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THE POISEUILLE TYPE SOLUTION FOR THE NON-STATIONARY STOKES PROBLEM IN THE INFINITE PERIODIC PIPE

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Abstract. Fluids dynamic is one of the main applications of PDE in physics. In the paper, the non-stationary Stokes problem with a given flux condition is considered. The problem is analyzed in the three dimensional infinite domain (pipe) which is periodic with respect to one of the axis. The aim of the article is to prove the existence and uniqueness of the Poiseuille type solution. For solving the problem, the Galerkin approximation method is chosen, and the convergence of the constructed series and the uniqueness of the solution is proved. Under the flux condition, the unique existence of the axial pressure drop function is proved.

Key words and phrases: existence and uniqueness of solution, infinite domain, Galerkin method, non-stationary Stokes equation, periodic pipe, Poiseuille type solution, Volterra integral equation.

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1. Introduction

In this paper, we study the mathematical model of the incompressible, homogeneous flow in the 3-dimensional infinite Lipschitz domain, periodic with respect to x_3 axis,

$$\Pi = \{x = (x', x_3) \in \mathbb{R}^3 \mid (x', x_3) \in \sigma_z \times \mathbb{R},\ z \in \mathbb{R}\},$$

where $x' = (x_1, x_2)$, and $\sigma_z = \{x \in \Pi \mid x_3 = z\}$ denotes the cross-section of the tube, which can be different for each z. Let us denote by L the period of the pipe and by Π_L the area between the plots $x_3 = -L$ and $x_3 = L$, i.e.,

$$\Pi_L = \{x \mid x = (x', x_3) \in \sigma_z \times [-L, L],\ z \in [-L, L]\}.$$

For simplicity, denote by σ the cross section which plot is the biggest, i.e.,

$$\sigma = \max_{z\in[-L,L]} |\sigma_z|,$$

where $|\sigma_z|$ means the plot of the cross section. For the external boundary of the periodicity cell Π_L, we use the notation $S_L = \partial\Pi_L \backslash (\sigma_{-L} \cup \sigma_L)$.

In the prescribed periodic domain, we solve the non-stationary Stokes problem with homogenous conditions

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0 \quad \text{in} \quad \Pi, \\ \operatorname{div}\, \mathbf{u} = 0 \quad \text{in} \quad \Pi, \\ \mathbf{u}|_{\partial\Pi} = 0, \quad \mathbf{u}(x,0) = 0 \quad \text{in} \quad \Pi, \\ \mathbf{u}(x', x_3 - L, t) = \mathbf{u}(x', x_3 + L, t) \quad \text{in} \quad \Pi \end{cases} \tag{1}$$

with the prescribed flux

$$\int_\sigma u_3(x,t)\mathrm{d}x' = F(t). \tag{2}$$

Let us remark that the flux $F(t) \in W_2^1(0,t)$ is prescribed and is the same for all cross sections σ_z because of the divergence free condition. In problem (1), the vector-function $\mathbf{u} = (u_1, u_2, u_3)$ means the velocity, p describes the pressure field, and ν is the viscosity coefficient.

The motion of the Stokes and Navier-Stokes fluid in an infinite straight pipe with the constant cross section is one of the most studied problems in the fluid dynamics, e.g., [2, 5, 8, 9, 14]. One of the fundamental motions is the so-called Poiseuille type flow. It is well known that the pressure field for such a flow is characterized by an axial gradient $q(t)$. In the infinite cylinders, the Poiseuille type flow was considered in several papers, e.g., [6, 10, 11, 12]. In the periodic infinite tubular domains, the existence and uniqueness of the Poiseuille type solution for the stationary Stokes and Navier-Stokes equations where proved in [7].

This article, by its structure and used methodology, is similar to that in [6, 10]. However, the periodicity of the domain requires to define special function spaces and to be careful integrating by parts. Moreover, the totally different basis has to be used looking for the solutions.

In this paper, we are looking for the Poiseuille type solutions and searching for the pressure function $p(x,t)$ having the form

$$p(x,t) = -q(t)x_3 + p_0(t) + \widetilde{p}(x,t), \tag{3}$$

where $p_0(t)$ is an arbitrary function, and $\widetilde{p}(x,t)$ is a L-periodic function with respect to x_3 axis.

Substituting expression (3) into (1), we get

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\widetilde{p} = qe_3 \quad \text{in} \quad \Pi, \\ \text{div } \mathbf{u} = 0 \quad \text{in} \quad \Pi, \\ \mathbf{u}|_{\partial\Pi} = 0, \quad \mathbf{u}(x,0) = 0 \quad \text{in} \quad \Pi, \\ \mathbf{u}(x', x_3 - L, t) = \mathbf{u}(x', x_3 + L, t) \quad \text{in} \quad \Pi, \\ \int_\sigma u_3(x,t)\mathrm{d}x' = F(t). \end{cases} \tag{4}$$

Now we obtain the problem with two unknown functions $(\mathbf{u}(x,t), q(t))$. In the first step, we will find the solutions for problem (4) in one periodicity cell Π_L. Later, we will show that the solution can be extending to the whole infinite periodic pipe.

For the formulation of the weak solution, define two function spaces

$$S(\Pi_L) = \{\boldsymbol{\eta} \in W_2^1(\Pi_L) \mid \text{div } \boldsymbol{\eta} = 0, \ \boldsymbol{\eta}(\mathrm{x}', -\mathrm{L}) = \boldsymbol{\eta}(\mathrm{x}', \mathrm{L}), \ \boldsymbol{\eta}|_{\mathrm{S_L}} = 0\}$$

and

$$P_2(\Pi_L) = \left\{\mathbf{u} \in L_2(0,T; S(\Pi_L)) \mid \frac{\partial \mathbf{u}}{\partial t} \in L_2(0,T; L_2(\Pi_L))\right\}.$$

By a weak solution of problem (4) in a cell Π_L, we understand a pair of functions $(\mathbf{u}, q) \in P_2(\Pi_L) \times L_2(0,T)$ for all $t \in [0,T]$ satisfying the integral identity

$$\int_0^t \int_{\Pi_L} \frac{\partial}{\partial t}\mathbf{u} \cdot \boldsymbol{\eta}\mathrm{d}x\mathrm{d}\tau + \nu \int_0^t \int_{\Pi_L} \nabla\mathbf{u} \cdot \nabla\boldsymbol{\eta}\mathrm{d}x\mathrm{d}\tau = 2L \int_0^t q(\tau) \int_\sigma \eta_3 \mathrm{d}x'\mathrm{d}\tau \tag{5}$$

for all $\boldsymbol{\eta} \in L_2(0,T; S(\Pi_L))$, and the flux condition (2).

The right-hand side of integral equation (5) is obtained using the fact that $\boldsymbol{\eta}$ are solenoidal functions.

2. The existence of the unique approximate solution

We apply the Galerkin approximation method using eigenvalues $\mathbf{v}_k(x)$ of the following periodic boundary value problem:

$$\begin{cases} -\nu\Delta\mathbf{v}_k(x) + \nabla p_k = \lambda_k \mathbf{v}_k(x) \quad \text{in} \quad \Pi_L, \\ \text{div } \mathbf{v}_\mathrm{k} = 0 \quad \text{in} \quad \Pi_\mathrm{L}, \quad \mathbf{v}_\mathrm{k}|_{\mathrm{S_L}} = 0, \\ \mathbf{v}_k(x', -L) = \mathbf{v}_k(x', L) \quad \text{in} \quad \Pi_L, \\ p_k(x', -L) = p_k(x', L) \quad \text{in} \quad \Pi_L. \end{cases} \tag{6}$$

THEOREM 1. *Let $\Pi_L \in \mathbb{R}^3$ be a Lipschitz domain. Then*

- *(6) defines a countable set of eigenvalues $\lambda_k > 0$, $\lambda_k \to \infty$, $k = 1, 2, \ldots$; the corresponding eigenfunctions $\mathbf{v}_k$ constitute a basis $\{\mathbf{v}_k\}_{k \geqslant 1}$ in $S(\Pi_L)$.*

- *The eigenfunctions* $\mathbf{v}_k$ *can be orthonormalized:*

$$\int_{\Pi_L} \mathbf{v}_k \cdot \mathbf{v}_\ell \mathrm{d}x = \begin{cases} 1, & k = \ell, \\ 0, & k \neq \ell. \end{cases} \tag{7}$$

- *Moreover,*

$$\int_{\Pi_L} \nabla \mathbf{v}_k \cdot \nabla \mathbf{v}_\ell \mathrm{d}x = \begin{cases} \frac{\lambda_k}{\nu}, & k = \ell, \\ 0, & k \neq \ell. \end{cases} \tag{8}$$

For the proof of this theorem, see [8] or [9].

We look for an approximate solution $(\mathbf{u}^{(N)}(x,t), q^{(N)}(t))$ of problem (4) in the form

$$\mathbf{u}^{(N)}(x,t) = \sum_{k=1}^{N} y_k^{(N)}(t) \mathbf{v}_k(x),$$

where the coefficients $y_k^{(N)}(t)$ are found from the differential equations

$$\int_{\Pi_L} \frac{\partial}{\partial t} \mathbf{u}^{(N)} \cdot \mathbf{v}_k \mathrm{d}x + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla \mathbf{v}_k \mathrm{d}x = 2Lq^{(N)} \int_\sigma v_{k3} \mathrm{d}x, \tag{9}$$

$k = 1, 2, \dots, N$, and the initial conditions $\mathbf{u}^{(N)}(x, 0) = 0$. Here v_{k3} means the third component of the vector function $\mathbf{v}_k$. It is important to mention that the pressure drop functions $q^{(N)}(t)$ have to be found such that the flux condition are satisfied

$$\int_\sigma u_3^{(N)}(x,t) \mathrm{d}x' = F(t). \tag{10}$$

Using the properties of the basis $\{\mathbf{v}_k(x)\}$ (see Theorem 1) from the equality (9), we derive the Cauchy problem for the system of linear ordinary differential equations

$$\begin{cases} \mathbf{Y}^{(N)\prime}(t) + \mathbb{J}^{(N)} \mathbf{Y}^{(N)}(t) = L\boldsymbol{\beta}(t) q^{(N)}(t), \\ \mathbf{Y}^{(N)}(0) = 0, \end{cases} \tag{11}$$

where $\mathbf{Y}^{(N)}(t) = (y_1^{(N)}, \dots, y_N^{(N)})$, $\boldsymbol{\beta}^{(N)} = \left(\int_\sigma v_{13} \mathrm{d}x', \dots, \int_\sigma v_{N3} \mathrm{d}x'\right)$, $\mathbb{J}^{(N)} = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$.

For any fixed N, the elements of the matrices $\mathbb{J}^{(N)}$ are bounded. From the theory of the ordinary differential equations (see, for example, [15]), it follows that if $q^{(N)} \in L_2(0,T)$, $T \in (0, \infty)$, then there exists a unique solution $\mathbf{Y}^{(N)} \in W_2^1(0,T)$ of the problem (11).

The solution $\mathbf{Y}^{(N)}(t)$ of the problem (11) can be represented in the form

$$\mathbf{Y}^{(N)}(t) = L \int_0^t \exp^{-\mathbb{J}^{(N)}(t-\tau)} \boldsymbol{\beta}^{(N)} q^{(N)}(\tau) \mathrm{d}\tau. \tag{12}$$

As the pressure drop functions $q^{(N)}(t)$ have to satisfy the flux condition (10), we substitute the expression of $\mathbf{u}^{(N)}(x,t)$ into (10)

$$\begin{aligned} F(t) &= \int_\sigma u_3^{(N)}(x,t)\mathrm{d}x' = \sum_{k=1}^{N} y_k^{(N)}(t)\int_\sigma v_{k3}\mathrm{d}x' = \mathbf{Y}^{(N)}(t)\cdot\boldsymbol{\beta}^{(N)} \\ &= 2L\boldsymbol{\beta}^{(N)}\cdot\int_0^t \exp^{-\mathbb{J}^{(N)}(t-\tau)}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)\mathrm{d}\tau. \end{aligned}$$

Thus, $q^{(N)}(t)$ has to be found as the solution of the Volterra integral equation of the first kind

$$2L\boldsymbol{\beta}^{(N)}\cdot\int_0^t \exp^{-\mathbb{J}^{(N)}(t-\tau)}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)\mathrm{d}\tau = F(t).$$

Differentiating both sides of the last equation, we reduce it to the Volterra integral equation of the second kind

$$q^{(N)}(t) - \int_0^t \mathcal{K}^{(N)}(t,\tau)q^{(N)}(\tau)\mathrm{d}\tau = \frac{F'(t)}{2L|\boldsymbol{\beta}^{(N)}|^2} \tag{13}$$

with the kernel

$$\mathcal{K}^{(N)}(t,\tau) = \frac{\boldsymbol{\beta}^{(N)}}{|\boldsymbol{\beta}^{(N)}|^2}\cdot\mathbb{J}^{(N)}\exp^{-\mathbb{J}^{(N)}(t-\tau)}\boldsymbol{\beta}^{(N)}.$$

For any fixed N, the kernel $\mathcal{K}^{(N)}(t,\tau)$ is bounded, hence, $\mathcal{K}^{(N)}\in L_2(\mathbb{Q}^T)$ with $\mathbb{Q}^T = (0,T)\times(0,T)$. Therefore, for any $\frac{\mathrm{d}}{\mathrm{d}t}F\in L_2(0,T)$, there exists a unique solution $q^{(N)}\in L_2(0,T)$ of the integral equation (13) and the following estimate

$$\|q^{(N)}\|_{L_2(0,T)} \leqslant C_N\left\|\frac{\mathrm{d}}{\mathrm{d}t}F\right\|_{L_2(0,T)} \tag{14}$$

holds (see, for example, [16]). The constant C_N in (14) depends on the kernel $\mathcal{K}^{(N)}(t,\tau)$, and we cannot say in advance that C_N stays bounded as $N\to\infty$.

3. A priory estimates for the approximate solution

THEOREM 2. *Suppose that $F\in W_2^1(0,t)$. Then, for the approximate weak solution $(\boldsymbol{u}^{(N)}(x,t), q^{(N)}(t))$ of the problem (4), the estimate*

$$\|\boldsymbol{u}^{(N)}\|^2_{W_2^1(\Pi_L)} + \left\|\frac{\partial}{\partial t}\boldsymbol{u}^{(N)}\right\|^2_{L_2(0,t;L_2(\Pi_L))} + \|q^{(N)}\|^2_{L_2(0,t)} \leqslant c\|F\|^2_{W_2^1(0,t)} \tag{15}$$

holds.

Proof. Multiplying equality (9) by $y_k^{(N)}(t)$ and summing up by k from 1 to N, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi_L}|\mathbf{u}^{(N)}|^2\mathrm{d}x+\nu\int_{\Pi_L}|\nabla\mathbf{u}^{(N)}|^2\mathrm{d}x=2Lq^{(N)}\int_\sigma u_3^{(N)}\mathrm{d}x'.$$

Combining the obtained equation with the flux condition (10) and integrating by t, we derive the following estimate

$$\frac{1}{2}\int_{\Pi_L}|\mathbf{u}^{(N)}|^2\mathrm{d}x+\nu\int_0^t\int_{\Pi_L}|\nabla\mathbf{u}^{(N)}|^2\mathrm{d}x\mathrm{d}\tau\leqslant c\varepsilon\int_0^t|q^{(N)}|^2\mathrm{d}\tau+\frac{c}{\varepsilon}\int_0^t|F|^2\mathrm{d}\tau. \quad (16)$$

Let us multiply the equality (9) by $\frac{\mathrm{d}}{\mathrm{d}t}y_k^{(N)}(t)$ and sum up by k from 1 to N. This gives

$$\int_{\Pi_L}|\frac{\partial}{\partial t}\mathbf{u}^{(N)}|^2\mathrm{d}x+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi_L}|\nabla\mathbf{u}^{(N)}|^2\mathrm{d}x=2Lq^{(N)}\int_\sigma\frac{\partial}{\partial t}u_3^{(N)}\mathrm{d}x'.$$

Using the Cauchy-Schwartz and Young inequalities and integrating in time t, we obtain the estimate

$$\begin{aligned}&\int_0^t\int_{\Pi_L}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{u}^{(N)}\right|^2\mathrm{d}x\mathrm{d}\tau+\frac{\nu}{2}\int_{\Pi_L}|\nabla\mathbf{u}^{(N)}|^2\mathrm{d}x\\&\quad\leqslant c\varepsilon\int_0^t|q^{(N)}|^2\mathrm{d}\tau+\frac{c}{\varepsilon}\int_0^t\left|\frac{\mathrm{d}}{\mathrm{d}\tau}F\right|^2\mathrm{d}\tau.\end{aligned} \quad (17)$$

The next aim is to get an estimate for the pressure drop functions $q^{(N)}(t)$. For this, as a test function, we use the weak solution $\boldsymbol{\omega}\in W_2^1(\Pi_L)$ of the problem

$$\begin{cases}-\nu\Delta\boldsymbol{\omega}(x)+\nabla r(x)=0,\\ \mathrm{div}\ \boldsymbol{\omega}(x)=0,\quad \boldsymbol{\omega}(x)|_{\partial\Pi_L}=0,\\ \boldsymbol{\omega}(x',x_3-L)=\boldsymbol{\omega}(x',x_3+L),\\ r(x',x_3-L)-r(x',x_3+L))=2L.\end{cases} \quad (18)$$

The existence of the solution of the problem (18) is given in [7]. We look for $r(x)=-x_3+\tilde{r}(x)$. The function $\boldsymbol{\omega}(x)$ satisfies the integral identity

$$\nu\int_{\Pi_L}\nabla\boldsymbol{\omega}\cdot\nabla\boldsymbol{\eta}\mathrm{d}x-2L\int_\sigma\eta_3\mathrm{d}x'=0,$$

where $\boldsymbol{\eta}\in S(\Pi_L)$. If $\boldsymbol{\eta}=\boldsymbol{\omega}$, we obtain

$$\nu\int_{\Pi_L}|\nabla\boldsymbol{\omega}|^2\mathrm{d}x=2L\int_\sigma\omega_3\mathrm{d}x'=\xi_0\neq 0.$$

Since $\boldsymbol{\omega}\in S(\Pi_L)$, we have that $\boldsymbol{\omega}(x)=\sum_{k=1}^\infty\gamma_k\mathbf{v}_k$, where $\gamma_k=\int_{\Pi_L}\boldsymbol{\omega}\cdot\mathbf{v}_k\mathrm{d}x$. Multiplying equation (9) by γ_k and summing up from 1 to M, we find that

$$\int_{\Pi_L}\frac{\partial}{\partial t}\mathbf{u}^{(N)}\cdot\boldsymbol{\omega}^{(M)}\mathrm{d}x+\nu\int_{\Pi_L}\nabla\mathbf{u}^{(N)}\cdot\nabla\boldsymbol{\omega}^{(M)}\mathrm{d}x=2Lq^{(N)}(t)\int_\sigma\omega_3^{(M)}\mathrm{d}x', \quad (19)$$

where $\boldsymbol{\omega}^{(M)}(x) = \sum_{k=1}^{M} \gamma_k \mathbf{v}_k$.

In virtue of (18), the periodicity condition and the flux condition (10), we have

$$\begin{aligned}
&\nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla \boldsymbol{\omega}^{(M)} \mathrm{d}x \\
&= \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla \boldsymbol{\omega} \mathrm{d}x + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(M)} - \boldsymbol{\omega}) \mathrm{d}x \\
&= -\nu \int_{\Pi_L} \mathbf{u}^{(N)} \cdot \Delta \boldsymbol{\omega} \mathrm{d}x + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(M)} - \boldsymbol{\omega}) \mathrm{d}x \\
&= - \int_{\Pi_L} \mathbf{u}^{(N)} \cdot \nabla \tilde{r} \mathrm{d}x + 2L \int_{\sigma} u_3^{(N)} \mathrm{d}x + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(M)} - \boldsymbol{\omega}) \mathrm{d}x \\
&= \int_{\Pi_L} \operatorname{div} \mathbf{u}^{(N)} \tilde{r} \mathrm{d}x + 2LF(t) + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(M)} - \boldsymbol{\omega}) \mathrm{d}x \\
&= 2LF(t) + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(M)} - \boldsymbol{\omega}) \mathrm{d}x,
\end{aligned}$$

as well as

$$\begin{aligned}
2L \int_{\sigma} \omega_3^{(M)} \mathrm{d}x' &= 2L \int_{\sigma} \omega_3 \mathrm{d}x' + 2L \int_{\sigma} (\omega_3^{(M)} - \omega_3) \mathrm{d}x' \\
&= \xi_0 + 2L \int_{\sigma} (\omega_3^{(M)} - \omega_3) \mathrm{d}x'.
\end{aligned}$$

Therefore, equation (19) can be rewritten as

$$\begin{aligned}
&\int_{\Pi_L} \frac{\partial}{\partial t} \mathbf{u}^{(N)} \cdot \boldsymbol{\omega}^{(M)} \mathrm{d}x + 2LF(t) + \nu \int_{\Pi_L} \nabla \mathbf{u}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(M)} - \boldsymbol{\omega}) \mathrm{d}x \\
&\quad = q^{(N)}(t) \xi_0 + 2L q^{(N)}(t) \int_{\sigma} (\omega_3^{(M)} - \omega_3) \mathrm{d}x'.
\end{aligned}$$

Taking in mind the orthogonality of the eigenfunctions $\mathbf{v}_k$ implies that

$$\xi_0 q^{(N)}(t) = 2L q^{(N)}(t) \int_{\sigma} (\omega_3 - \omega_3^{(M)}) \mathrm{d}x' + \int_{\Pi_L} \frac{\partial}{\partial t} \mathbf{u}^{(N)} \cdot \boldsymbol{\omega}^{(M)} \mathrm{d}x + 2LF(t).$$

Integrating the last equation by t, we derive

$$\begin{aligned}
&\int_0^t \xi_0^2 |q^{(N)}|^2 \mathrm{d}\tau \\
&\quad \leqslant c \Big(\int_0^t |q^{(N)}(\tau)|^2 \mathrm{d}\tau \int_{\Pi_L} |\boldsymbol{\omega} - \boldsymbol{\omega}^{(M)}|^2 \mathrm{d}x \\
&\qquad + \int_0^t \int_{\Pi_L} |\frac{\partial}{\partial \tau} \mathbf{u}^{(N)}|^2 \mathrm{d}x \mathrm{d}\tau \int_{\Pi_L} |\boldsymbol{\omega}^{(M)}|^2 \mathrm{d}x + \int_0^t |F|^2 \mathrm{d}\tau \Big).
\end{aligned}$$

Since $\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(M)}\|_{L_2(\Pi_L)} \to 0$ as $M \to \infty$, we obtain the estimate for the functions $q^{(N)}(t)$,

$$\int_0^t |q^{(N)}(\tau)|^2 \mathrm{d}\tau$$

$$\leqslant c\Big(\int_0^t \|\frac{\partial}{\partial\tau}\mathbf{u}^{(N)}\|^2_{L_2(\Pi_L)}\mathrm{d}\tau + \int_0^t \|\nabla\mathbf{u}^{(N)}\|^2_{L_2(\Pi_L)}\mathrm{d}\tau + \|F\|^2_{L_2(0,t)}\Big). \quad (20)$$

On the other hand, from (16) and (17), we establish

$$\begin{aligned}&\|\mathbf{u}^{(N)}\|^2_{W_2^1(\Pi_L)} + \int_0^t \|\frac{\partial}{\partial\tau}\mathbf{u}^{(N)}\|^2_{L_2(\Pi_L)}\mathrm{d}\tau + \int_0^t \|\nabla\mathbf{u}^{(N)}\|^2_{L_2(\Pi_L)}\mathrm{d}\tau\\ &\quad\leqslant \frac{c}{\varepsilon}\|F\|^2_{W_2^1(0,t)} + c\varepsilon\int_0^t |q^{(N)}|^2\mathrm{d}\tau. \end{aligned} \quad (21)$$

For sufficiently small ε, we get from (20) and (21) that

$$\int_0^t |q^{(N)}(\tau)|^2\mathrm{d}\tau \leqslant c\|F\|^2_{W_2^1(0,t)}. \quad (22)$$

Note that the constant c does not depend on N. Estimate (15) is a consequence of (21) and (22).

4. The existence of the unique weak solution

THEOREM 3. *Suppose that $F \in W_2^1(0,T)$, $T \in (0,\infty)$, and the compatibility condition $F(0) = 0$ holds. Then problem* (4) *admits a unique weak solution $(\boldsymbol{u}, q) \in L_2(0,T;H(\Pi)) \times L_2(0,T)$, and the estimate*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}\|^2_{W_2^1(\Pi)} + \|\frac{\partial}{\partial t}\boldsymbol{u}\|^2_{L_2(0,T;L_2(\Pi))} + \|q\|^2_{L_2(0,T)} \leqslant c\|F\|^2_{W_2^1(0,T)}. \quad (23)$$

holds. Here $H(\Pi) = \{\boldsymbol{u} \in W_2^1(\Pi),\ \frac{\partial \boldsymbol{u}}{\partial t} \in L_2(\Pi)\ \mid \mathrm{div}\,\boldsymbol{\eta} = 0,\ \boldsymbol{\eta}(\mathrm{x}', \mathrm{x}_3 - \mathrm{L}) = \boldsymbol{\eta}(\mathrm{x}', \mathrm{x}_3 + \mathrm{L}),\ \boldsymbol{\eta}|_{\partial\Pi} = 0\}$.

Proof. From the Theorem 2, we have that $\{\mathbf{u}^{(N)}(x,t)\}$ and $\{\frac{\partial}{\partial t}\mathbf{u}^{(N)}(x,t)\}$ are bounded in the spaces $L_2(0,T;W_2^1(\Pi_L))$ and $L_2(0,T;L_2(\Pi_L))$, respectively. Hence, there exist subsequences $\{\mathbf{u}^{(N_\ell)}(x,t)\}$ and $\{\frac{\partial}{\partial t}\mathbf{u}^{(N_l)}(x,t)\}$ which converge weakly to $\mathbf{u}(x,t)$ in the space $L_2(0,T;W_2^1(\Pi_L))$ and $\frac{\partial}{\partial t}\mathbf{u}(x,t)$ in the space $L_2(0,T;L_2(\Pi_L))$, respectively. Moreover, $\{q^{(N)}(t)\}$ are bounded in the space $L_2(0,T)$, hence, there exists a subsequence $\{q^{(N_\ell)}(t)\}$ which converges weakly to $q(t)$ in the space $L_2(0,T)$.

For the approximate solution, we have the integral identity

$$\int_0^t \int_{\Pi_L} \frac{\partial}{\partial\tau}\mathbf{u}^{(N_\ell)} \cdot \boldsymbol{\eta}\mathrm{d}x\mathrm{d}\tau + \nu\int_0^t \int_{\Pi_L} \nabla\mathbf{u}^{(N_\ell)} \cdot \nabla\boldsymbol{\eta}\mathrm{d}x\mathrm{d}\tau = 2L\int_0^t q^{(N_\ell)} \int_\sigma \eta_3\mathrm{d}x\mathrm{d}\tau$$

for $\boldsymbol{\eta} \in L_2(0,T;H(\Pi_L))$. Passing to the limit as $N_\ell \to \infty$, we get

$$\int_0^t \int_{\Pi_L} \frac{\partial}{\partial\tau}\mathbf{u} \cdot \boldsymbol{\eta}\mathrm{d}x\mathrm{d}\tau + \nu\int_0^t \int_{\Pi_L} \nabla\mathbf{u} \cdot \nabla\boldsymbol{\eta}\mathrm{d}x\mathrm{d}\tau = 2L\int_0^t q \int_\sigma \eta_3\mathrm{d}x\mathrm{d}\tau,$$

and so the existence of the weak solution of the problem (4) in the cell Π_L is shown.

To prove the uniqueness of the solution, we consider $\boldsymbol{\eta}(x,t) = \mathbf{u}(x,t) = \mathbf{u}_1(x,t) -\mathbf{u}_2(x,t)$ and $q(t) = q_1(t) - q_2(t)$, where $(\mathbf{u}_i, q_i)$ are weak solutions of the problem (4). Since both the solutions have to fulfill the flux condition, we obtain that

$$\int_\sigma u_3 \mathrm{d}x' = \int_\sigma u_{1\ 3} - u_{2\ 3} \mathrm{d}x' = F(t) - F(t) = 0,$$

and so

$$\frac{1}{2}\int_{\Pi_L} |\mathbf{u}|^2 \mathrm{d}x + \nu \int_0^t \int_{\Pi_L} |\nabla \mathbf{u}|^2 \mathrm{d}x \mathrm{d}\tau = 2L \int_0^t q \int_\sigma u_3 \mathrm{d}x' \mathrm{d}\tau = 0.$$

It is obvious that $\mathbf{u}(x,t) = 0$, therefore, $\mathbf{u}_1 = \mathbf{u}_2$. Since we have already shown the existence of the problem (4) and $\mathbf{v} = 0$ is true, we obtain that

$$0 = 2L \int_0^t q(\tau) \int_\sigma \eta_3 \mathrm{d}x' \mathrm{d}\tau$$

for all possible $\boldsymbol{\eta}$, which implies $q = 0$ and so $q_1 = q_2$. This fact completes the proof of the uniqueness.

We can take a sequence of the periodicity cells $\Pi_L, \Pi_{2L}, \ldots, \Pi_{kL}, \ldots$ such that

$$\Pi_L \subset \Pi_{2L} \subset \cdots \subset \Pi_{kL} \subset \cdots \quad \text{and exhausts} \quad \Pi \quad \text{as} \quad k \to \infty. \tag{24}$$

In every periodicity cell, we proved the unique existence of the solution $(\mathbf{u}(x,t), q(t))$. According to the condition (24), we proved the existence of the unique solution of problem (4) in the whole pipe Π.

For the proof of the estimate (23), compare the proof of the Theorem 2.

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