \leqslant b(\nu)$, then (1.3) yields

$$\min_{|t|\leqslant T} |\widehat{w_\delta}(t)|^2 \int_{-T}^{T} \Big|\sum_\nu s(\nu)e(\nu y)\Big|^2 \mathrm{d}y \leqslant \int_{\mathbb{R}} |\sum_\nu b(\nu)w_\delta(x-\nu)|^2 \mathrm{d}x.$$

Indeed, by applying the inequality $s(\nu_1)\overline{s(\nu_2)} + s(\nu_2)\overline{s(\nu_1)} \leqslant 2|s(\nu_1)||s(\nu_2)|$ one has

$$\begin{aligned}\int_{\mathbb{R}} \Big|\sum_\nu s(\nu)w_\delta(x-\nu)\Big|^2 \mathrm{d}x &= \sum_{\nu_1,\nu_2} s(\nu_1)\overline{s(\nu_2)}(w_\delta * w_\delta)(\nu_2-\nu_1)\\ &\leqslant \sum_{\nu_1,\nu_2} b(\nu_1)b(\nu_2)(w_\delta * w_\delta)(\nu_2-\nu_1)\\ &= \int_{\mathbb{R}} |\sum_\nu b(\nu)w_\delta(x-\nu)|^2 \mathrm{d}x.\end{aligned}$$

Note that, by an application of the Hardy-Littlewood majorant principle (see [13], Chapter 7, or [12]) combined with (1.3) for $b(\nu)$, one would get the factor 3 in the right-hand side of the above inequality.

2. Gallagher's original (1.1) is recovered from (1.3) by taking the weight $\delta^{-1}u_\delta$ associated to the unit step function

$$u(t) \overset{\mathrm{def}}{=} \begin{cases} 1 & \text{if} \quad t > 0,\\ 0 & \text{otherwise,}\end{cases}$$

with $\delta = \theta T^{-1}$ for $T > 0$ and $\theta \in (0,1)$ fixed.

In the following, we exploit the particular case of the Cesàro weight

$$C_\delta(y) \stackrel{\rm def}{=} \max(1-\delta^{-1}|y|,0),$$

for which (1.3) yields the asymptotic inequality

$$\|S\|_{2,T}^2 \ll_\theta \delta^{-2}\int_{\mathbb{R}}\Big|\sum_\nu s(\nu)C_\delta(\nu-x)\Big|^2 \mathrm{d}x, \tag{1.5}$$

that was already established in the unpublished manuscript [6]. More explicitly, recalling that the well-known Fourier transform of C_δ is (see also Section 3)

$$\widehat{C_\delta}(y) = \begin{cases} \frac{\sin^2(\pi\delta y)}{\pi^2\delta y^2} & \text{if} \quad y\neq 0,\\ \delta & \text{if} \quad y=0,\end{cases}$$

the inequality (1.3) becomes

$$\|S\|_{2,T}^2 \leqslant \frac{\pi^4\theta^2T^2}{\sin^4(\pi\theta)}\int_{\mathbb{R}}\Big|\sum_{|\nu-x|\leqslant\theta/T}\Big(1-\frac{T|\nu-x|}{\theta}\Big)s(\nu)\Big|^2\mathrm{d}x. \tag{1.6}$$

By applying (1.5), in Section 2 we prove also the following modified version of (1.2) for Dirichlet polynomials, but it may be easily generalized to absolutely convergent Dirichlet series.

Theorem 1. *For every Dirichlet polynomial $\sum_n a_n n^{it}$, one has, for $T\to\infty$,*

$$\Big\|\sum_n a_n n^{it}\Big\|_{2,T}^2 \ll T^2\int_1^{+\infty}\Big|\sum_n C_{y/T}(n-y)a_n\Big|^2\frac{\mathrm{d}y}{y}+\int_1^{+\infty}\Big(\sum_{y-\Delta\leqslant n\leqslant y+\Delta}|a_n|\Big)^2\frac{\mathrm{d}y}{y}, \tag{1.7}$$

where $\Delta=\Delta(y,T)\stackrel{\rm def}{=}\frac{y}{T}+\mathrm{O}\big(\frac{y}{T^2}\big)$.

Now let us consider the special case of Dirichlet polynomials approximating Dirichlet series on the critical line $\frac12+it$, namely

$$\sum_{N_1\leqslant n\leqslant N_2}\frac{w(n)b(n)}{n^{1/2+it}},$$

where N_1, N_2 are positive integers, w is uniformly bounded and supported in $[N_1,N_2]$, and b is an essentially bounded arithmetic function, that is $|b(n)|\ll_\varepsilon n^\varepsilon$ for $\varepsilon>0$. In Section 2, we show the following consequence of Theorem 1 by applying (1.7) with $a_n=w(n)b(n)n^{-1/2}$.

Corollary. *For every $\varepsilon > 0$, as $T \to \infty$, one has*

$$\Big\| \sum_{N_1 \leqslant n \leqslant N_2} \frac{w(n)b(n)}{n^{1/2+it}} \Big\|_{2,T}^2 \ll_\varepsilon T^2 \int_{N_1/2}^{3N_2/2} \Big| \sum_n C_{y/T}(n-y) \frac{w(n)b(n)}{n^{1/2}} \Big|^2 \frac{\mathrm{d}y}{y} + \frac{N_2^{1+\varepsilon}}{T^2}.$$

Beyond the possible interest of our results within the general context of the exponential sums and the possible further generalizations involving more Harmonic Analysis, here we wish to focus on another motivation for exploiting the particular case of (1.3) with the Cesàro weight. First, let us recall that, taking inspiration from the classical method introduced by Selberg [14] to study the distribution of the prime numbers in short intervals, the so-called Selberg integral of an arithmetic function f has been defined as (see [2])

$$\int_{hN^\varepsilon}^{N} \Big| \sum_{x<n\leqslant x+h} f(n) - M_f(x,h) \Big|^2 \mathrm{d}x,$$

where N is an arbitrarily large integer, the real number $\varepsilon > 0$ is arbitrarily small and $M_f(x,h)$ is the expected mean value of the inner sum in the short interval $(x, x+h]$ (as $h = \mathrm{o}(x)$ for $x \in [hN^\varepsilon, N]$). By a dyadic argument, it is easily seen that the interval $[hN^\varepsilon, N]$ can be replaced by $[N, 2N]$ and that negligible remainder terms are generated when the resulting integral on $[N, 2N]$ is substituted by the discrete mean square (compare [5]),

$$J_f(N,h) \stackrel{\mathrm{def}}{=} \sum_{x\sim N} \Big| \sum_{x<n\leqslant x+h} f(n) - M_f(x,h) \Big|^2,$$

where $x \sim N$ means that $x \in (N, 2N] \cap \mathbb{N}$. So that one still refers to $J_f(N,h)$ as the Selberg integral of f.

In this sense, if the coefficients $s(n)$ are assigned by an arithmetic function s supported over $(N, 2N] \cap \mathbb{N}$, then the right-hand side of the Gallagher inequality (1.1) is de facto the Selberg integral of s, whenever we assume that $M_s(x,h)$ vanishes identically (in this case, we say that s is balanced). Indeed,

$$J_s(N,\delta) = \sum_{x\sim N} \Big| \sum_{x<n\leqslant x+\delta} s(n) \Big|^2 \quad \text{with} \quad \delta = \mathrm{o}(N).$$

Analogously, it transpires that the mean square

$$\sum_{x\sim N} \Big| \sum_n C_\delta(n-x) s(n) \Big|^2$$

emulates the integral on the right-hand side of (1.5). Moreover, we showed that the inequality (see [5], Section 4)

$$\sum_{x\sim N} \Big| \sum_n C_\delta(n-x) s(n) \Big|^2 \ll \sum_{x\sim N} \Big| \sum_{x<n\leqslant x+\delta} s(n) \Big|^2 + \delta^3 \max_{N-\delta<n\leqslant 2N+\delta} |s(n)|^2$$

holds for every real and balanced function s. From this point of view, the inequality (1.5) can be proposed as a sort of refinement of Gallagher's inequality (1.1) (see Section 3 for further discussions in this direction).

More in general, let us point out that the same mean value $M_f(x,h)$ appears in both $J_f(N,h)$ and the *modified* Selberg integral of f, i.e.,

$$\widetilde{J}_f(N,h) \stackrel{\text{def}}{=} \sum_{x\sim N} \Big| \sum_n C_h(n-x)f(n) - M_f(x,h) \Big|^2.$$

According to Ivić [10], if the Dirichlet series $F(s)$ generated by f is meromorphic in $\mathbb{C}$ and absolutely convergent in the half-plane $\operatorname{Re} s > 1$ at least, then the mean value takes the analytic form

$$M_f(x,h) = hp_f(\log x),$$

where $p_f(\log x) \stackrel{\text{def}}{=} \operatorname{Res}_{s=1} F(s)x^{s-1}$ is the so-called logarithmic polynomial of f (compare [5]).

In [7], we exhibited the following length-inertia property for the Selberg integral, that allows to preserve non-trivial bounds as the length of the short interval increases: for $H \geqslant h$ with $h \to \infty$, $H = \mathrm{o}(N)$ when $N \to \infty$ one has

$$J_f(N,H) \ll \left(\frac{H}{h}\right)^2 J_f(N,h) + J_f\left(N, H - h\left[\frac{H}{h}\right]\right) + H^3\left(\|f\|_\infty^2 + (\log N)^{2c}\right),$$

where $[\cdot]$ denotes the integer part, $\|f\|_\infty \stackrel{\text{def}}{=} \max_{[N-H,2N+H]} |f|$ and c is the degree of the polynomial p_f.

In order to establish an analogous property for the modified Selberg integral of a real arithmetic function, we need to apply (1.5), while we underline that in [7] no Gallagher type inequality has been used in the proof of the length-inertia property for $J_f(N,H)$. Indeed, the last application of (1.5) in the present paper is devoted to prove the following result (see Section 2).

Theorem 2. *Let f be a real arithmetic function for which the logarithmic polynomial $p_f(\log n)$ is defined. For $H \geqslant h$ with $h \to \infty$, $H = \mathrm{o}(N)$ when $N \to \infty$ one has*

$$\widetilde{J}_f(N,H) \ll H^2 h^{-2} \widetilde{J}_f(N,h) + \left(Nh^4H^{-2} + H^3\right)\|f\|_\infty^2 + H^3(\log N)^{2c}.$$

After next section, that includes the proofs of the above results, in Section 3, we first describe the smoothing process of a weight via self-convolutions and compare it with the analogous process performed trough the so-called correlations, which have been introduced in [5]. Then we analyze the possible repercussions of such processes within the study of the weighted Selberg integral, with particular emphasis on the cases of the Cesàro weight and its relatives generated by iterations of the self-convolution. Finally, in Section 4, a comparison argument is introduced in view of refinements on the right-hand side of the Lemma inequalities, according to different instances of the weight w.

2. Proofs of the main results

Proof of Lemma. First, note that there is nothing to prove when the series

$$W_{\delta,s}(x) \stackrel{\text{def}}{=} \sum_{\nu} s(\nu) w_\delta(x-\nu)$$

is not square-integrable on $\mathbb{R}$, because the integral on the right-hand side of (1.3) is not finite. Otherwise, since by Lebesgue's dominated convergence theorem the Fourier transform of $W_{\delta,s}$ is

$$\begin{aligned}\widehat{W}_{\delta,s}(y) &\stackrel{\text{def}}{=} \int_{\mathbb{R}} W_{\delta,s}(x) e(-xy)\mathrm{d}x = \sum_{\nu} s(\nu) \int_{\mathbb{R}} w_\delta(x-\nu)e(-xy)\mathrm{d}x \\ &= \sum_{\nu} s(\nu)e(-\nu y) \int_{\mathbb{R}} w_\delta(t)e(-ty)\mathrm{d}t = S(-y)\widehat{w}_\delta(y),\end{aligned}$$

then (1.3) follows immediately by Plancherel's theorem

$$\int_{\mathbb{R}} |W_{\delta,s}(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\widehat{W}_{\delta,s}(y)|^2 \mathrm{d}y,$$

it being plain that

$$\int_{\mathbb{R}} |S(-y)\widehat{w}_\delta(y)|^2 \mathrm{d}y \geqslant \min_{|t|\leqslant T} |\widehat{w_\delta}(t)|^2 \int_{-T}^{T} |S(y)|^2 \mathrm{d}y.$$

Now, let us prove (1.4). We can clearly assume that

$$\sum_{\nu} s(\nu) w(x-\nu) \in L^2(\mathbb{R}),$$

and apply (1.3) with $\delta = n \in \mathbb{N}$ to write

$$\min_{|y|\leqslant T} |\widehat{w}_n(y)|^2 \int_{-T}^{T} |S(t)|^2 \mathrm{d}t \leqslant \int_{\mathbb{R}} \Big| \sum_{\nu} s(\nu) w_n(x-\nu) \Big|^2 \mathrm{d}x.$$

Since Fourier transforms are continuous functions, there exists $y_n \in [-T, T]$ such that

$$\min_{|y|\leqslant T} |\widehat{w}_n(y)|^2 = |\widehat{w_n}(y_n)|^2 = \Big| \int_{-n}^{n} w(t)e(-ty_n)dt \Big|^2.$$

We can also assume that y_n converges to some $y_0 \in [-T, T]$ (this is surely true for some subsequence of y_n). Consequently, $w_n(t)e(-ty_n)$ converges to $w(t)e(-ty_0)$, while it is plain that $|w_n(t)e(-ty_n)| \leqslant |w(t)|$. Since $w \in L^1(\mathbb{R})$, then the dominated convergence theorem yields

$$\lim_n \min_{|y|\leqslant T} |\widehat{w}_n(y)|^2 = \Big| \int_{\mathbb{R}} w(t)e(-ty_0)\mathrm{d}t \Big|^2 = |\widehat{w}(y_0)|^2 \geqslant \min_{|t|\leqslant T} |\widehat{w}(t)|^2.$$

On the other side, the same theorem implies that

$$\lim_n \int_{\mathbb{R}} \Big| \sum_\nu s(\nu) w_n(x-\nu) \Big|^2 \mathrm{d}x = \int_{\mathbb{R}} \Big| \sum_\nu s(\nu) w(x-\nu) \Big|^2 \mathrm{d}x.$$

Hence, (1.4) follows from the previous inequalities after passage to the limit as $n \to \infty$.

Proof of Theorem 1. Let us apply (1.5) to

$$\sum_n a_n n^{it} = \sum_\nu s(\nu) e(\nu t) \quad \text{with} \quad \nu = (2\pi)^{-1} \log n, \quad s(\nu) = a_{e(\nu/i)},$$

by taking $\theta = (2\pi)^{-1}$, $T = \theta\delta^{-1}$ and $x = \theta \log y$. Thus, we get

$$\begin{aligned}
&\Big\| \sum_n a_n n^{it} \Big\|_{2,T}^2 \\
&\quad \ll \delta^{-2} \int_{\mathbb{R}} \Big| \sum_{|\nu - x| \leqslant \delta} (1 - |\nu - x|\delta^{-1}) s(\nu) \Big|^2 \mathrm{d}x \\
&\quad \ll T^2 \int_0^{+\infty} \Big| \sum_{|\log n - \log y| \leqslant 1/T} (1 - T|\log n - \log y|) a_n \Big|^2 \frac{\mathrm{d}y}{y} \\
&\quad \ll T^2 \int_1^{+\infty} \Big| \sum_{y - (1-1/\tau)y \leqslant n \leqslant y + (\tau-1)y} \Big(1 - T\Big|\log\Big(1 + \frac{n-y}{y}\Big)\Big|\Big) a_n \Big|^2 \frac{\mathrm{d}y}{y},
\end{aligned}$$

where we have set $\tau \overset{\text{def}}{=} \mathrm{e}^{1/T} > 1$ (note that $\tau \to 1$ as $T \to \infty$, so the sum over n is empty for $0 < y < 1$).

Since Taylor expansion yields

$$y - \big(1 - \frac{1}{\tau}\big)y = y - \frac{y}{T} + \mathrm{O}\Big(\frac{y}{T^2}\Big) \quad \text{and} \quad y + (\tau - 1)y = y + \frac{y}{T} + \mathrm{O}\Big(\frac{y}{T^2}\Big),$$

then the Cesàro weight, $1 - T|\log(1 + (n-y)y^{-1})|$, is bounded for the present range of n, while we have

$$1 - T\Big|\log\Big(1 + \frac{n-y}{y}\Big)\Big| \ll \frac{1}{T}$$

in both ranges

$$0 \leqslant |n - (y - \frac{y}{T})| \ll \frac{y}{T^2} \quad \text{and} \quad 0 \leqslant |n - (y + \frac{y}{T})| \ll \frac{y}{T^2}.$$

Accordingly, we write

$$\Big\| \sum_n a_n n^{it} \Big\|_{2,T}^2$$

$$\ll T^2 \int_1^{+\infty} \Big| \sum_{y-y/T \leqslant n \leqslant y+y/T} \Big(1 - T\Big|\log\Big(1+\frac{n-y}{y}\Big)\Big|\Big) a_n \Big|^2 \frac{\mathrm{d}y}{y}$$
$$+ \int_1^{+\infty} \Big(\sum_{0 \leqslant |n-(y-y/T)| \ll y/T^2} |a_n| + \sum_{0 \leqslant |n-(y+y/T)| \ll y/T^2} |a_n| \Big)^2 \frac{\mathrm{d}y}{y}.$$

Then (1.7) follows, since by Taylor expansion again we have

$$T\Big|\log\Big(1+\frac{n-y}{y}\Big)\Big| - \frac{|n-y|}{y/T} \ll \frac{T(n-y)^2}{y^2} \ll \frac{1}{T}$$

for $y - \frac{y}{T} \leqslant n \leqslant y + \frac{y}{T}$, and this yields

$$T^2 \int_1^{+\infty} \Big| \sum_{y-y/T \leqslant n \leqslant y+y/T} \Big(1 - T\Big|\log\Big(1+\frac{n-y}{y}\Big)\Big|\Big) a_n \Big|^2 \frac{\mathrm{d}y}{y}$$
$$\ll T^2 \int_1^{+\infty} \Big| \sum_{y-y/T \leqslant n \leqslant y+y/T} \Big(1 - \frac{|n-y|}{y/T}\Big) a_n \Big|^2 \frac{\mathrm{d}y}{y}$$
$$+ \int_1^{+\infty} \Big(\sum_{-y/T \leqslant n-y \leqslant y/T} |a_n| \Big)^2 \frac{\mathrm{d}y}{y},$$

whence

$$\Big\| \sum_n a_n n^{it} \Big\|_{2,T}^2$$
$$\ll T^2 \int_1^{+\infty} \Big| \sum_{y-y/T \leqslant n \leqslant y+y/T} \Big(1 - \frac{|n-y|}{y/T}\Big) a_n \Big|^2 \frac{\mathrm{d}y}{y}$$
$$+ \int_1^{+\infty} \Big(\sum_{-\Delta \leqslant n-y \leqslant \Delta} |a_n| \Big)^2 \frac{\mathrm{d}y}{y},$$

where recall that $\Delta = \frac{y}{T} + \mathrm{O}\big(\frac{y}{T^2}\big)$.

Proof of Corollary. Let us apply Theorem 1 to $\sum_n a_n n^{-it}$ with $a_n = w(n)b(n)n^{-1/2}$ and write

$$\Big\| \sum_{N_1 \leqslant n \leqslant N_2} \frac{w(n)b(n)}{n^{1/2+it}} \Big\|_{2,T}^2$$
$$\ll_\varepsilon T^2 \int_1^{+\infty} \Big| \sum_{y-y/T \leqslant n \leqslant y+y/T} \Big(1 - \frac{|n-y|}{y/T}\Big) \frac{w(n)b(n)}{n^{1/2}} \Big|^2 \frac{\mathrm{d}y}{y}$$
$$+ N_2^\varepsilon \int_1^{+\infty} \Big(\sum_{y-\Delta \leqslant n \leqslant y+\Delta} \frac{1}{\sqrt{n}} \Big)^2 \frac{\mathrm{d}y}{y}.$$

Since w has the support in $[N_1, N_2]$, then by using $\Delta = \Delta(y,T) \ll \frac{y}{T}$ one has

$$\Big\| \sum_{N_1 \leqslant n \leqslant N_2} \frac{w(n)b(n)}{n^{1/2+it}} \Big\|_{2,T}^2$$

$$\begin{aligned}&\ll_\varepsilon T^2\int_{N_1/2}^{3N_2/2}\Big|\sum_{y-y/T\leqslant n\leqslant y+y/T}\Big(1-\frac{|n-y|}{y/T}\Big)\frac{w(n)b(n)}{n^{1/2}}\Big|^2\frac{\mathrm{d}y}{y}\\&+N_2^\varepsilon\int_{N_1/2}^{3N_2/2}\frac{\Delta(y,T)^2}{y^2}\mathrm{d}y\\&\ll_\varepsilon T^2\int_{N_1/2}^{3N_2/2}\Big|\sum_{y-y/T\leqslant n\leqslant y+y/T}\Big(1-\frac{|n-y|}{y/T}\Big)\frac{w(n)b(n)}{n^{1/2}}\Big|^2\frac{\mathrm{d}y}{y}+N_2^{1+\varepsilon}\frac{1}{T^2}.\end{aligned}$$

Hence, Corollary is proved.

Before going to the proof of Theorem 2, let us prove an auxiliary proposition.

PROPOSITION 1. *Let $w:\mathbb{R}\to\mathbb{C}$ be an uniformly bounded weight and let f be an arithmetic function for which the logarithmic polynomial $p_f(\log n)$ is defined. For every fixed real number $\delta>0$, one has*

$$\begin{aligned}&\sum_{x\sim N}\Big|\sum_n w_\delta(n-x)f(n)-p_f(\log x)\sum_n w_\delta(n-x)\Big|^2\\&\quad\ll\sum_{x\sim N}\Big|\sum_n w_\delta(n-x)\tilde f(n)\Big|^2+N^{-1}\delta^4(\log N)^{2c-2},\end{aligned}$$

where $\widetilde{f}(n)\overset{\mathrm{def}}{=}f(n)-p_f(\log n)$ assigns the balanced part of f, and c is the degree of the polynomial p_f.

Proof. Since

$$\begin{aligned}&\sum_{x\sim N}\Big|\sum_n w_\delta(n-x)f(n)-p_f(\log x)\sum_n w_\delta(n-x)\Big|^2\\&\quad\ll\sum_{x\sim N}\Big|\sum_n w_\delta(n-x)\tilde f(n)\Big|^2\\&\qquad+\sum_{x\sim N}\Big|\sum_{0\leqslant|n-x|\leqslant\delta}w(n-x)(p_f(\log n)-p_f(\log x))\Big|^2,\end{aligned}$$

then the conclusion follows immediately from the mean value theorem applied to $p_f(\log n)-p_f(\log x)$.

Proof of Theorem 2. First, we recall here that, for every $x\in\mathbb{R}$, one has

$$p_f(\log x)\sum_n C_h(n-x)=hp_f(\log x)=M_f(x,h).$$

For a balanced function s supported over $(N,2N]\cap\mathbb{N}$, by taking $\delta=h$ and $T=\frac{1}{2h}$ in (1.6) one sees that

$$\int_{-\frac{1}{2h}}^{\frac{1}{2h}}\Big|\sum_{n\sim N}s(n)e(n\alpha)\Big|^2\mathrm{d}\alpha\ \leqslant\ \frac{\pi^4}{16h^2}\int_{\mathbb{R}}\Big|\sum_n C_h(n-x)s(n)\Big|^2\mathrm{d}x$$

$$\ll \frac{1}{h^2}\widetilde{J}_s(N,h) + h\,\|s\|_\infty^2 .$$

On the other side, by taking $w_\delta = C_\delta$ with $\delta = h \leqslant H$ in the previous proposition and recalling that $\tilde{f}$ is the balanced part of f, one gets

$$\widetilde{J}_f(N,h) - \widetilde{J}_{\tilde{f}}(N,h) \ll N^{-1}h^4(\log N)^{2c-2} \ll H^3(\log N)^{2c}.$$

Thus, without loss of generality we can assume that f is balanced. Since by hypothesis f is also real, we can apply the second formula[1] of (59) in [5] and write

$$\widetilde{J}_f(N,H) \ll \frac{1}{H^2}\int_{-1/2}^{1/2}\Big|\sum_{n\sim N} f(n)e(n\alpha)\Big|^2\Big|\sum_{1\leqslant n\leqslant H} e(n\alpha)\Big|^4 d\alpha + H^3\,\|f\|_\infty^2 .$$

Let us recall that such exponential sums satisfy the well-known properties

$$\sum_{1\leqslant n\leqslant H} e(n\alpha) \ll \min\left(H, \frac{1}{|\alpha|}\right) \quad \text{for } |\alpha| \leqslant \frac{1}{2},$$

and

$$\int_{|\alpha|\leqslant 1/2}\Big|\sum_{n\sim N} f(n)e(n\alpha)\Big|^2 d\alpha \ll \sum_{n\sim N} f(n)^2.$$

Now from all the previous inequalities it follows

$$\begin{aligned}
\widetilde{J}_f(N,H) &\ll H^2\int_{-1/(2h)}^{1/(2h)}\Big|\sum_{n\sim N} f(n)e(n\alpha)\Big|^2 \mathrm{d}\alpha \\
&\quad + \frac{h^4}{H^2}\int_{1/(2h)<|\alpha|\leqslant 1/2}\Big|\sum_{n\sim N} f(n)e(n\alpha)\Big|^2 \mathrm{d}\alpha + H^3\,\|f\|_\infty^2 \\
&\ll \frac{H^2}{h^2}\left(\widetilde{J}_f(N,h) + h^3\,\|f\|_\infty^2\right) \\
&\quad + \frac{h^4}{H^2}\int_{|\alpha|\leqslant 1/2}\Big|\sum_{n\sim N} f(n)e(n\alpha)\Big|^2 \mathrm{d}\alpha + H^3\,\|f\|_\infty^2 \\
&\ll \frac{H^2}{h^2}\left(\widetilde{J}_f(N,h) + h^3\,\|f\|_\infty^2\right) + \frac{h^4}{H^2}N\,\|f\|_\infty^2 + H^3\,\|f\|_\infty^2 ,
\end{aligned}$$

which implies the desired conclusion.

3. Two parallel ways of smoothing weights: self-convolution and autocorrelation

For every locally summable weight $w : \mathbb{R} \to \mathbb{C}$ and for some real number $\delta > 0$, let us consider the normalized self-convolution of w_δ given as

$$\widetilde{w}_\delta(x) \stackrel{\text{def}}{=} \frac{1}{2\delta}(w_\delta * w_\delta)(x) = \frac{1}{2\delta}\int_{\mathbb{R}} w_\delta(t)w_\delta(x-t)\mathrm{d}t.$$

[1] In [5] formulæ (59) are given in the context of a discussion about real and balanced functions that are supposed to be also essentially bounded. However, it is easy to see that the latter assumption is in fact redundant for such formulæ.

For example, the Cesàro weight C_δ is the normalized self-convolution of the restriction to $\left[-\frac{\delta}{2}, \frac{\delta}{2}\right]$ of $\mathbf{1}$, the constantly 1 function:

$$C_\delta(x) = \frac{1}{\delta} \int_{\substack{|t|\leqslant \delta/2 \\ |x-t|\leqslant \delta/2}} \mathrm{d}t = \frac{1}{\delta}(\mathbf{1}_{\delta/2} * \mathbf{1}_{\delta/2})(x) = \widetilde{\mathbf{1}}_{\delta/2}(x).$$

It is well known that the iteration of the self-convolution gives rise to a process of smoothing (see [1]). Moreover, the support of $\widetilde{w}_\delta$ is doubled with respect to the support of w_δ in the sense that it is a subset of $\left[-2\delta, 2\delta\right]$. Because of the normalizing factor $(2\delta)^{-1}$, that takes into account the length of the integration interval, the magnitude of w_δ is not altered much by the normalized self-convolution. More precisely, if one has $w_\delta \asymp 1$, i.e., $1 \ll w_\delta \ll 1$, in an interval of length $\gg \delta$, then there exists an interval of length $\gg \delta$ (not necessarily the same) where $\widetilde{w}_\delta \asymp 1$. From another well-known property of the convolution, it follows that the Fourier transform of $\widetilde{w}_\delta$ is

$$\widehat{\widetilde{w}_\delta}(y) = \frac{\widehat{w}_\delta(y)^2}{2\delta}.$$

In particular, from

$$\widehat{\mathbf{1}}_\delta(y) = \int_{-\delta}^{\delta} e(-ty)\mathrm{d}t = 2\delta\mathrm{sinc}(2\delta y) = \begin{cases} \frac{\sin(2\pi\delta y)}{\pi y} & \text{if} \quad y \neq 0, \\ 2\delta & \text{if} \quad y = 0, \end{cases}$$

we find that

$$\widehat{C}_\delta(y) = \widehat{\widetilde{\mathbf{1}}}_{\delta/2}(y) = \frac{\widehat{\mathbf{1}}_{\delta/2}(y)^2}{\delta} = \delta\mathrm{sinc}^2(\delta y).$$

Starting with $\mathbf{1}$, the normalized self-convolution generates recursively the family of Cesàro weights:

$$C_\delta^{(j)}(x) \stackrel{\text{def}}{=} \widetilde{C}_{\delta/2}^{(j-1)}(x), \quad j \geqslant 1,$$

with the base steps $C_\delta^{(0)}(x) \stackrel{\text{def}}{=} \mathbf{1}_\delta(x)$ and $C_\delta^{(1)}(x) \stackrel{\text{def}}{=} C_\delta(x)$.

Correspondingly, we have an inductive formula for the Fourier trasforms of these Cesàro weights.

Proposition 2. *For every $j \geqslant 0$ and every real number $\delta > 0$,*

$$\widehat{C}_\delta^{(j)}(y) = \frac{4\delta}{2^{2^j}}\mathrm{sinc}^{2^j}\left(\frac{\delta y}{2^{j-1}}\right).$$

Consequently, $\widehat{C}_\delta^{(j)}(y) \asymp_j \delta$ at least for $0 \leqslant |y| \leqslant 2^{j-2}\delta^{-1}$.

Proof. The case $j = 0$ is the above formula $\widehat{\mathbf{1}}_\delta$. For $j \geqslant 1$, note that

$$\widehat{C}_\delta^{(j)}(y) = \widehat{\widetilde{C}}_{\delta/2}^{(j-1)}(y) = \delta^{-1}\widehat{C}_{\delta/2}^{(j-1)}(y)^2,$$

and more in general for $j \geqslant k \geqslant 0$, it is easily seen that

$$\widehat{C}_\delta^{(j)}(y) = \frac{2^{a_k}}{\delta^{b_k}} \widehat{C}_{\delta/2^k}^{(j-k)}(y)^{2^k},$$

where

$$a_k \overset{\text{def}}{=} \sum_{i=1}^{k-1} i2^i = k2^k - 2(2^k - 1) \quad \text{and} \quad b_k \overset{\text{def}}{=} \sum_{i=0}^{k-1} 2^i = 2^k - 1.$$

By taking $k = j-1$ and setting $J = 2^k = 2^{j-1}$ for brevity, we obtain the stated formula

$$\widehat{C}_\delta^{(j)}(y) = \frac{J^J}{(4\delta)^{J-1}} \widehat{C}_{\delta/J}(y)^J = \frac{\delta}{4^{J-1}} \operatorname{sinc}^{2J}\Big(\frac{\delta}{J} y\Big).$$

The remaining part of the statement is a straightforward consequence of this formula.

Such a process of continuous smoothing through the self-convolution of a weight w_δ has a discrete counterpart given by the autocorrelation[2] of w_δ:

$$\mathfrak{C}_{w_\delta}(a) \overset{\text{def}}{=} \sum_{\substack{n \\ n-m=a}} \sum_{m} w_\delta(n)\overline{w_\delta(m)}.$$

For example, since it turns out that

$$C_\delta(t) = \frac{1}{\delta} \sum_{a \leqslant \delta - |t|} 1 = \frac{1}{\delta} \sum_{\substack{a,b \leqslant \delta \\ b-a=t}} \sum 1 = \frac{\mathfrak{C}_{u_\delta}(t)}{\delta},$$

then the Cesàro weight is the normalized correlation of the unit step weight $u_\delta = \mathbf{1}_{[1,\delta]}$. Note that the Cesàro weight is generated by both type of smoothing from a constantly one function. Moreover, through an iteration of the normalized correlation one might parallel the self-convolution process to generate the whole family of Cesàro weights $C_\delta^{(j)}$ with $j \geqslant 1$.

An important aspect is that the exponential sums, whose coefficients are correlations of a weight w, are non-negative. More precisely,

$$\sum_h \mathfrak{C}_{w_\delta}(h) e(h\alpha) = \sum_h \sum_{n-m=h} \sum w_\delta(n)\overline{w_\delta(m)} e(h\alpha) = \Big| \sum_n w_\delta(n) e(n\alpha) \Big|^2.$$

A particularly well-known case is the Fejér kernel

$$\delta \sum_h C_\delta(h) e(h\alpha) = \sum_h \mathfrak{C}_{u_\delta}(h) e(h\alpha) = \Big| \sum_{1 \leqslant n \leqslant \delta} e(n\alpha) \Big|^2.$$

[2]Consistently with [5], since no confusion can arise in the following, we will use the simpler term correlation.

Such a positivity property is the complete analogous of the aforementioned fact that the Fourier transform of a self-convolution is a square. For this reason, in [5] by abuse of notation we write

$$\widehat{w}_\delta(\alpha) = \sum_n w_\delta(n)e(n\alpha) \quad \text{and} \quad \widehat{\mathfrak{C}}_{w_\delta}(\alpha) = \sum_h \mathfrak{C}_{w_\delta}(h)e(h\alpha),$$

and refer to such exponential sums as the discrete Fourier transform of w_δ and $\mathfrak{C}_{w_\delta}$, respectively.

By arguing as [5] (compare formulæ (59) for the weighted Selberg integral $J_{w,f}$ of a real and balanced function f), the positivity property provides the following alternative viewpoint of Gallagher's inequality:

$$\begin{aligned} J_{w,f}(N,H) &\stackrel{\text{def}}{=} \sum_{x\sim N} \Big| \sum_n w_H(n-x)f(n) \Big|^2 \\ &= \int_{-1/2}^{1/2} \Big| \sum_{n\sim N} f(n)e(n\alpha) \Big|^2 |\widehat{w}_H(\alpha)|^2 \,\mathrm{d}\alpha + \mathrm{O}\left(H^3 \|f\|_\infty^2\right), \end{aligned}$$

where $\|f\|_\infty \stackrel{\text{def}}{=} \max_{N-H<n\leqslant 2N+H} |f(n)|$ and $\widehat{w}_H$ is the discrete Fourier transform of w_H.

Trivially, this formula yields the inequality

$$\min_{|y|\leqslant 1/(2H)} |\widehat{w}_H(y)|^2 \int_{-1/(2H)}^{1/(2H)} \Big| \sum_{n\sim N} f(n)e(n\alpha) \Big|^2 \mathrm{d}\alpha \leqslant J_{w,f}(N,H) + \mathrm{O}\left(H^3 \|f\|_\infty^2\right).$$

In particular, recalling Proposition 2, we write

$$\int_{-1/(2H)}^{1/(2H)} \Big| \sum_{n\sim N} f(n)e(n\alpha) \Big|^2 \mathrm{d}\alpha \ll_j H^{-2} \tilde{J}_f^{(j)}(N,H) + H \|f\|_\infty^2,$$

where

$$J_f^{(j)}(N,H) \stackrel{\text{def}}{=} \sum_{x\sim N} \Big| \sum_n C_H^{(j)}(n-x)f(n) \Big|^2$$

is what we might call the jth modified Selberg integral of the balanced function f.

Such Gallagher type inequalities open the possibility of improving on the right-hand side term by suitably picking a smoother weight from the Cesàro family, while Proposition 2 ensures that the order of magnitude of $\min_{|y|\leqslant 1/(2H)} \big|\widehat{C}_H^{(j)}(y)\big|^2 \asymp_j H^2$ is substantially unaffected by different choices of j. In this sense, our results of [5] have already shed a light on the relation between the cases $j = 0$ (the Selberg integral) and $j = 1$ (the modified Selberg integral). Finally, an additional feature offered by the discrete smoothing via correlations of a weight w is a wider number theoretical perspective since correlations can be plainly interpreted as a weighted count of solutions $n, m \sim N$ of the diophantine equation $n - m = a$.

4. Comparing weights in view of Gallagher's generalized inequality

Bearing in mind the considerations of the previous section, here we compare weights in view of possible refinements of the right-hand side of Gallagher's inequality (1.3), here written as (see the proof of Lemma)

$$\int_{-T}^{T} |S(t)|^2 \mathrm{d}t \leqslant m_{\delta,T}^{-1} \int_{\mathbb{R}} |S(-y)\widehat{w}_\delta(y)|^2 \mathrm{d}y,$$

after assuming that $m_{\delta,T} \stackrel{\mathrm{def}}{=} \min_{|t|\leqslant T} |\widehat{w}_\delta(t)|^2 \neq 0$. To this end, we give the following definition.

Definition. Suppose that (1.3) holds for weights w_δ and v_δ such that

$$m_{\delta,T} \stackrel{\mathrm{def}}{=} \min_{|t|\leqslant T} |\widehat{w}_\delta(t)|^2 \neq 0 \quad \text{and} \quad r_{\delta,T} \stackrel{\mathrm{def}}{=} \min_{|t|\leqslant T} |\widehat{v}_\delta(t)|^2 \neq 0,$$

respectively. We say that v_δ is T-better than w_δ if

$$\frac{r_{\delta,T}}{m_{\delta,T}} \geqslant \frac{|\widehat{v_\delta}(y)|^2}{|\widehat{w_\delta}(y)|^2}$$

for all $y \in \mathbb{R}$.

If so, it is plain that v_δ yields a refinement of (1.3) with respect to w_δ. Further, assuming that both weights are also positive, when $|y| \leqslant T$, one has the following upper bound for the gain

$$\begin{aligned} \frac{|\widehat{w}_\delta(y)|^2}{m_{\delta,T}} - \frac{|\widehat{v}_\delta(y)|^2}{r_{\delta,T}} &= \frac{(|\widehat{w}_\delta(y)|^2 - m_{\delta,T})|\widehat{v}_\delta(y)|^2}{m_{\delta,T} r_{\delta,T}} \\ &\leqslant \Big(\frac{||w_\delta||_1^2}{m_{\delta,T}} - 1\Big)\frac{||v_\delta||_1^2}{r_{\delta,T}} = \Big(\frac{|\widehat{w_\delta}(0)|^2}{m_{\delta,T}} - 1\Big)\frac{|\widehat{v}_\delta(0)|^2}{r_{\delta,T}}, \end{aligned}$$

where $\|\cdot\|_1$ denotes the L^1 norm.

Example 1 (the Cesàro weights). Let us start by comparing $\mathbf{1}_\delta$ and C_δ. Accordingly to the previous definition, C_δ is T-better than $\mathbf{1}_\delta$ if

$$\frac{|\widehat{\mathbf{1}}_\delta(y)|^2}{|\widehat{C}_\delta(y)|^2} \geqslant \frac{\min_{|t|\leqslant T} |\widehat{\mathbf{1}}_\delta(t)|^2}{\min_{|t|\leqslant T} |\widehat{C}_\delta(t)|^2} \quad \text{for every} \quad y \in \mathbb{R}, \tag{4.8}$$

that is trivially true when $\widehat{\mathbf{1}}_\delta(y) = 0$ for some $y \in [-T, T]$. Therefore, let us assume $\theta \stackrel{\mathrm{def}}{=} \delta T \in \big(0, \frac{1}{2}\big)$ for $\delta, T > 0$, so that (see Section 3)

$$\min_{|t|\leqslant T} |\widehat{\mathbf{1}}_\delta(t)|^2 = \frac{\sin^2(2\pi\delta T)}{\pi^2 T^2} \neq 0 \quad \text{and} \quad \min_{|t|\leqslant T} |\widehat{C}_\delta(t)|^2 = \frac{\sin^4(\pi\delta T)}{\pi^4 T^4 \delta^2} \neq 0.$$

Note that (4.8) is true when $y = 0$, since, for $\delta T \in \left(0, \frac{1}{2}\right)$, one sees that

$$\frac{\min_{|t|\leqslant T}|\widehat{\mathbf{1}}_\delta(t)|^2}{\min_{|t|\leqslant T}|\widehat{C}_\delta(t)|^2} = \frac{4(\pi\delta T)^2}{\tan^2(\pi\delta T)} \leqslant 4 = \frac{|\widehat{\mathbf{1}}_\delta(0)|^2}{|\widehat{C}_\delta(0)|^2}.$$

Thus, we consider the case $y \neq 0$ and set $x \stackrel{\text{def}}{=} \frac{y}{T}$ in (4.8), that becomes

$$G_\theta(x) \stackrel{\text{def}}{=} \frac{|\widehat{C}_\delta(xT)|^2}{|\widehat{\mathbf{1}}_\delta(xT)|^2} = \frac{\tan^2(\pi\theta x)}{(2\pi\theta x)^2} \leqslant G_\theta(1) \quad \text{for every} \quad x \in \mathbb{R}\setminus\{0\}.$$

It is easy to see that $G_\theta(x)$ satisfies the following properties:

(1) $G_\theta(x)$ is even with respect to both x and θ;

(2) $G_\theta(x)$ is strictly increasing with respect to $x \in (0, 1]$;

(3) $\lim_{x\to 0} G_\theta(x) = \frac{1}{4}$ for all $\theta \in \left(0, \frac{1}{2}\right)$;

(4) $G_\theta(1)$ is strictly increasing with respect to $\theta \in \left(0, \frac{1}{2}\right)$;

(5) $\lim_{\theta\to 0} G_\theta(1) = \frac{1}{4}$, $\lim_{\theta\to 1/2} G_\theta(1) = +\infty$;

(6) $G_\theta(x) = 0$ if and only if $x = \frac{k}{\theta}$ for all $k \in \mathbb{Z}\setminus\{0\}$ (note that for all $k \in \mathbb{Z}\setminus\{0\}$ and for all $\theta \in \left(0, \frac{1}{2}\right)$ one has $\frac{|k|}{\theta} > 2|k| \geqslant 2$);

(7) $G_\theta(x) \to +\infty$ as $x \to (2k+1)(2\theta)^{-1}$ for all $k \in \mathbb{Z}$ (note that for all $k \in \mathbb{Z}$ and for all $\theta \in \left(0, \frac{1}{2}\right)$ one has $|2k+1|(2\theta)^{-1} > |2k+1| \geqslant 1$).

From properties (1)–(5), it follows that the above inequality for G_θ is true when $x \in [-1, 1]\setminus\{0\}$, while properties (1), (6) and (7) imply that it is true for $|x| \in \mathcal{I}_{\theta,k} \stackrel{\text{def}}{=} [(2k+1)(2\theta)^{-1} + \Delta'_k, (2k+3)(2\theta)^{-1} - \Delta''_k]$ for all $k \in \mathbb{N}$, where both Δ'_k and Δ''_k tend to 0^+ as $k \to +\infty$. Since $\mathcal{I}_{\theta,k}$ tends to cover the whole interval $[2k+1, 2k+3]$ as $\theta \to \frac{1}{2}$ and $k \to +\infty$, then we deduce that (4.8) is true for all $|y| \leqslant T$, and almost everywhere for $|y| > T$ as $\delta T \to \frac{1}{2}$. Thus, we say that C_δ is almost T-better than $\mathbf{1}_\delta$ when $\delta T \to \frac{1}{2}$.

More generally, according to Proposition 2 of Section 3, for every $j \geqslant 0$, one has

$$\frac{|\widehat{C}_\delta^{(j+1)}(y)|^2}{|\widehat{C}_\delta^{(j)}(y)|^2} = \left(\frac{\tan(\pi\delta y/2^j)}{\delta\pi y/2^{j-1}}\right)^{2^{j+1}} = \left(\frac{\tan(\pi\theta x)}{2\pi\theta x}\right)^{2^{j+1}} = G_\theta(x)^{2^j},$$

where we have set $\theta = \frac{\delta T}{2^j}$, $x = \frac{y}{T}$ and $G_\theta(x) = \frac{\tan^2(\pi\theta x)}{(2\pi\theta x)^2}$ as before.

Hence, we conclude that $C_\delta^{(j+1)}$ is almost T-better than $C_\delta^{(j)}$ for every $j \geqslant 0$, whenever $\delta T \to 2^{j-1}$.

Remark. An effective use of (1.3) with $C_\delta^{(j)}$ requires finding explicit expressions of such weights. For example, the so-called Jackson-de La Vallé Poussin weight $C_\delta^{(2)}$

(given by the normalized self-convolution of $C_{\delta/2}$) is the following cubic spline (see [1], Problem 5.1.2 (v)):

$$\delta^{-1}C_{\delta/2}*C_{\delta/2}(t)=\begin{cases}\frac{6|t|^3-6\delta t^2+\delta^3}{3\delta^3} & \text{if } \ |t|\leqslant \frac{\delta}{2},\\ \frac{2(\delta-|t|)^3}{3\delta^3} & \text{if } \ \frac{\delta}{2}<|t|\leqslant\delta,\\ 0 & \text{if } \ |t|>\delta.\end{cases}$$

Note that the support of $C_\delta^{(2)}$ is $[-\delta,\delta]$, as expected. Evidently one could push forward the process by comparing arbitrary powers of $\widehat{\mathbf{1}}_\delta$. However, the comparison between odd and even powers seems to be cumbersome.

Example 2. Given real numbers $\delta\geqslant\Delta>0$, the Lanczos weight[3] is defined as (see [1], Problem 5.1.2 (v))

$$\mathcal{L}_{\delta,\Delta}(x)\overset{\text{def}}{=}\frac{1}{\Delta}(\mathbf{1}_{\delta-\Delta/2}*\mathbf{1}_{\Delta/2})(x)=\begin{cases}1 & \text{if } \ |x|\leqslant\delta-\Delta,\\ \frac{\delta-|x|}{\Delta} & \text{if } \ \delta-\Delta<|x|\leqslant\delta,\\ 0 & \text{if } \ |x|>\delta,\end{cases}$$

whose Fourier transform is

$$\widehat{\mathcal{L}}_{\delta,\Delta}(y)=(2\delta-\Delta)\mathrm{sinc}(\Delta y)\mathrm{sinc}((2\delta-\Delta)y)=\begin{cases}\frac{\sin(\pi\Delta y)\sin(2\pi\delta y-\pi\Delta y)}{(\pi y)^2\Delta} & \text{if } \ y\neq 0,\\ 2\delta-\Delta & \text{if } \ y=0.\end{cases}$$

Since $\mathcal{L}_{\delta,\delta}=C_\delta$, then we can assume that $\delta>\Delta$. Let us compare the weights $\mathcal{L}_{\delta,\Delta}$ and $\mathbf{1}_\delta$ by verifying the inequality

$$\frac{|\widehat{\mathbf{1}}_\delta(y)|^2}{|\widehat{\mathcal{L}}_{\delta,\Delta}(y)|^2}=\frac{(2\delta)^2\mathrm{sinc}^2(2\delta y)}{(2\delta-\Delta)^2\mathrm{sinc}^2(\Delta y)\mathrm{sinc}^2((2\delta-\Delta)y)}\geqslant\frac{\min_{|t|\leqslant T}|\widehat{\mathbf{1}}_\delta(t)|^2}{\min_{|t|\leqslant T}|\widehat{\mathcal{L}}_{\delta,\Delta}(t)|^2},$$

for every real y, and by assuming that $0<\Delta T<\delta T<\frac{1}{2}$. Then, it is easy to see that such inequality is satisfied by $y=0$. Therefore, for $y\neq 0$, we can write the left-hand side as

$$\frac{|\widehat{\mathbf{1}}_\delta(y)|^2}{|\widehat{\mathcal{L}}_{\delta,\Delta}(y)|^2}=\frac{(\Delta\pi y)^2\sin^2(2\delta\pi y)}{\sin^2(\Delta\pi y)\sin^2((2\delta-\Delta)\pi y)}=\Big(\frac{\Delta\pi y}{\tan((2\delta-\Delta)\pi y)}+\frac{\Delta\pi y}{\tan(\Delta\pi y)}\Big)^2.$$

Since $0<\delta T<\frac{1}{2}$, then

$$\min_{|t|\leqslant T}|\widehat{\mathbf{1}}_\delta(t)|^2=\frac{\sin^2(2\pi\delta T)}{(\pi T)^2},$$

which goes to 0 as $\delta T\to\frac{1}{2}$. Hence, we conclude that for $0<\Delta T<\delta T<\frac{1}{2}$ the Lanczos weight $\mathcal{L}_{\delta,\Delta}$ is almost T-better than $\mathbf{1}_\delta$ when $\delta T\to\frac{1}{2}$. In a complete analogous way, we also see that, under the same conditions, the Cesàro weight C_δ is almost T-better than $\mathcal{L}_{\delta,\Delta}$.

[3] In the literature, the Fourier transform $\widehat{\mathcal{L}}_{\delta,\Delta}$ is known as Lanczos kernel. The diagram of $\mathcal{L}_{\delta,\Delta}$ is an isosceles trapezium. Note that $\mathbf{1}_\delta\geqslant\mathcal{L}_{\delta,\Delta}\geqslant C_\delta=\mathcal{L}_{\delta,\delta}$ for any $\delta\geqslant\Delta>0$.

5. Final considerations

Because of the averaging over the inner short sum coming from Cesàro weights, one could expect that, under suitable conditions on f, the modified Selberg integral $\widetilde{J}_f(N,h)$ should be more easily approachable than $J_f(N,h)$ (compare [5]). The process of smoothing described in Section 3 and the comparison study in Section 4 make us to foretell that such a relaxing behavior might be hopefully shared by every jth modified Selberg integral. Noteworthy, with the aid of (1.5) the first author [4] has recently found a way to deduce upper bounds for $J_f(N,h)$ from hypothetical estimates for $\widetilde{J}_f(N,h)$ by assuming that f is balanced and essentially bounded. An application of such a method leads to a non-trivial estimate, $J_3(N,h) \ll N^{1+\varepsilon}h^{6/5}$, for the Selberg integral of the divisor function d_3, whenever one assumes that the sharp bound for the modified Selberg integral, $\widetilde{J}_3(N,h) \ll N^{1+\varepsilon}h$, holds for every positive integer $h \ll N^{1/3}$ and for every real number $\varepsilon > 0$. This has to be compared with the unconditional lower bound $Nh\log^4 N \ll J_3(N,h)$ for $h \ll N^{1/3-\varepsilon}$ proved in [3]. In [5], we analyzed such conjecture on $\widetilde{J}_3(N,h)$ and analogous hypothesis on the modified Selberg integral $\widetilde{J}_k(N,h)$ of the divisor function d_k, where $d_k(n)$ is the number of ways to write n as a product of k positive integer factors. The importance of investigating about alternative ways of estimating the Selberg integral $J_k(N,h)$ of d_k relies mainly on its strict connection with the $2k$th moment of the Riemann zeta on the critical line (see [2]). In particular, under the aforementioned conjectural bound of $\widetilde{J}_3(N,h)$, such an approach leads to the so-called weak sixth moment for the Riemann zeta function (see [5], section 8). Finally, in the next future we are going to extend the results of the present paper to the general case of a weight w that satisfies the hypothesis of the Lemma, mainly in view of possible applications to the study of the weighted Selberg integral $J_{w,f}(N,H)$ under suitable conditions on the function f.

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