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ZETA-FUNCTIONS OF WEIGHT LATTICES OF COMPACT CONNECTED SEMISIMPLE LIE GROUPS

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Abstract. We define zeta-functions of weight lattices of compact connected semisimple Lie groups. If the group is simply-connected, these zeta-functions coincide with ordinary zeta-functions of root systems of associated Lie algebras. In this paper, we consider the general connected (but not necessarily simply-connected) case, prove the explicit form of Witten's volume formulas for these zeta-functions, and further prove functional relations among them which include their volume formulas. Also, we give new examples of zeta-functions for which parity results hold.

Key words and phrases: Lie group, multiple zeta values, root systems, Witten's zeta-functions.

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1. The background and the motivation

Let M be a compact 2-dimensional manifold, G a compact connected semisimple Lie group acting as a gauge group, and E a G-bundle over M. Motivated by 2-dimensional quantum gauge theories, Witten [37] evaluated the volume of the moduli space $\mathcal{M}$ of flat connections on E up to gauge transformations. Such a result can be regarded as a limit of Verlinde's formula [35] when M is orientable, but Witten developed a more elementary method, based on the decomposition of M into three-holed spheres. The main result in [37] is now called Witten's volume formula, which expresses the volume of $\mathcal{M}$ in terms of special values of the Dirichlet series

$$\zeta_W(s;G)=\sum_{\psi}(\dim\psi)^{-s}, \tag{1.1}$$

where ψ runs over all isomorphism classes of finite dimensional irreducible representations of G.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G, and define

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.2}$$

where the summation runs over all isomorphism classes of finite dimensional irreducible representations φ of $\mathfrak{g}$. When G is simply-connected, then there is a one-to-one correspondence between φ and ψ. In fact, each φ is the differential of a certain ψ, and so

$$\zeta_W(s;\mathfrak{g}) = \zeta_W(s;G). \tag{1.3}$$

Zagier [38] formulated the series (1.2) and called them Witten's zeta-functions (see also Gunnells-Sczech [5]). Witten's volume formula especially implies

$$\zeta_W(2k;\mathfrak{g}) = C_W(2k,\mathfrak{g})\pi^{2kn} \tag{1.4}$$

for $k \in \mathbb{N}$, where n is the number of all positive roots of $\mathfrak{g}$ and $C_W(2k,\mathfrak{g})$ is a rational number.

Before proceeding further, here we fix several notations. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the rational number field, $\mathbb{R}$ the real number field, $\mathbb{C}$ the complex number field, respectively.

Let Δ be the set of all roots of $\mathfrak{g}$, Δ_+ the set of all positive roots of $\mathfrak{g}$ (hence $n = |\Delta_+|$), $\Psi = \{\alpha_1,\ldots,\alpha_r\}$ the fundamental system of Δ, $\alpha_j^\vee$ the coroot of α_j, $1 \leqslant j \leqslant r$. Let $\lambda_1,\ldots,\lambda_r$ be the fundamental weights satisfying $\langle \alpha_i^\vee, \lambda_j\rangle = \lambda_j(\alpha_i^\vee) = \delta_{ij}$ (Kronecker's delta).

Witten's zeta-function corresponding to $\mathfrak{g} = \mathfrak{sl}(2)$ is nothing but the Riemann zeta-function $\zeta(s)$ and (1.4) implies Euler's well-known formula for $\zeta(2k)$. For more general $\mathfrak{g}$, Szenes [25, 26], and also Gunnells and Sczech [5], introduced certain methods (different from each other) of computing $C_W(2k,\mathfrak{g})$.

In [12, 17], the authors introduced the multi-variable version of Witten's zeta-function

$$\zeta_r(\mathbf{s};\mathfrak{g}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \ldots + m_r\lambda_r\rangle^{-s_\alpha}, \tag{1.5}$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^n$. When $\mathfrak{g}$ is of type X_r, where $X = A, B, C, D, E, F$, or G, we call (1.5) the zeta-function of the root system of type X_r, and denote it by $\zeta_r(\mathbf{s}; X_r)$. Putting

$$K(\Delta) = \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_1 + \ldots + \lambda_r\rangle, \tag{1.6}$$

and using [17, (1.5) and (1.7)], we see that

$$K(\Delta)^s \zeta_r(s,\ldots,s;\mathfrak{g}) = \zeta_W(s;\mathfrak{g}). \tag{1.7}$$

In [18], the authors introduced a root-system theoretic generalization of Bernoulli numbers and periodic Bernoulli functions, and express $C_W(2k,\mathfrak{g})$ explicitly in terms of generalized periodic Bernoulli functions $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$. Therefore, we now have sufficiently explicit information on formula (1.4). Moreover, in [12, 14, 18, 19, 22, 33], we proved various functional relations among zeta-functions (1.2), which include evaluation formulas like (1.4) as special cases.

However, the group G is not necessarily simply-connected in Witten's paper [37]. (In fact, this point is emphasized by Witten himself on p. 182 of [37].) When G is not simply-connected, relation (1.3) does not hold. It is the aim of the present paper to consider such situation; that is, to study the zeta-functions and volume formulas in the sense of original formulation of Witten [37].

For this purpose, we introduce the multi-variable version of $\zeta_W(s;G)$. From (1.5) we see that the multi-variable version of $\zeta_W(s;\mathfrak{g})$ can be regarded as the zeta-function of the weight lattice of $\mathfrak{g}$. Similarly, in the present paper we will define a multi-variable zeta-function of the weight lattice of G. Actually, this zeta-function, defined in Section 3, is a partial sum of $\zeta_r(\mathbf{s};\mathfrak{g})$. The volume formula for this zeta-function is given as Theorem 3.2, which gives an explicit formula for the values of this zeta-function at $\mathbf{s}=2\mathbf{k}$, where $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{N}^n$ satisfying $k_\alpha=k_\beta$ if α and β are of the same length. As explicit examples, in Section 4, we consider the cases of types A_r, B_r and C_r, $r\leqslant 3$, and evaluate the associated zeta-functions in these cases.

Since Theorem 3.2 is a formula for $\mathbf{s}=2\mathbf{k}$, it is not useful when we consider the values at odd integer points. In order to study such cases, in Section 5, we give some functional relations among zeta-functions of types A_2 and $C_2(\simeq B_2)$. Those relations produce explicit formulas for special values of zeta-functions at some points of the form $\mathbf{s}=\mathbf{l}=(l_\alpha)_{\alpha\in\Delta_+}$, where $l_\alpha\in\mathbb{N}$ and some of them are odd. Those results include not only evaluation formulas given in Section 4 but also another type of evaluation formulas which can be regarded as certain extensions of the previous results in [14, 27, 30, 33]. In Section 6, we consider so-called parity results for zeta values of types A_2 and C_2. We prove that parity results hold for the zeta-functions associated with the groups $PU(3)$ and $PSp(2)$.

The present paper was already posted to the arXiv in 2010 (arXiv:math/1011.0323). A continuation of the present paper, in which the details of the case of type A_3 are discussed, was separately published in [20] in 2012.

2. A general form of zeta-functions

We begin our theory with the definition of rather general form of zeta-functions. We use the same notation as in [15, 17, 18] (see also [12, 13, 16, 19]). For the details of basic facts about root systems and Weyl groups, see [3, 6, 7].

Let V be an r-dimensional real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. The norm $\|\cdot\|$ is defined by $\|v\|=\langle v,v\rangle^{1/2}$. The dual space V^* is identified with V via the inner product of V. Let Δ be a finite reduced root system which may not be irreducible, and $\Psi=\{\alpha_1,\dots,\alpha_r\}$ its fundamental system. We fix Δ_+

and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \coprod \Delta_-$. Let $Q = Q(\Delta)$ be the root lattice, $Q^\vee$ the coroot lattice, $P = P(\Delta)$ the weight lattice, $P^\vee$ the coweight lattice, P_+ the set of integral dominant weights and P_{++} the set of integral strongly dominant weights, respectively defined by

$$Q = \bigoplus_{i=1}^{r} \mathbb{Z}\,\alpha_i, \quad Q^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\,\alpha_i^\vee, \tag{2.1}$$

$$P = \bigoplus_{i=1}^{r} \mathbb{Z}\,\lambda_i, \quad P^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\,\lambda_i^\vee, \tag{2.2}$$

$$P_+ = \bigoplus_{i=1}^{r} \mathbb{N}_0\,\lambda_i, \quad P_{++} = \bigoplus_{i=1}^{r} \mathbb{N}\,\lambda_i, \tag{2.3}$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of $\Psi^\vee$ and Ψ satisfying $\langle\alpha_i^\vee, \lambda_j\rangle = \delta_{ij}$ and $\langle\lambda_i^\vee, \alpha_j\rangle = \delta_{ij}$ respectively. A coweight $\mu \in P^\vee$ is said to be minuscule if $0 \leqslant \langle\mu, \alpha\rangle \leqslant 1$ for all $\alpha \in \Delta$. It is known that a minuscule coweight is one of fundamental coweights and as a system of representatives for $P^\vee/Q^\vee$, we can take $\{0\} \cup \{\lambda_j^\vee\}_{j\in J}$, where J is the set of all indices of minuscule coweights.

Let

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha = \sum_{j=1}^{r} \lambda_j \tag{2.4}$$

be the lowest strongly dominant weight. Then $P_{++} = P_+ + \rho$. Let σ_α be the reflection with respect to a root $\alpha \in \Delta$ defined as

$$\sigma_\alpha : V \to V, \qquad \sigma_\alpha : v \mapsto v - \langle\alpha^\vee, v\rangle\alpha. \tag{2.5}$$

For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha \in A$. In particular, $W = W(\Delta)$ is the Weyl group, and $\{\sigma_j = \sigma_{\alpha_j} \mid 1 \leqslant j \leqslant r\}$ generates W. For $w \in W$, denote $\Delta_w = \Delta_+ \cap w^{-1}\Delta_-$.

Let $\mathrm{Aut}(\Delta)$ be the subgroup of all the automorphisms $\mathrm{GL}(V)$ which stabilizes Δ. Then the Weyl group W is a normal subgroup of $\mathrm{Aut}(\Delta)$ and there exists a subgroup $\Omega \subset \mathrm{Aut}(\Delta)$ such that $\mathrm{Aut}(\Delta) = \Omega \ltimes W$. The subgroup Ω is isomorphic to the group $\mathrm{Aut}(\Gamma)$ of automorphisms of the Dynkin diagram Γ (see [6, Section 12.2]).

For a set X, denote by $\mathfrak{F}(X)$ the set of all complex valued functions on X. For a function $f \in \mathfrak{F}(P)$, we define a subset

$$H_f = \{\lambda \in P \mid f(\lambda) = 0\} \tag{2.6}$$

and for a subset A of $\mathfrak{F}(P)$, define $H_A = \bigcup_{f\in A} H_f$. Note that an action of W is induced on $\mathfrak{F}(P)$ as $(wf)(\lambda) = f(w^{-1}\lambda)$.

Let $f \in \mathfrak{F}(P/Q)$. Since $P^\vee/Q^\vee$ is regarded as the dual of P/Q over $\mathbb{Q}/\mathbb{Z}$, f can be expanded as follows:

$$f(\lambda) = \sum_{\mu\in P^\vee/Q^\vee} \widehat{f}(\mu)\mathrm{e}^{2\pi i\langle\mu,\lambda\rangle}, \tag{2.7}$$

where $\langle\ ,\ \rangle$ is regarded as an inner product on $P^\vee/Q^\vee$, and $\widehat{f}: P^\vee/Q^\vee \to \mathbb{C}$ is given by

$$\widehat{f}(\mu) = \frac{1}{|P/Q|} \sum_{\lambda \in P/Q} f(\lambda) \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle}, \tag{2.8}$$

because for $\nu \in P^\vee/Q^\vee$ we have

$$\sum_{\lambda \in P/Q} \mathrm{e}^{2\pi i \langle \nu, \lambda \rangle} = |P/Q| \delta_{\nu,0}. \tag{2.9}$$

Note that f is automatically W-invariant because for $\lambda \in P$ we have $\sigma_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \equiv \lambda \pmod{Q}$.

For $\mathbf{s} = (s_\alpha) \in \mathbb{C}^n$, $\mathbf{y} \in V$ and $f \in \mathfrak{F}(P/Q)$, we define

$$\zeta_r(\mathbf{s}, \mathbf{y}, f; \Delta) = \sum_{\lambda \in P_{++}} f(\lambda) \mathrm{e}^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}. \tag{2.10}$$

Note that $\zeta_r(\mathbf{s}, \mathbf{y}, 1; \Delta)$ was already studied in our previous work (see [15, Section 3], [18, Section 4]). When $\Delta = \Delta(X_r) = \Delta(\mathfrak{g})$ is the root system attached to $\mathfrak{g}$ of type X_r, then $\zeta_r(\mathbf{s}, \mathbf{0}, 1; \Delta)$ coincides with $\zeta_r(\mathbf{s}; \mathfrak{g})$ (see (1.5)). For $w \in \mathrm{Aut}(\Delta)$, define the action of w on ζ_r by

$$(w\zeta_r)(\mathbf{s}, \mathbf{y}, f; \Delta) = \zeta_r(w^{-1}\mathbf{s}, w^{-1}\mathbf{y}, w^{-1}f; \Delta), \tag{2.11}$$

where $w^{-1}(\mathbf{s}) = (s_{w\alpha})_{\alpha \in \Delta_+}$ (if $w\alpha \in \Delta_-$, we identify it with $-w\alpha$). Then it is easy to see that for $w \in \mathrm{Aut}(\Gamma)$,

$$(w\zeta_r)(\mathbf{s}, \mathbf{y}, f; \Delta) = \zeta_r(\mathbf{s}, \mathbf{y}, f; \Delta). \tag{2.12}$$

Remark 1. Here we discuss the reasons why we include the exponential factor $e^{2\pi i \langle \mathbf{y}, \lambda \rangle}$ in the definition (2.10). This is analogous to the Lerch zeta-function

$$\phi(s, \alpha) = \sum_{m=1}^{\infty} \frac{\mathrm{e}^{2\pi i m \alpha}}{m^s}. \tag{2.13}$$

It is clear that the form (2.10) with an exponential factor is useful in the study of multiple series with twisting factors, such as the series discussed in Example 4.1 and Remark 4, or multiple L-functions with Dirichlet characters [15]. Moreover, the existence of this exponential factor simplifies our argument in various places. In the argument below, (2.14), (2.19) etc. are impossible to show without using this factor.

Remark 2. It is to be noted that we may regard $\mathbf{y} \in V/Q^\vee$ in (2.10). This is because when $a \in Q^\vee$ we have $\langle a, \lambda \rangle \in \mathbb{Z}$, hence $\mathrm{e}^{2\pi i \langle \mathbf{y}+a, \lambda \rangle} = \mathrm{e}^{2\pi i \langle \mathbf{y}, \lambda \rangle}$, for any $\lambda \in P_{++}$.

Proposition 2.1. *The function $\zeta_r(\mathbf{s}, \mathbf{y}, f; \Delta)$, as a function in $\mathbf{s}$, can be continued meromorphically to the whole space $\mathbb{C}^n$.*

Proof. By use of the expression (2.7), noting Remark 2, we obtain

$$\begin{aligned}\zeta_r(\mathbf{s},\mathbf{y},f;\Delta) &= \sum_{\lambda\in P_{++}}\sum_{\mu\in P^\vee/Q^\vee}\widehat{f}(\mu)\mathrm{e}^{2\pi i\langle\mu,\lambda\rangle}\mathrm{e}^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}\\ &= \sum_{\mu\in P^\vee/Q^\vee}\widehat{f}(\mu)\sum_{\lambda\in P_{++}}\mathrm{e}^{2\pi i\langle\mathbf{y}+\mu,\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}\\ &= \sum_{\mu\in P^\vee/Q^\vee}\widehat{f}(\mu)\zeta_r(\mathbf{s},\mathbf{y}+\mu,1;\Delta). \qquad (2.14)\end{aligned}$$

In [15, Section 8], we showed that $\zeta_r(\mathbf{s},\mu,1;\Delta)$ can be continued meromorphically to the whole space, hence, so can be $\zeta_r(\mathbf{s},\mathbf{0},f;\ \Delta)$ from (2.14). More generally the recent result of the first-named author in [11] gives that $\zeta_r(\mathbf{s},\mathbf{y},1;\Delta)$, $\mathbf{y}\in V$, can be continued meromorphically, so can be $\zeta_r(\mathbf{s},\mathbf{y},f;\ \Delta)$ from (2.14). This completes the proof of Proposition 2.3.

Let

$$S(\mathbf{s},\mathbf{y},f;\Delta)=\sum_{\lambda\in P\setminus H_{\Delta^\vee}}f(\lambda)\mathrm{e}^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}. \qquad (2.15)$$

Then, in the same way as in the case of zeta-functions, we obtain

$$S(\mathbf{s},\mathbf{y},f;\Delta)=\sum_{\mu\in P^\vee/Q^\vee}\widehat{f}(\mu)S(\mathbf{s},\mathbf{y}+\mu,1;\Delta). \qquad (2.16)$$

Here we recall the generalized periodic Bernoulli functions $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ associated with Δ as follows (for the details, see [18, Section 4]). For $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{N}_0^n$ and $\mathbf{y}\in V$ (or $\in V/Q^\vee$), we define

$$\begin{aligned}&\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)\\ &\quad=\int_0^1\cdots\int_0^1\Bigl(\prod_{\alpha\in\Delta_+\setminus\Psi}B_{k_\alpha}(x_\alpha)\Bigr)\\ &\quad\times\Bigl(\prod_{i=1}^r B_{k_{\alpha_i}}\Bigl(\Bigl\{\langle\mathbf{y},\lambda_i\rangle-\sum_{\alpha\in\Delta_+\setminus\Psi}x_\alpha\langle\alpha^\vee,\lambda_i\rangle\Bigr\}\Bigr)\Bigr)\prod_{\alpha\in\Delta_+\setminus\Psi}\mathrm{d}x_\alpha, \qquad (2.17)\end{aligned}$$

where $\{B_k(x)\}$ are the classical Bernoulli polynomials defined by

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t-1}=\sum_{k=0}^\infty B_k(x)\frac{t^k}{k!}.$$

We have already obtained

$$S(\mathbf{k},\mathbf{y},1;\Delta)=(-1)^n\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\Bigr)\mathcal{P}(\mathbf{k},\mathbf{y};\Delta) \qquad (2.18)$$

for $\mathbf{k}\in(\mathbb{N}_{\geqslant 2})^n$ (see [18, (4.19)]). This function $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ may be regarded as a generalization of the periodic Bernoulli function and $\mathcal{B}_{\mathbf{k}}(\Delta)=\mathcal{P}(\mathbf{k},0;\Delta)$ the Bernoulli number.

Note that Szenes [25, 26] also studied generalizations of Bernoulli polynomials from the viewpoint of the theory of arrangement of hyperplanes, which include $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ mentioned above.

Suggested by (2.16), we define generalized Bernoulli functions associated with f and Δ by

$$\mathcal{P}(\mathbf{k},\mathbf{y},f;\Delta)=\sum_{\mu\in P^\vee/Q^\vee}\widehat{f}(\mu)\mathcal{P}(\mathbf{k},\mathbf{y}+\mu;\Delta). \tag{2.19}$$

Then, by (2.16), (2.18) and (2.19), we have

$$S(\mathbf{k},\mathbf{y},f;\Delta)=(-1)^n\biggl(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\biggr)\mathcal{P}(\mathbf{k},\mathbf{y},f;\Delta) \tag{2.20}$$

for $\mathbf{k}\in(\mathbb{N}_{\geqslant 2})^n$.

In [16, Section 9] and [18, Section 3], we constructed the generating function of $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$, which is

$$F(\mathbf{t},\mathbf{y};\Delta)=\sum_{\mathbf{k}\in\mathbb{N}_0^n}\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)\prod_{\alpha\in\Delta_+}\frac{t^{k_\alpha}}{k_\alpha!}. \tag{2.21}$$

Since $F(\mathbf{t},\mathbf{y};\Delta)$ can be evaluated explicitly ([15, Theorem 4.1]), we can evaluate $\mathcal{P}(\mathbf{k},\mathbf{y}+\mu;\Delta)$ from the expansion of $F(\mathbf{t},\mathbf{y};\Delta)$. In particular, we find that $\mathcal{P}(\mathbf{k},\mu;\Delta)\in\mathbb{Q}$ for any $\mu\in P^\vee/Q^\vee$.

THEOREM 2.2. *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{N}_{\geqslant 2}^n$, $\mathbf{y}\in V$ *and* $f\in\mathfrak{F}(P/Q)$,

$$\begin{aligned}\sum_{w\in W}\biggl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}&(-1)^{k_\alpha}\biggr)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y},f;\Delta)\\&=(-1)^n\biggl(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\biggr)\mathcal{P}(\mathbf{k},\mathbf{y},f;\Delta).\end{aligned} \tag{2.22}$$

Proof. Since $P\setminus H_{\Delta^\vee}=\bigcup_{w\in W}w(P_{++})$, we have

$$\begin{aligned}S(\mathbf{s},\mathbf{y},f;\Delta)&=\sum_{\lambda\in P\setminus H_{\Delta^\vee}}f(\lambda)e^{2\pi i\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}}\\&=\sum_{w\in W}\sum_{\lambda\in P_{++}}f(w\lambda)e^{2\pi i\langle\mathbf{y},w\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,w\lambda\rangle^{s_\alpha}}\\&=\sum_{w\in W}\sum_{\lambda\in P_{++}}(w^{-1}f)(\lambda)e^{2\pi i\langle w^{-1}\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle w^{-1}\alpha^\vee,\lambda\rangle^{s_\alpha}}\\&=\sum_{w\in W}\biggl(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{-s_\alpha}\biggr)\zeta_r(w^{-1}\mathbf{s},w^{-1}\mathbf{y},w^{-1}f;\Delta),\end{aligned} \tag{2.23}$$

where the last equality follows by rewriting α to $w\alpha$, and when $\alpha\in-\Delta_w=w^{-1}\Delta_+\cap\Delta_-$ further replacing α by $-\alpha$ (see the proof of Theorem 4.3 in [18]).

Combining (2.20) and the W-invariance of f, we obtain the assertion of the theorem.

In the following sections, we treat some special cases. Let $\mathcal{A} \subset P$ with $\mathcal{A}+Q = \mathcal{A}$. Let $\iota_{\mathcal{A}} : P \to \{0,1\}$ be the characteristic function of $\mathcal{A}$ defined by

$$\iota_{\mathcal{A}}(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \in \mathcal{A}, \\ 0 & \text{if} \quad \lambda \notin \mathcal{A}. \end{cases} \tag{2.24}$$

Then $\iota_{\mathcal{A}}$ can be regarded as a function on P/Q. Hence, (2.7) and (2.8) imply that

$$\iota_{\mathcal{A}}(\lambda) = \sum_{\mu \in P^\vee/Q^\vee} \widehat{\iota_{\mathcal{A}}}(\mu) \mathrm{e}^{2\pi i \langle \mu, \lambda \rangle}, \tag{2.25}$$

where $\widehat{\iota_{\mathcal{A}}} : P^\vee/Q^\vee \to \mathbb{C}$ is given by

$$\widehat{\iota_{\mathcal{A}}}(\mu) = \frac{1}{|P/Q|} \sum_{\lambda \in P/Q} \iota_{\mathcal{A}}(\lambda) \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle} = \frac{1}{|P/Q|} \sum_{\lambda \in \mathcal{A}/Q} \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle}. \tag{2.26}$$

3. Zeta-functions of weight lattices of Lie groups

Now we define zeta-functions of weight lattices of Lie groups. Let $\widetilde{G}$ be a simply-connected compact semisimple Lie group, and $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$. There is a one-to-one correspondence between a connected compact semisimple Lie group G whose universal covering group is $\widetilde{G}$, and a lattice L with $Q(\Delta(\mathfrak{g})) \subset L \subset P(\Delta(\mathfrak{g}))$ up to automorphisms (see Remark 3) by taking $L = L(G)$ as the weight lattice of G. Let $L_+ = P_+ \cap L$.

We define the zeta-function of the weight lattice $L = L(G)$ of the semisimple Lie group G by

$$\zeta_r(\mathbf{s}, \mathbf{y}; G) = \zeta_r(\mathbf{s}, \mathbf{y}; L; \Delta) := \sum_{\lambda \in L_+ + \rho} \mathrm{e}^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}. \tag{3.1}$$

This is the case $f = \iota_{\mathcal{A}}$, $\mathcal{A} = L + \rho$ and Δ of (2.10), and so, by Proposition 2.1, we see that this zeta-function can be continued meromorphically to $\mathbb{C}^n$. When $\mathbf{y} = \mathbf{0}$, we sometimes write this zeta-function as $\zeta_r(\mathbf{s}; G)$ or $\zeta_r(\mathbf{s}; L; \Delta)$ for brevity. It is to be noted that as a generalization of (1.7), we have

$$K(\Delta)^s \zeta_r(s, \ldots, s; G) = \zeta_W(s; G) \tag{3.2}$$

and, if $G = \widetilde{G}$, then $L = P$ and $\zeta_r(\mathbf{s}; \widetilde{G})$ coincides with $\zeta_r(\mathbf{s}; \mathfrak{g})$ defined in Section 1.

For any lattice M, we define $M^* = \mathrm{Hom}(M, \mathbb{Z})$. Since $Q^* = P^\vee$ and $P^* = Q^\vee$, from $Q \subset L \subset P$ we obtain

$$P^\vee = Q^* \supset L^* \supset P^* = Q^\vee. \tag{3.3}$$

We define

$$\delta_{L^*/Q^\vee}(\mu) = \begin{cases} 1 & \text{if} \quad \mu \in L^*/Q^\vee, \\ 0 & \text{if} \quad \mu \notin L^*/Q^\vee. \end{cases} \tag{3.4}$$

Proposition 3.1. *Let L be a lattice satisfying $Q \subset L \subset P$. For $\mu \in P^\vee/Q^\vee$, we have*

$$\widehat{\iota_{L+\rho}}(\mu) = \frac{(-1)^{\langle \mu, 2\rho \rangle}}{|P/L|} \delta_{L^*/Q^\vee}(\mu) \in \mathbb{Q}. \tag{3.5}$$

Proof. We have

$$\sum_{\lambda \in (L+\rho)/Q} \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle} = (-1)^{\langle \mu, 2\rho \rangle} \sum_{\lambda \in L/Q} \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle}.$$

Note that $(-1)^{\langle \mu, 2\rho \rangle} \in \{1, -1\}$ due to $\rho \in Q/2$. We obtain

$$\sum_{\lambda \in L/Q} \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle} = \begin{cases} |L/Q| & \text{if} \quad \mu \in L^*/Q^\vee, \\ 0 & \text{if} \quad \mu \notin L^*/Q^\vee. \end{cases}$$

Therefore, (2.26) gives

$$\widehat{\iota_{L+\rho}}(\mu) = (-1)^{\langle \mu, 2\rho \rangle} \frac{|L/Q|}{|P/Q|} \delta_{L^*/Q^\vee}(\mu).$$

This completes the proof of Proposition 3.1.

In particular,

$$\widehat{\iota_{P+\rho}}(\mu) = \frac{1}{|P/Q|} \sum_{\lambda \in (P+\rho)/Q} \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle} = \delta_{\mu,0} \tag{3.6}$$

and

$$\widehat{\iota_{Q+\rho}}(\mu) = \frac{1}{|P/Q|} \sum_{\lambda \in (Q+\rho)/Q} \mathrm{e}^{-2\pi i \langle \mu, \lambda \rangle} = \frac{(-1)^{\langle \mu, 2\rho \rangle}}{|P/Q|}. \tag{3.7}$$

We define

$$\mathcal{P}(\mathbf{k}, \mathbf{y}; L; \Delta) = \mathcal{P}(\mathbf{k}, \mathbf{y}, \iota_{L+\rho}; \Delta). \tag{3.8}$$

Note that since $P/L \simeq L^*/Q^\vee \simeq \pi_1(G)$, this can also be written as

$$\mathcal{P}(\mathbf{k}, \mathbf{y}; L; \Delta) = \frac{1}{|\pi_1(G)|} \sum_{\mu \in \pi_1(G)} (-1)^{\langle \mu, 2\rho \rangle} \mathcal{P}(\mathbf{k}, \mathbf{y} + \mu; \Delta) \tag{3.9}$$

by (2.19) and Proposition 3.1.

We can compute $\mathcal{P}(\mathbf{k}, \mathbf{y}; L; \Delta)$ explicitly by (2.19). In particular, combining with (3.5) we have for $\nu \in P^\vee/Q^\vee$,

$$\mathcal{P}(\mathbf{k}, \nu; L; \Delta) \in \mathbb{Q}. \tag{3.10}$$

From this fact, we can deduce the following.

Theorem 3.2. *For a compact connected semisimple Lie group G, let $\Delta = \Delta(G)$ be its root system, and $L = L(G)$ be its weight lattice. Let $\mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{N}^n$, $n = |\Delta_+|$, satisfying $k_\alpha = k_\beta$ whenever $\|\alpha\| = \|\beta\|$. Let $\kappa = \sum_{\alpha\in\Delta_+} 2k_\alpha$. Then, for $\nu \in P^\vee/Q^\vee$, we have*

$$\begin{aligned}&\zeta_r(2\mathbf{k},\nu;G)\\ &\quad=\zeta_r(2\mathbf{k},\nu;L;\Delta)\\ &\quad=\frac{(-1)^n}{|W|}\biggl(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!}\biggr)\mathcal{P}(2\mathbf{k},\nu;L;\Delta)\in\mathbb{Q}\cdot\pi^\kappa. \qquad (3.11)\end{aligned}$$

Proof. By Theorem 2.2 with $\mathbf{s} = 2\mathbf{k}$ and $\mathbf{y} = \nu$, we obtain

$$\sum_{w\in W}\zeta_r(w^{-1}(2\mathbf{k}),w^{-1}\nu;L;\Delta)=(-1)^n\biggl(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\biggr)\mathcal{P}(\mathbf{k},\mathbf{y};L;\Delta).$$

Since roots of the same length form a single Weyl-orbit and $w^{-1}\nu \equiv \nu \pmod{Q^\vee}$, the left-hand side of the above is

$$\sum_{w\in W}\zeta_r(2\mathbf{k},\nu;L;\Delta)=|W|\zeta_r(2\mathbf{k},\nu;L;\Delta).$$

The assertion of the theorem follows from this and (3.10).

This theorem is the explicit form of the volume formula for the zeta-function of the lattice $L = L(G)$. In the case when $L = P$, (3.11) coincides with our previous result in [18, Theorem 4.6].

Remark 3. The correspondence between Lie groups and lattices is a well-known fact, but here we sketch the demonstration for the convenience of readers. Let G be a compact connected semisimple Lie group, whose universal covering group is $\widetilde{G}$. There is a one-to-one correspondence between isomorphism classes of finite dimensional irreducible representations of G and dominant analytically integral forms (for example, [10, Theorem 5.110]). The set of dominant analytically integral forms produces a sublattice $L = L(G)$ of the weight lattice (or the lattice of algebraically integral forms) P of $\mathfrak{g}$, and L includes the root lattice Q ([10, (4.63)]). In particular, $L(\widetilde{G}) = P$. Conversely, let $\widetilde{G}$ be a simply-connected Lie group, $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$, and let L be a lattice satisfying $Q = Q(\Delta(\mathfrak{g})) \subset L \subset P = P(\Delta(\mathfrak{g}))$. Then (3.3) holds. Since $P^\vee/Q^\vee$ is isomorphic to the center $\widetilde{Z}$ of $\widetilde{G}$ by the mapping

$$\Phi\ :\ P^\vee \ni \mu \mapsto \exp_{\widetilde{G}}(2\pi i\mu) \in \widetilde{Z}$$

(where $\exp_{\widetilde{G}}$ means the exponential mapping associated with $\widetilde{G}$), we may regard $L^*/Q^\vee$ as a subgroup of $\widetilde{G}$. Define $G = \widetilde{G}/(L^*/Q^\vee)$. We show that the lattice corresponding to G is L. Write $L_1 = L(G)$. Take a maximal torus K of G, and $\mathfrak{k} = \mathrm{Lie}(G)$. Let $\lambda \in L$ and $H \in \mathfrak{k}$ with $\exp_G(H) = 1$. The last condition implies $\exp_{\widetilde{G}}(H) \in \Phi(L^*)$, so $H \in 2\pi i L^*$, $\lambda(H) \in 2\pi i\mathbb{Z}$. Therefore $\lambda \in L_1$ by [10, Proposition 4.58], hence $L \subset L_1$. On the other hand, $(L_1^* : Q^\vee) = (L_1^* : P^*) = (P : L_1)$, but the right-hand side is equal to $(L^* : Q^\vee)$ by [10, Proposition 4.67]. Therefore, $(L_1^* : Q^\vee) = (L^* : Q^\vee)$, and we can conclude that $L_1 = L$.

4. Explicit forms of zeta-functions

In this section, we will give several examples of explicit forms of zeta-functions defined by (3.1). When $L = P$, the zeta-function is nothing but $\zeta_r(\mathbf{s};\mathfrak{g})$, so our main concern is the case $P \supsetneqq L \supset Q$.

Example 4.1. We first study the case of A_2 type. Let $\Delta = \Delta(A_2)$ with $\Psi = \{\alpha_1, \alpha_2\}$, $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$, $P = \mathbb{Z}\lambda_1+\mathbb{Z}\lambda_2$, $Q = \mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2$, and $\rho = \lambda_1+\lambda_2$. It is known that $(P : Q) = 3$ (see [3, Planche I]). Therefore, the only lattice L with $P \supsetneqq L \supset Q$ is Q. Then $Q_+ = P_+ \cap Q$. We show

$$Q_+ + \rho = \{m_1\lambda_1 + m_2\lambda_2 \mid m_1, m_2 \in \mathbb{N},\ m_1 \equiv m_2 \pmod 3\}. \tag{4.1}$$

To show this, first note that

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2. \tag{4.2}$$

In fact, write $\lambda_1 = u\alpha_1 + v\alpha_2$. Since $\alpha_i^\vee = 2\alpha_i/\langle\alpha_i, \alpha_i\rangle$ and $\langle\alpha_i^\vee, \lambda_j\rangle = \delta_{ij}$, we have

$$1 = \langle\alpha_1^\vee, \lambda_1\rangle = \frac{2}{\langle\alpha_1, \alpha_1\rangle}(u\langle\alpha_1, \alpha_1\rangle + v\langle\alpha_1, \alpha_2\rangle) = 2u - v,$$

and $0 = \langle\alpha_2^\vee, \lambda_1\rangle = -u+2v$, from which we obtain $u = \frac{2}{3}$, $v = \frac{1}{3}$, so $\lambda_1 = \frac{2}{3}\alpha_1+\frac{1}{3}\alpha_2$. The case of λ_2 is similar.

Let $\lambda = m_1\lambda_1 + m_2\lambda_2 \in Q_+ + \rho$, $m_1, m_2 \in \mathbb{N}$. Then $\lambda - \rho = n_1\lambda_1 + n_2\lambda_2 \in Q_+$, where $n_j = m_j - 1$, $j = 1, 2$. From (4.2) we have

$$n_1\lambda_1 + n_2\lambda_2 = \frac{2n_1 + n_2}{3}\alpha_1 + \frac{n_1 + 2n_2}{3}\alpha_2.$$

Since this belongs to Q, we have $\frac{2n_1+n_2}{3} \in \mathbb{Z}$ and $\frac{n_1+2n_2}{3} \in \mathbb{Z}$, which are equivalent to $n_1 \equiv n_2 \pmod 3$. This implies (4.1).

The simply-connected group $\widetilde{G}$ in the A_2 case is $SU(3)$. Let $\widetilde{Z}$ be the center of $\widetilde{G}$. The group corresponding to Q is $\widetilde{G}/\widetilde{Z}$, which is the projective unitary group $PU(3)$. The zeta-function corresponding to P is

$$\begin{aligned}\zeta_2((s_1, s_2, s_3), \mathbf{y}; SU(3)) &= \zeta_2((s_1, s_2, s_3), \mathbf{y}; P; A_2) \\ &= \sum_{m,n=1}^{\infty} \frac{e^{2\pi i\langle\mathbf{y}, m\lambda_1+n\lambda_2\rangle}}{m^{s_1}n^{s_2}(m+n)^{s_3}},\end{aligned} \tag{4.3}$$

which, when $\mathbf{y} = \mathbf{0}$, is the classical Mordell-Tornheim double sum. In the case $\mathbf{y} = \lambda_1^\vee = \frac{2}{3}\alpha_1^\vee + \frac{1}{3}\alpha_2^\vee$, we have

$$\zeta_2((s_1, s_2, s_3), \lambda_1^\vee; SU(3)) = \sum_{m,n=1}^{\infty} \frac{\varrho^{2m+n}}{m^{s_1}n^{s_2}(m+n)^{s_3}}, \tag{4.4}$$

where $\varrho = \mathrm{e}^{2\pi i/3}$ is the cube root of unity. The zeta-function corresponding to Q is, by (3.1) and (4.1),

$$\begin{aligned}\zeta_2((s_1,s_2,s_3),\mathbf{y};PU(3)) &= \zeta_2((s_1,s_2,s_3),\mathbf{y};Q;A_2)\\ &= \sum_{\lambda\in Q_+ + \rho} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y},\lambda\rangle}}{\langle\alpha_1^\vee,\lambda\rangle^{s_1}\langle\alpha_2^\vee,\lambda\rangle^{s_2}\langle\alpha_1^\vee+\alpha_2^\vee,\lambda\rangle^{s_3}}\\ &= \sum_{\substack{m,n=1\\ m\equiv n \ (\mathrm{mod}\ 3)}}^{\infty} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y},m\lambda_1+n\lambda_2\rangle}}{m^{s_1}n^{s_2}(m+n)^{s_3}}. \end{aligned} \tag{4.5}$$

This can be regarded as a kind of "partial zeta-function" of A_2 type, the double analogue of the partial (Riemann) zeta-function.

For $\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee$, we can compute $\mathcal{P}(\mathbf{k},\mathbf{y};A_2)$ from their generating function $F(\mathbf{t},\mathbf{y};A_2)$, whose explicit form is given as [16, (9.12)]. For example, the result for $\mathcal{P}((2,2,2),\mathbf{y};A_2)$ is explicitly written as [16, (9.13)], from which and (2.19), when $\mathbf{y} = \mathbf{0}$, it follows that

$$\mathcal{P}((2,2,2),\mathbf{0};Q;A_2) = \frac{187}{2755620}.$$

Therefore, we obtain from Theorem 3.2 that

$$\zeta_2((2,2,2);PU(3)) = \sum_{\substack{m,n=1\\ m\equiv n \ (\mathrm{mod}\ 3)}}^{\infty} \frac{1}{m^2n^2(m+n)^2} = \frac{187}{688905}\pi^6. \tag{4.6}$$

Similarly, we can compute

$$\zeta_2((4,4,4);PU(3)) = \frac{3279473}{48475988686125}\pi^{12}, \tag{4.7}$$

$$\zeta_2((6,6,6);PU(3)) = \frac{53109402098}{302027554315710345625}\pi^{18}, \tag{4.8}$$

$$\zeta_2((8,8,8);PU(3)) = \frac{178778564412743}{3909780002479478774489029687 5}\pi^{24}. \tag{4.9}$$

Also, in the case $\mathbf{y} = \lambda_1^\vee = \frac{2}{3}\alpha_1^\vee + \frac{1}{3}\alpha_2^\vee$ (see (4.4)), that is, $(y_1,y_2) = \left(\frac{2}{3},\frac{1}{3}\right)$, we can similarly obtain

$$\zeta_2((2,2,2),\lambda_1^\vee;SU(3)) = \frac{53}{229635}\pi^6, \tag{4.10}$$

$$\zeta_2((4,4,4),\lambda_1^\vee;SU(3)) = \frac{1078771}{16158662895375}\pi^{12}, \tag{4.11}$$

$$\zeta_2((6,6,6),\lambda_1^\vee;SU(3)) = \frac{88392335894}{503379257192850576037 5}\pi^{18}, \tag{4.12}$$

$$\zeta_2((8,8,8),\lambda_1^\vee;SU(3)) = \frac{1012923518531597}{221554200140503797221045015625}\pi^{24}. \tag{4.13}$$

Note that from the definition, we can confirm

$$\begin{aligned}\zeta_2((2p,2p,2p),\lambda_1^\vee;SU(3)) &= \zeta_2((2p,2p,2p),\lambda_2^\vee;SU(3)), \quad p\in\mathbb{N},\\ \zeta_2((2p,2p,2p),\lambda_1^\vee;PU(3)) &= \zeta_2((2p,2p,2p),\lambda_2^\vee;PU(3))\\ &= \zeta_2((2p,2p,2p),\mathbf{0};PU(3)), \quad p\in\mathbb{N}.\end{aligned}$$

In the next section, we will prove certain functional relations for $\zeta_2(\mathbf{s},\mathbf{0};\ PU(3))$ including (4.6)–(4.9).

Remark 4. In [24, Section 5], Subbarao and Sitaramachandrarao proposed a problem of evaluating the double series

$$\sum_{m,n=1}^{\infty}\frac{(-1)^{m-1}}{m^k n^k (m+n)^k},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^k n^k (m+n)^k},\quad k\in\mathbb{N}.$$

As for the case of odd k, the third-named author evaluated each series in terms of values of $\zeta(s)$ (see [28, 29]). The case of even k is still open. It follows from (4.4) that the above formulas (4.10)–(4.13) imply certain answers to a problem analogous to that of Subbarao and Sitaramachandrarao.

Example 4.2. We consider the A_3 type. Let $\Delta=\Delta(A_3)$ with $\Psi=\{\alpha_1,\alpha_2,\alpha_3\}$, $\Delta_+=\{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$, $P=\sum_{j=1}^{3}\mathbb{Z}\lambda_j$ and $Q=\sum_{j=1}^{3}\mathbb{Z}\alpha_j$. Analogously to (4.2), we have

$$\lambda_1=\frac{3}{4}\alpha_1+\frac{1}{2}\alpha_2+\frac{1}{4}\alpha_3,\quad \lambda_2=\frac{1}{2}\alpha_1+\alpha_2+\frac{1}{2}\alpha_3,\quad \lambda_3=\frac{1}{4}\alpha_1+\frac{1}{2}\alpha_2+\frac{3}{4}\alpha_3. \tag{4.14}$$

It is known that $P/Q\simeq\mathbb{Z}/4\mathbb{Z}$ (see [3]). Therefore, there is a unique intermediate lattice L_1 with $P\supsetneq L_1\supsetneq Q$, satisfying $(L_1:Q)=2$. The group corresponding to P (respectively Q) is $SU(4)$ (respectively $PU(4)$). The group $G=G(L_1)$ is $SU(4)/\{\pm1\}$, which is known to be isomorphic to $SO(6)$.

We know (see [17]) that

$$\begin{aligned}&\zeta_3(\mathbf{s},\mathbf{y};SU(4))=\zeta_3(\mathbf{s},\mathbf{y};P;A_3)\\ &=\sum_{m_1,m_2,m_3=1}^{\infty}\frac{e^{2\pi i\langle \mathbf{y},m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}}.\end{aligned}\tag{4.15}$$

Note that $\zeta_3(\mathbf{s},\mathbf{0};SU(4))=\zeta_3(\mathbf{s};A_3)$ (see [22]). Similarly to the A_2 case (see Example 4.1), from the generating function which was already given in [15, Example 2], we can compute $\mathcal{P}((2,2,2,2,2,2),\mathbf{y};A_3)$, though it is too long to write it here. Hence, we can obtain by (2.19) that

$$\mathcal{P}((2,2,2,2,2,2),\lambda_1^\vee;P;A_3)=-\frac{19329337}{14283291230208000},$$

where $\lambda_1^\vee=\frac{3}{4}\alpha_1^\vee+\frac{1}{2}\alpha_2^\vee+\frac{1}{4}\alpha_3^\vee$. Therefore, we obtain from Theorem 3.2 that

$$\zeta_3((2,2,2,2,2,2),\lambda_1^\vee;SU(4))=\zeta_3((2,2,2,2,2,2),\lambda_1^\vee;P;A_3)$$

$$\begin{aligned}&= \sum_{m_1,m_2,m_3=1}^{\infty} \frac{i^{3l+2m+n}}{m_1^2m_2^2m_3^2(m_1+m_2)^2(m_2+m_3)^2(m_1+m_2+m_3)^2}\\ &= -\frac{19329337}{2678117105664000}\pi^{12}.\end{aligned} \tag{4.16}$$

Concerning L_1 and Q, similarly to (4.1), we can show

$$(L_1)_+ + \rho = \left\{ \sum_{j=1}^{3} m_j\lambda_j \,\middle|\, (m_j) \in \mathbb{N}^3, m_1 \equiv m_3 \ (\mathrm{mod}\ 2) \right\}, \tag{4.17}$$

$$Q_+ + \rho = \left\{ \sum_{j=1}^{3} m_j\lambda_j \,\middle|\, (m_j) \in \mathbb{N}^3, m_1+2m_2+3m_3 \equiv 2 (\mathrm{mod}\ 4) \right\}. \tag{4.18}$$

In fact, letting $\lambda = \sum_{j=1}^{3} m_j\lambda_j \in Q_+ + \rho$, $m_j \in \mathbb{N}$, we have $\lambda - \rho = \sum_{j=1}^{3} n_j\lambda_j \in Q_+$, where $n_j = m_j - 1$, $1 \leqslant j \leqslant 3$. From (4.14), we have

$$\sum_{j=1}^{3} n_j\lambda_j = \frac{3n_1+2n_2+n_3}{4}\alpha_1 + \frac{n_1+2n_2+n_3}{2}\alpha_2 + \frac{n_1+2n_2+3n_3}{4}\alpha_3,$$

which belongs to Q. Therefore,

$$\begin{aligned}&\text{(i)} \quad 3n_1+2n_2+n_3 \equiv 0 \ (\mathrm{mod}\ 4),\\ &\text{(ii)} \quad n_1+2n_2+n_3 \equiv 0 \ (\mathrm{mod}\ 2),\\ &\text{(iii)} \quad n_1+2n_2+3n_3 \equiv 0 \ (\mathrm{mod}\ 4).\end{aligned}$$

We see that (iii) implies

$$\text{(iv)}\ n_1 \equiv n_3 \ (\mathrm{mod}\ 2),$$

which automatically implies (ii). Moreover, we find that (iii) and (iv) imply (i). Therefore, the only essential condition is (iii), which is equivalent to the congruence condition in (4.18).

Next, define the homomorphism $\eta : P \cong \mathbb{Z}^3 \to \mathbb{Z}/4\mathbb{Z}$ by

$$\eta(n_1,n_2,n_3) = n_1 + 2n_2 + 3n_3 \ (\mathrm{mod}\ 4).$$

Then from the above argument we find that $Q = \mathrm{Ker}\ \eta$. Let L_1^* be the set of all (n_1,n_2,n_3) satisfying $n_1 \equiv n_3 \ (\mathrm{mod}\ 2)$. Then $\{0\} \subsetneq \eta(L_1^*) \subsetneq \mathbb{Z}/4\mathbb{Z}$, hence $Q \subsetneq L_1^* \subsetneq P$. Therefore L_1^* should be equal to L_1, which implies (4.17).

Let $\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee + y_3\alpha_3^\vee$. From (4.17) and (4.18), we obtain

$$\begin{aligned}&\zeta_3(\mathbf{s},\mathbf{y};SO(6)) = \zeta_3(\mathbf{s},\mathbf{y};L_1;A_3)\\ &= \sum_{\substack{m_1,m_2,m_3=1\\ m_1\equiv m_3 (\mathrm{mod} 2)}}^{\infty} \frac{e^{2\pi i\langle \mathbf{y},m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}}, \quad (4.19)\\ &\zeta_3(\mathbf{s},\mathbf{y};PU(4)) = \zeta_3(\mathbf{s},\mathbf{y};Q;A_3)\end{aligned}$$

$$= \sum_{\substack{m_1,m_2,m_3=1 \\ m_1+2m_2+3m_3\equiv 2(\mathrm{mod} 4)}}^{\infty} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y}, m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}}. \quad (4.20)$$

EXAMPLE 4.3. We consider the case of B_r and of C_r types. The simply-connected group $\widetilde{G}$ in the B_r case is the spinor group $Spin(2r+1)$, and $\widetilde{G}/\widetilde{Z} = SO(2r+1)$, where $\widetilde{Z}$ is the center of $\widetilde{G}$. In the C_r case $\widetilde{G} = Sp(r)$, and $\widetilde{G}/\widetilde{Z}$ is the projective symplectic group $PSp(r)$. The explicit forms of zeta-functions for $r = 2, 3$ are

$$\begin{aligned}
&\zeta_2(\mathbf{s},\mathbf{y};Spin(5)) = \zeta_2(\mathbf{s},\mathbf{y};P;B_2) \\
&= \sum_{m_1,m_2=1}^{\infty} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y}, m_1\lambda_1+m_2\lambda_2\rangle}}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}(2m_1+m_2)^{s_4}}, \\
&\zeta_2(\mathbf{s},\mathbf{y};Sp(2)) = \zeta_2(\mathbf{s},\mathbf{y};P;C_2) \\
&= \sum_{m_1,m_2=1}^{\infty} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y}, m\lambda_1+n\lambda_2\rangle}}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}(m_1+2m_2)^{s_4}}, \\
&\zeta_3(\mathbf{s},\mathbf{y};Spin(7)) = \zeta_3(\mathbf{s},\mathbf{y};P;B_3) \\
&= \sum_{m_1,m_2,m_3=1}^{\infty} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y}, m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(2m_2+m_3)^{s_6}} \\
&\quad \times \frac{1}{(m_1+m_2+m_3)^{s_7}(m_1+2m_2+m_3)^{s_8}(2m_1+2m_2+m_3)^{s_9}}, \\
&\zeta_3(\mathbf{s},\mathbf{y};Sp(3)) = \zeta_3(\mathbf{s},\mathbf{y};P;C_3) \\
&= \sum_{m_1,m_2,m_3=1}^{\infty} \frac{\mathrm{e}^{2\pi i\langle \mathbf{y}, m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_2+2m_3)^{s_6}} \\
&\quad \times \frac{1}{(m_1+m_2+m_3)^{s_7}(m_1+m_2+2m_3)^{s_8}(m_1+2m_2+2m_3)^{s_9}}
\end{aligned}$$

(see [14, Sections 8 and 9], [17, Section 6], where only the formulas in the special case when $\mathbf{y} = \mathbf{0}$ are stated). We know that $(P : Q) = 2$ in the case of B_r and of C_r types (see [3]). Therefore, the lattice L with $P \supset L \supset Q$ coincides with P or Q. We show

$$\begin{aligned}
Q_+(B_2)+\rho &= \left\{ \sum_{j=1}^{2} m_j\lambda_j \,\middle|\, (m_j) \in \mathbb{N}^2,\ m_2 \equiv 1 \ (\mathrm{mod}\ 2) \right\}, \\
Q_+(C_2)+\rho &= \left\{ \sum_{j=1}^{2} m_j\lambda_j \,\middle|\, (m_j) \in \mathbb{N}^2,\ m_1 \equiv 1 \ (\mathrm{mod}\ 2) \right\}, \\
Q_+(B_3)+\rho &= \left\{ \sum_{j=1}^{3} m_j\lambda_j \,\middle|\, (m_j) \in \mathbb{N}^3,\ m_3 \equiv 1 \ (\mathrm{mod}\ 2) \right\},
\end{aligned}$$

$$Q_+(C_3)+\rho \quad = \quad \left\{ \sum_{j=1}^{3} m_j\lambda_j \;\middle|\; (m_j)\in\mathbb{N}^3,\ m_1\equiv m_3 \ (\mathrm{mod}\ 2) \right\}.$$

In fact, we consider, for example, the case of B_3. Analogously to Examples 4.1 and 4.2, we have

$$\lambda_1=\alpha_1+\alpha_2+\alpha_3, \quad \lambda_2=\alpha_1+2\alpha_2+2\alpha_3, \quad \lambda_3=\frac{1}{2}\alpha_1+\alpha_2+\frac{3}{2}\alpha_3. \tag{4.21}$$

Let $\lambda=\sum_{j=1}^{3} m_j\lambda_j\in Q_+ +\rho$, $m_j\in\mathbb{N}$. Then $\lambda-\rho=\sum_{j=1}^{3} n_j\lambda_j\in Q_+$, where $n_j=m_j-1$, $1\leqslant j\leqslant 3$. It follows from (4.21) that

$$\lambda-\rho=\frac{2n_1+2n_2+n_3}{2}\alpha_1+(n_1+2n_2+n_3)\alpha_2+\frac{2n_1+4n_2+3n_3}{2}\alpha_3,$$

which belongs to Q. Therefore, $n_3\equiv 0$ (mod 2), that is, $m_3\equiv 1$ (mod 2). The cases of B_2, C_2 and C_3 can be similarly treated.

Therefore, we obtain

$$\begin{aligned}\zeta_2(\mathbf{s},\mathbf{y};SO(5)) &= \zeta_2(\mathbf{s},\mathbf{y};Q;B_2)\\ &= \sum_{\substack{m_1,m_2=1\\ m_2\equiv 1\ (\mathrm{mod}\ 2)}}^{\infty} \frac{e^{2\pi i\langle \mathbf{y},m_1\lambda_1+m_2\lambda_2\rangle}}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}(2m_1+m_2)^{s_4}},\end{aligned} \tag{4.22}$$

$$\begin{aligned}\zeta_2(\mathbf{s},\mathbf{y};PSp(2)) &= \zeta_2(\mathbf{s},\mathbf{y};Q;C_2)\\ &= \sum_{\substack{m_1,m_2=1\\ m_1\equiv 1\ (\mathrm{mod}\ 2)}}^{\infty} \frac{e^{2\pi i\langle \mathbf{y},m_1\lambda_1+m_2\lambda_2\rangle}}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}(m_1+2m_2)^{s_4}},\end{aligned} \tag{4.23}$$

$$\begin{aligned}&\zeta_3(\mathbf{s},\mathbf{y};SO(7)) = \zeta_3(\mathbf{s},\mathbf{y};Q;B_3)\\ &\quad= \sum_{\substack{m_1,m_2,m_3=1\\ m_3\equiv 1\ (\mathrm{mod}\ 2)}}^{\infty} \frac{e^{2\pi i\langle \mathbf{y},m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(2m_2+m_3)^{s_6}}\\ &\qquad\times\frac{1}{(m_1+m_2+m_3)^{s_7}(m_1+2m_2+m_3)^{s_8}(2m_1+2m_2+m_3)^{s_9}},\end{aligned} \tag{4.24}$$

$$\begin{aligned}&\zeta_3(\mathbf{s},\mathbf{y};PSp(3)) = \zeta_3(\mathbf{s},\mathbf{y};Q;C_3)\\ &\quad= \sum_{\substack{m_1,m_2,m_3=1\\ m_1\equiv m_3\ (\mathrm{mod}\ 2)}}^{\infty} \frac{e^{2\pi i\langle \mathbf{y},m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_2+2m_3)^{s_6}}\\ &\qquad\times\frac{1}{(m_1+m_2+m_3)^{s_7}(m_1+m_2+2m_3)^{s_8}(m_1+2m_2+2m_3)^{s_9}}.\end{aligned} \tag{4.25}$$

Now we evaluate special values. Consider the case $G=PSp(2)$. Theorem 3.2 of C_2 type with $\nu=\mathbf{y}=\mathbf{0}$ gives that

$$\zeta_2((2k,2l,2l,2k);PSp(2))\in\mathbb{Q}\cdot\pi^{4(k+l)} \tag{4.26}$$

for $k, l \in \mathbb{N}$. Actually we have already given the generating function of C_2 type (see [15, Examples 1]) and also the generalized Bernoulli function $\mathcal{P}((2,2,2,2),\mathbf{y};C_2)$ (see [15, Examples 3]). Similarly we can give explicit forms of $\mathcal{P}((2k,2l,2l,2k),\mathbf{y};C_2)$, $k, l \in \mathbb{N}$, though it is too complicated to describe them here. Using these results, we can obtain the following explicit formulas:

$$\begin{aligned}
\zeta_2((2,2,2,2);PSp(2)) &= \frac{\pi^8}{322560},\\
\zeta_2((2,4,4,2);PSp(2)) &= \frac{29}{3832012800}\pi^{12},\\
\zeta_2((4,2,2,4);PSp(2)) &= \frac{13}{3832012800}\pi^{12},\\
\zeta_2((4,4,4,4);PSp(2)) &= \frac{479}{55794106368000}\pi^{16}.
\end{aligned}$$

We will give another type of evaluation formulas in the following sections (see Examples 5.7 and 6.5).

Remark 5. From (1.4) and (1.7) we see that the original volume formula of Witten is restricted to the case $\mathbf{s} = 2(k_\alpha)_{\alpha\in\Delta_+}$, where all the k_αs are the same. Our Theorem 3.2 covers a wider class of special values, such as the $(2,4,4,2)$ and $(4,2,2,4)$ cases in the above.

5. Functional relations and various evaluation formulas

In the preceding section, we gave explicit forms of several zeta-functions of Lie groups, and especially gave some evaluation formulas in the cases of A_2, C_2 and A_3 types at even integer points, by computing generating functions of their values. However, it seems a difficult problem to evaluate zeta-functions of Lie groups at arbitrary positive integer points by that method. In this section, we give various evaluation formulas for zeta values in the cases of A_2 and $C_2(\simeq B_2)$ types, by proving certain functional relations among them which are analogues of our previous results given in [12, 14, 15, 22, 33]. The advantage of the method in this section is that it may treat the special values at $\mathbf{s} = \mathbf{l} = (l_\alpha)_{\alpha\in\Delta_+}$, $l_\alpha \in \mathbb{N}$ and some of them are odd.

First we consider $\zeta_2(\mathbf{s};PU(3)) = \zeta_2(\mathbf{s},\mathbf{0};Q;A_2)$ and prove the following theorem, where $\phi(s,\alpha)$ is the Lerch zeta-function defined by (2.13).

Theorem 5.1. *For* $p, q \in \mathbb{N}$,

$$\begin{aligned}
3\Big\{&\zeta_2((p,q,s);PU(3)) + (-1)^p\zeta_2((p,s,q);PU(3))\\
&+(-1)^q\zeta_2((q,s,p);PU(3))\Big\}\\
&= -\sum_{\tau=0}^{p}\binom{p+q-\tau-1}{q-1}(-1)^\tau\frac{(2\pi i)^\tau}{\tau!}
\end{aligned}$$

$$\begin{aligned} &\times\sum_{a=0}^{2} B_\tau\left(\frac{a}{3}\right)\phi\Bigl(s+p+q-\tau,-\frac{a}{3}\Bigr) \\ &-\sum_{\tau=0}^{q}\binom{p+q-\tau-1}{p-1}\frac{(2\pi i)^\tau}{\tau!} \\ &\times\sum_{a=0}^{2} B_\tau\left(\frac{a}{3}\right)\phi\Bigl(s+p+q-\tau,\frac{a}{3}\Bigr) \end{aligned} \tag{5.1}$$

holds for $s\in\mathbb{C}$ except for singularities of functions on the both sides.

Example 5.2. It should be emphasized that Theorem 5.1 gives evaluation formulas for $\zeta_2((a,b,c);PU(3))$ when $a+b+c$ is odd. For example, putting $(p,q,s)=(1,1,1)$ in (5.1), we have

$$\begin{aligned} 3\zeta_2((1,1,1);PU(3)) &= \sum_{\tau=0}^{1}(-1)^\tau\frac{(2\pi i)^\tau}{\tau!}\sum_{a=0}^{2}B_\tau\left(\frac{a}{3}\right)\sum_{m=1}^{\infty}\frac{\varrho^{-ma}}{m^{3-\tau}} \\ &\quad+\sum_{\tau=0}^{1}\frac{(2\pi i)^\tau}{\tau!}\sum_{a=0}^{2}B_\tau\left(\frac{a}{3}\right)\sum_{m=1}^{\infty}\frac{\varrho^{ma}}{m^{3-\tau}}. \end{aligned} \tag{5.2}$$

We can easily check that

$$\sum_{a=0}^{2}\varrho^{la}B_1\left(\frac{a}{3}\right)=\begin{cases}-\frac{1}{2} & \text{if}\quad l\equiv 0 \pmod 3,\\ -\frac{1}{2}-\frac{1}{2\sqrt{3}}i & \text{if}\quad l\equiv 1 \pmod 3,\\ -\frac{1}{2}+\frac{1}{2\sqrt{3}}i & \text{if}\quad l\equiv 2 \pmod 3.\end{cases} \tag{5.3}$$

Then (5.2) can be rewritten to

$$\zeta_2((1,1,1);PU(3))=\frac{2}{27}\zeta(3)+\frac{2\pi}{3\sqrt{3}}L(2,\chi_3), \tag{5.4}$$

where we denote by χ_3 the primitive Dirichlet character of conductor 3. This is an analogue of $\zeta_2((1,1,1);SU(3))=2\zeta(3)$ (see [27]). Similarly, setting $(p,q,s)=(1,2,2)$ in (5.1), and using the relations (5.3) and

$$\sum_{a=0}^{2}\varrho^{la}B_2\left(\frac{a}{3}\right)=\begin{cases}\frac{1}{18} & \text{if}\quad l\equiv 0 \pmod 3,\\ \frac{2}{9} & \text{if}\quad l\equiv 1,2 \pmod 3,\end{cases} \tag{5.5}$$

we can obtain

$$\zeta_2((2,2,1);PU(3))=-\frac{1}{81}\zeta(5)+\frac{35\pi^2}{243}\zeta(3)-\frac{2\pi}{3\sqrt{3}}L(4,\chi_3). \tag{5.6}$$

The above formulas (5.4) and (5.6) can also be deduced by using

$$\phi\Bigl(s,\frac{1}{3}\Bigr)-\phi\Bigl(s,\frac{2}{3}\Bigr)=2i\sum_{m=1}^{\infty}\frac{\sin(2\pi m/3)}{m^s}=\sqrt{3}iL(s,\chi_3) \tag{5.7}$$

instead of (5.3), (5.5). A more general result will be given in the next section (see Theorem 6.1).

By the partial fraction decomposition, we have

$$\zeta_2((1,1,1);PU(3)) = \sum_{\substack{m,n=1\\ m\equiv n \pmod 3}}^{\infty} \frac{1}{mn(m+n)} = 2\sum_{\substack{m,n=1\\ m\equiv n \pmod 3}}^{\infty} \frac{1}{m(m+n)^2}.$$

Hence, combining with (5.4) we obtain

$$\sum_{\substack{m,n=1\\ m\equiv n \pmod 3}}^{\infty} \frac{1}{m(m+n)^2} = \frac{1}{27}\zeta(3) + \frac{\pi}{3\sqrt{3}}L(2,\chi_3). \tag{5.8}$$

This can be regarded as a formula for a partial sum of the double zeta value, analogously to the well-known result given by Euler (cf. [9]):

$$\sum_{m,n=1}^{\infty} \frac{1}{m(m+n)^2} = \zeta(3).$$

Remark 6. Setting $(p,q,s)=(2k,2k,2k)$ in Theorem 5.1 and using the fact

$$B_j(x) = -\frac{j!}{(2\pi i)^j} \lim_{M\to\infty} \sum_{\substack{m=-M\\ m\neq 0}}^{M} \frac{\mathrm{e}^{2\pi i m x}}{m^j}, \quad j\in\mathbb{N}, \quad 0\leqslant x<1, \tag{5.9}$$

(see [1, p. 266]), we obtain

$$\begin{aligned}&\zeta_2((2k,2k,2k);PU(3))\\ &\quad = \frac{(2\pi i)^{6k}}{9}\sum_{\tau=0}^{2k}\binom{4k-\tau-1}{2k-1}\\ &\quad\quad \times\sum_{a=0}^{2}\frac{B_\tau(a/3)}{\tau!}\frac{B_{6k-\tau}(a/3)}{(6k-\tau)!}, \quad k\in\mathbb{N},\end{aligned} \tag{5.10}$$

which is an explicit form of (3.2) for $PU(3)$ and includes (4.6)–(4.9).

Now we give the proof of Theorem 5.1. We first prepare the following lemma which can be proved by the same method as introduced in [14]. In fact, this lemma in the case when p and q are even has already been proved in [14, (7.55)]. We use the notation $\phi(s) := \phi(s,\frac{1}{2}) = (2^{1-s}-1)\zeta(s)$ and $\varepsilon_m := \frac{1+(-1)^m}{2}$ for $m\in\mathbb{Z}$.

Lemma 5.3. *For $p\in\mathbb{N}$, $s\in\mathbb{R}$ with $s>1$ and $x\in\mathbb{C}$ with $|x|\leqslant 1$,*

$$\sum_{\substack{l\neq 0,\, m\geqslant 1\\ l+m\neq 0}} \frac{(-1)^{l+m}x^m \mathrm{e}^{i(l+m)\theta}}{l^p m^s (l+m)^q}$$

$$\begin{aligned}
&-2\sum_{j=0}^{p}\ \phi(p-j)\varepsilon_{p-j}\sum_{\xi=0}^{j}\binom{q-1+j-\xi}{q-1}\\
&\qquad\times(-1)^{j-\xi}\sum_{m=1}^{\infty}\frac{(-1)^m x^m \mathrm{e}^{im\theta}}{m^{s+q+j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\
&+2\sum_{j=0}^{q}\ \phi(q-j)\varepsilon_{q-j}\sum_{\xi=0}^{j}\binom{p-1+j-\xi}{p-1}\\
&\qquad\times(-1)^{p-1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p+j-\xi}}\frac{(i\theta)^\xi}{\xi!}=0 \qquad (5.11)
\end{aligned}$$

holds for $\theta \in [-\pi,\pi]$.

Proof. For $p \in \mathbb{N}$, it is known that (see, for example, [14, (4.31), (4.32)])

$$\lim_{L\to\infty}\sum_{\substack{-L\leqslant l\leqslant L\\ l\neq 0}}\frac{(-1)^l\mathrm{e}^{il\theta}}{l^p}=2\sum_{j=0}^{p}\ \phi(p-j)\varepsilon_{p-j}\frac{(i\theta)^j}{j!},\quad \theta\in(-\pi,\pi). \qquad (5.12)$$

Note that the left-hand side is uniformly convergent for $\theta \in (-\pi,\pi)$ (see [36, § 3.35]), and is also absolutely convergent for $\theta \in [-\pi,\pi]$ when $p \geqslant 2$. First we assume $p \geqslant 2$. Then, for $\theta \in [-\pi,\pi]$, it follows from (5.12) that

$$\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l\mathrm{e}^{il\theta}}{l^p}-2\sum_{j=0}^{p}\ \phi(p-j)\varepsilon_{p-j}\frac{(i\theta)^j}{j}\right)\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathrm{e}^{im\theta}}{m^s}=0, \qquad (5.13)$$

where the left-hand side is absolutely and uniformly convergent for $\theta \in [-\pi,\pi]$. Therefore, we have

$$\begin{aligned}
&\sum_{\substack{l\in\mathbb{Z},l\neq 0\\ m\geqslant 1\\ l+m\neq 0}}\frac{(-1)^{l+m}x^m\mathrm{e}^{i(l+m)\theta}}{l^p m^s}\\
&\qquad -2\sum_{j=0}^{p}\ \phi(p-j)\varepsilon_{p-j}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathrm{e}^{im\theta}}{m^s}\right\}\frac{(i\theta)^j}{j!}\\
&\qquad =(-1)^{p+1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p}} \qquad (5.14)
\end{aligned}$$

for $\theta \in [-\pi,\pi]$. Now we apply [14, Lemma 6.2] with $d = q \in \mathbb{N}$. Then we obtain (5.11) for $p \geqslant 2$.

Next we prove the case $p = 1$. As we proved above, (5.11) in the case $p = 2$ holds. Replacing x by $-x\mathrm{e}^{i\theta}$ in this case, we have

$$\sum_{\substack{l\neq 0,\, m\geqslant 1\\ l+m\neq 0}}\frac{(-1)^l x^m\mathrm{e}^{il\theta}}{l^2 m^s (l+m)^q}$$

$$
\begin{aligned}
&-2\sum_{j=0}^{2}\ \phi(2-j)\varepsilon_{2-j}\sum_{\xi=0}^{j}\binom{q-1+j-\xi}{q-1}\\
&\qquad\times(-1)^{j-\xi}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+q+j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\
&+2\sum_{j=0}^{q}\ \phi(q-j)\varepsilon_{q-j}\sum_{\xi=0}^{j}\binom{1+j-\xi}{1}\\
&\qquad\times(-1)^{1}\sum_{m=1}^{\infty}\frac{(-1)^m x^m \mathrm{e}^{-im\theta}}{m^{s+2+j-\xi}}\frac{(i\theta)^\xi}{\xi!}=0 \qquad (5.15)
\end{aligned}
$$

for $\theta\in[-\pi,\pi]$. We denote the first, the second and the third term on the left hand side of (5.15) by $I_1(\theta)$, $I_2(\theta)$ and $I_3(\theta)$, respectively. We differentiate these terms in θ. We can easily compute $I_1'(\theta)$ and $I_2'(\theta)$. As for $I_3'(\theta)$, we have

$$
\begin{aligned}
I_3'(\theta)=2\sum_{j=0}^{q}\phi(q-j)\varepsilon_{q-j}\Big\{&-i\sum_{\xi=0}^{j}(1+j-\xi)(-1)\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathrm{e}^{-im\theta}}{m^{s+1+j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\
&+i\sum_{\xi=1}^{j}(1+j-\xi)(-1)\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathrm{e}^{-im\theta}}{m^{s+2+j-\xi}}\frac{(i\theta)^{\xi-1}}{(\xi-1)!}\Big\}.
\end{aligned}
$$

Note that as for the second member in the curly brackets on the right-hand side, ξ may also run from 1 to $j+1$ because $1+j-(j+1)=0$ in the summand. Hence, by replacing $\xi-1$ by ξ, we have

$$
I_3'(\theta)=2i\sum_{j=0}^{q}\phi(q-j)\varepsilon_{q-j}\sum_{\xi=0}^{j}\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathrm{e}^{-im\theta}}{m^{s+1+j-\xi}}\frac{(i\theta)^\xi}{\xi!}.
$$

Thus, we see that $\frac{I_1'(\theta)+I_2'(\theta)+I_3'(\theta)}{i}$, replacing x by $-x\mathrm{e}^{i\theta}$, gives (5.11) in the case $p=1$. This completes the proof of Lemma 5.4.

Here we quote the following lemma given in [15, Lemma 9.1]. Note that the assertion in [15, Lemma 9.1] is stated only in the case that p is even. However, we can easily check that the assertion holds for any $p\in\mathbb{N}$ as follows.

LEMMA 5.4. *Let $t\in[0,2\pi)\subset\mathbb{R}$, and $h:\mathbb{N}_0\to\mathbb{C}$ be a function (which may depend on t). Then, for $p\in\mathbb{N}$,*

$$
\begin{aligned}
&\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\sum_{\xi=0}^{j}h(j-\xi)\frac{(i(t-\pi))^\xi}{\xi!}\\
&\qquad=-\frac{1}{2}\sum_{\xi=0}^{p}h(p-\xi)\frac{(2\pi i)^\xi}{\xi!}B_\xi\left(\left\{\frac{t}{2\pi}\right\}\right). \qquad (5.16)
\end{aligned}
$$

Put $\theta = t - \pi$, $0 \leqslant t < 2\pi$, in (5.11) and multiply by $(-1)^p$ the both sides. Then, using Lemma 5.4, we have the following. Note that this can also be derived by a certain transformation of a result of Nakamura [23, Theorem 3.1] when $|x| = 1$.

LEMMA 5.5. *For $p, q \in \mathbb{N}$, $s, t \in \mathbb{R}$ with $s > 1$ and $t \in [0, 2\pi)$, and $x \in \mathbb{C}$ with $|x| \leqslant 1$,*

$$\begin{aligned}
&\sum_{l,m=1}^{\infty} \frac{x^{l+m}\mathrm{e}^{imt}}{l^p m^q (l+m)^s} + (-1)^p \sum_{l,m=1}^{\infty} \frac{x^m \mathrm{e}^{i(l+m)t}}{l^p m^s (l+m)^q} + (-1)^q \sum_{l,m=1}^{\infty} \frac{x^m \mathrm{e}^{-ilt}}{l^q m^s (l+m)^p} \\
&= -\sum_{\tau=0}^{p} \binom{p+q-\tau-1}{q-1} (-1)^\tau \sum_{m=1}^{\infty} \frac{x^m \mathrm{e}^{imt}}{m^{s+p+q-\tau}} \frac{(2\pi i)^\tau}{\tau!} B_\tau\left(\left\{\frac{t}{2\pi}\right\}\right) \\
&\quad -\sum_{\tau=0}^{q} \binom{p+q-\tau-1}{p-1} \sum_{m=1}^{\infty} \frac{x^m}{m^{s+p+q-\tau}} \frac{(2\pi i)^\tau}{\tau!} B_\tau\left(\left\{\frac{t}{2\pi}\right\}\right). \qquad (5.17)
\end{aligned}$$

Using these results, we give the proof of Theorem 5.1 as follows.

Proof of Theorem 5.1. Let $x = \mathrm{e}^{-2it}$ and further let $t = 2\pi\frac{a}{3}$, $a = 0, 1, 2$, on the both sides of (5.17). Then, summing up with $a = 0, 1, 2$ and using the fact for $\varrho = \mathrm{e}^{2\pi i/3}$ that

$$\sum_{a=0}^{2} \varrho^{Na} = \begin{cases} 3 & \text{if } N \equiv 0 \pmod 3, \\ 0 & \text{if } N \not\equiv 0 \pmod 3, \end{cases}$$

we have

$$\begin{aligned}
&3\left\{ \sum_{\substack{l,m \geqslant 1 \\ l \equiv m \,(\mathrm{mod}\, 3)}} \frac{1}{l^p m^q (l+m)^s} + (-1)^p \sum_{\substack{l,m \geqslant 1 \\ l \equiv m \,(\mathrm{mod}\, 3)}} \frac{1}{l^p m^s (l+m)^q} \right. \\
&\qquad \left. + (-1)^q \sum_{\substack{l,m \geqslant 1 \\ l \equiv m \,(\mathrm{mod}\, 3)}} \frac{1}{l^q m^s (l+m)^p} \right\} \\
&= -\sum_{\tau=0}^{p} \binom{p+q-\tau-1}{q-1} (-1)^\tau \sum_{a=0}^{2} \sum_{m=1}^{\infty} \frac{\varrho^{-ma}}{m^{s+p+q-\tau}} \frac{(2\pi i)^\tau}{\tau!} B_\tau\left(\left\{\frac{a}{3}\right\}\right) \\
&\quad -\sum_{\tau=0}^{q} \binom{p+q-\tau-1}{p-1} \sum_{a=0}^{2} \sum_{m=1}^{\infty} \frac{\varrho^{ma}}{m^{s+p+q-\tau}} \frac{(2\pi i)^\tau}{\tau!} B_\tau\left(\left\{\frac{a}{3}\right\}\right).
\end{aligned}$$

Noting (4.5) and using Proposition 2.1, we complete the proof of Theorem 5.1.

Secondly, we consider the case of C_2 type, namely the zeta-function $\zeta_2(\mathbf{s}; PSp(2)) = \zeta_2(\mathbf{s}, \mathbf{0}; Q; C_2)$ defined by (4.23) with $\mathbf{y} = \mathbf{0}$. We already studied the zeta-function of C_2 type in [14, Section 8] and [15, Section 9]. In fact, using the same method as in the proof of [15, (9.8)], we can obtain

$$\sum_{\substack{l \geqslant 1 \\ m \geqslant 1}} \frac{\mathrm{e}^{ilt}}{l^p m^s (l+m)^q (l+2m)^r} + \sum_{\substack{l \geqslant 1 \\ m \geqslant 1 \\ l \neq m \\ l \neq 2m}} \frac{\mathrm{e}^{-ilt}}{(-l)^p m^s (-l+m)^q (-l+2m)^r}$$

$$
\begin{aligned}
&+\sum_{\xi=0}^{p}\sum_{\omega=0}^{p-\xi}\binom{\omega+r-1}{\omega}\binom{p+q-1-\xi-\omega}{q-1}\frac{(-1)^{p-\xi}}{2^{r+\omega}}\frac{(2\pi i)^{\xi}}{\xi!}\\
&\times\zeta(s+p+q+r-\xi)B_{\xi}\left(t/2\pi\right)\\
&+\sum_{\xi=0}^{q}\sum_{\omega=0}^{q-\xi}\binom{\omega+r-1}{\omega}\binom{p+q-1-\xi-\omega}{p-1}(-1)^{p-\omega}\frac{(2\pi i)^{\xi}}{\xi!}\\
&\times\phi(s+p+q+r-\xi,-t/2\pi)B_{\xi}\left(t/2\pi\right)\\
&+\sum_{\xi=0}^{r}\sum_{\omega=0}^{p-1}\binom{\omega+r-\xi}{\omega}\binom{p+q-2-\omega}{q-1}\frac{(-1)^{p}}{2^{r-\xi+\omega+1}}\frac{(2\pi i)^{\xi}}{\xi!}\\
&\times\phi(s+p+q+r-\xi,-t/\pi)B_{\xi}\left(t/2\pi\right)\\
&+\sum_{\xi=0}^{r}\sum_{\omega=0}^{q-1}\binom{\omega+r-\xi}{\omega}\binom{p+q-2-\omega}{p-1}(-1)^{p-\omega+1}\frac{(2\pi i)^{\xi}}{\xi!}\\
&\times\phi(s+p+q+r-\xi,-t/\pi)B_{\xi}\left(t/2\pi\right)=0 \qquad (5.18)
\end{aligned}
$$

for $p,q,r\in\mathbb{N}$ and $s,t\in\mathbb{R}$ with $s>1$ and $t\in[0,2\pi)$. Actually, this equation with replacing (p,q,r) by $(2p,2q,2p)$ coincides with [15, (9.8)] in the case $(\eta,\rho,\delta,\tau)=(t,0,0,0)$. Denote the second sum on the left-hand side of (5.18) by Σ_2. We split Σ_2 into two parts according to the conditions $l<m$ or $l>m$, and transform variables as $j=m-l(\geqslant 1)$ when $l<m$, and $j=l-m(\geqslant 1)$ when $l>m$. In the latter case we further split the sum according to $j<m$ or $j>m$ (that is, $l<2m$ or $l>2m$). Then we obtain

$$
\begin{aligned}
\Sigma_2 \;=\;& (-1)^{p}\sum_{\substack{l\geqslant 1\\ m\geqslant 1}}\frac{\mathrm{e}^{-ilt}}{l^{p}m^{q}(l+m)^{s}(l+2m)^{r}}\\
&+(-1)^{p+q}\sum_{\substack{l\geqslant 1\\ m\geqslant 1}}\frac{\mathrm{e}^{-i(l+2m)t}}{l^{r}m^{q}(l+m)^{s}(l+2m)^{p}}\\
&+(-1)^{p+q+r}\sum_{\substack{l\geqslant 1\\ m\geqslant 1}}\frac{\mathrm{e}^{-i(l+2m)t}}{l^{r}m^{s}(l+m)^{q}(l+2m)^{p}}. \qquad (5.19)
\end{aligned}
$$

Replacing Σ_2 by (5.19) on the left-hand side of (5.18), and denote the resulting left-hand side by $H(t)$. Then (5.18) implies $H(t)=0$ for any $t\in[0,2\pi)$. Let $t=0,\pi$. We note that $\mathrm{e}^{\pm il\pi}=\mathrm{e}^{-i(l+2m)\pi}=(-1)^{l}$ for $l,m\in\mathbb{N}$. Also we have $\phi(s,0)=\phi(s,-1)=\zeta(s)$, $\phi\big(s,-\frac{1}{2}\big)=\phi(s)=(2^{1-s}-1)\zeta(s)$. Therefore, considering $\frac{H(0)-H(\pi)}{2}=0$ and noting (4.23), similarly to Theorem 5.1, we have the following result.

Theorem 5.6. *For* $p,q,r\in\mathbb{N}$,

$$
\begin{aligned}
&\zeta_2((p,s,q,r);PSp(2))+(-1)^{p}\zeta_2((p,q,s,r);PSp(2))\\
&\quad+(-1)^{p+q}\zeta_2((r,q,s,p);PSp(2))+(-1)^{p+q+r}\zeta_2((r,s,q,p);PSp(2))\\
&\quad+\sum_{\xi=0}^{p}\sum_{\omega=0}^{p-\xi}\binom{\omega+r-1}{\omega}\binom{p+q-1-\xi-\omega}{q-1}\frac{(-1)^{p-\xi}}{2^{r+\omega}}\frac{(2\pi i)^{\xi}}{\xi!}
\end{aligned}
$$

$$
\begin{aligned}
&\times\zeta(s+p+q+r-\xi)\frac{B_\xi(0)-B_\xi(1/2)}{2}\\
&+\sum_{\xi=0}^{q}\sum_{\omega=0}^{q-\xi}\binom{\omega+r-1}{\omega}\binom{p+q-1-\xi-\omega}{p-1}(-1)^{p-\omega}\frac{(2\pi i)^\xi}{\xi!}\\
&\times\zeta(s+p+q+r-\xi)\frac{B_\xi(0)-\left(2^{1-s-p-q-r+\xi}-1\right)B_\xi(1/2)}{2}\\
&+\sum_{\xi=0}^{r}\sum_{\omega=0}^{p-1}\binom{\omega+r-\xi}{\omega}\binom{p+q-2-\omega}{q-1}\frac{(-1)^p}{2^{r-\xi+\omega+1}}\frac{(2\pi i)^\xi}{\xi!}\\
&\times\zeta(s+p+q+r-\xi)\frac{B_\xi(0)-B_\xi(1/2)}{2}\\
&+\sum_{\xi=0}^{r}\sum_{\omega=0}^{q-1}\binom{\omega+r-\xi}{\omega}\binom{p+q-2-\omega}{p-1}(-1)^{p-\omega+1}\frac{(2\pi i)^\xi}{\xi!}\\
&\times\zeta(s+p+q+r-\xi)\frac{B_\xi(0)-B_\xi(1/2)}{2}=0 \qquad (5.20)
\end{aligned}
$$

holds for $s\in\mathbb{C}$ except for singularities.

Example 5.7. By Theorem 5.6, we can evaluate $\zeta_2((a,b,c,d);PSp(2))$ in some case when $a+b+c+d$ is odd. For example, setting $(p,s,q,r)=(2,1,1,1)$ and $(2,3,3,5)$ in (5.20), we have

$$
\begin{aligned}
\zeta_2((2,1,1,1);PSp(2))&=\frac{3}{8}\zeta(2)\zeta(3)-\frac{31}{64}\zeta(5),\\
\zeta_2((2,3,3,5);PSp(2))&=-\frac{15}{16}\zeta(4)\zeta(9)-\frac{17379}{4096}\zeta(2)\zeta(11)+\frac{8191}{1024}\zeta(13).
\end{aligned}
$$

In general, it seems to be difficult to evaluate $\zeta_2((a,b,c,d);PSp(2))$ for arbitrary $a,b,c,d\in\mathbb{N}$. We will further consider this problem in the next section.

Remark 7. Putting $(p,s,q,r)=(2k,2l,2l,2k)$, $k,l\in\mathbb{N}$, in (5.20), we can see that

$$\zeta_2((2k,2l,2l,2k);PSp(2))$$

can be expressed as a polynomial in $\zeta(4k+4l-\xi)(i\pi)^\xi$ with $\mathbb{Q}$-coefficients. Since $\zeta_2((2k,2l,2l,2k);PSp(2))\in\mathbb{R}$, we see that the part consisting of the terms of $\zeta(4k+4l-\xi)(i\pi)^\xi$ for odd ξ vanish. On the other hand, for even ξ, each term belongs to $\mathbb{Q}\cdot\pi^{4(k+l)}$. Thus, we recover (4.26).

6. Parity results

In general, a parity result means a property that some multiple zeta value whose weight and depth are of different parity can be written in terms of multiple zeta values of lower depth. The first parity result is Euler's discovery that the double zeta value (of weight $p+q$)

$$\zeta_2(p,q)=\sum_{\substack{m\geqslant 1\\ n\geqslant 1}}\frac{1}{m^p(m+n)^q},\quad p,q\in\mathbb{N},\quad q\geqslant 2,$$

can be expressed as a polynomial in $\{\zeta(j+1)\,|\,j\in\mathbb{N}\}$ with $\mathbb{Q}$-coefficients (cf. [9]) if its weight $p+q$ is odd. This result has been generalized to the case of more general multiple zeta-values (see [8, 31]).

It is an interesting problem to ask what kind of multiple zeta values has this type of properties. Tornheim (see [27, Theorem 7]) proved that $\zeta_2((a,b,c),\mathbf{0};SU(3)) = \zeta_2((a,b,c),\mathbf{0};P,A_2)$ can be expressed as a polynomial in $\{\zeta(j+1)\,|\,j\in\mathbb{N}\}$ with $\mathbb{Q}$-coefficients if its weight $a+b+c$ is odd. This result has been generalized by the third-named author [32] to the case of multiple Mordell-Tornheim zeta values. Also, the third-named author (see [30]) proved that $\zeta_2((a,b,c,d),\mathbf{0};Sp(2)) = \zeta_2((b,a,c,d),\mathbf{0};Spin(5))$ has this property, which is an extension of the result of Apostol and Vu [2].

In this section, we first prove the following fact, which is a $PU(3)$ type analogue of Tornheim's result stated above.

Theorem 6.1. *Let $a,b,c\in\mathbb{N}$. If $a+b+c$ is odd then $\zeta_2((a,b,c);PU(3))$ can be expressed as a polynomial in $\{\phi(j;\frac{a}{3})\,|\,a\in\{0,1,2\},\ j\in\mathbb{N}\}$ with $\mathbb{Q}[\pi,i]$-coefficients.*

Proof. Denote by $\mathfrak{X}$ the set of polynomials in $\{\phi(j;\frac{a}{3})\,|\,a\in\{0,1,2\},\ j\in\mathbb{N}\}$ with $\mathbb{Q}[\pi,i]$-coefficients. Then we see that the right-hand side of (5.1) with $(p,q,s)=(c,a,b)$ is in $\mathfrak{X}$. First we consider the case a is odd and b,c is even, hence $a+b+c$ is odd. Then, by (5.1), we have

$$\zeta_2((c,a,b);PU(3))+\zeta_2((c,b,a);PU(3))-\zeta_2((a,b,c);PU(3))\in\mathfrak{X}.$$

Also, setting $(p,q,s)=(c,b,a)$ in (5.1), we have

$$\zeta_2((c,a,b);PU(3))+\zeta_2((c,b,a);PU(3))+\zeta_2((b,a,c);PU(3))\in\mathfrak{X}.$$

Note that $\zeta_2((p,q,r);PU(3)) = \zeta_2((q,p,r);PU(3))$. Hence, these imply the assertion $\zeta_2((a,b,c);PU(3))\in\mathfrak{X}$. As for other cases, we can similarly prove their assertions. This completes the proof of the theorem.

Example 6.2. Setting $(p,q,s)=(1,3,5)$ and $(1,5,3)$ in (5.1), we have

$$
\begin{aligned}
&\zeta_2((1,3,5);PU(3))-\zeta_2((1,5,3);PU(3))-\zeta_2((3,5,1);PU(3))\\
&\quad=-\frac{4}{3}\zeta(9)+\frac{\pi^2}{9}\zeta(7)-\frac{4}{3}\Bigl(\phi\Bigl(9,\frac{1}{3}\Bigr)+\phi\Bigl(9,\frac{2}{3}\Bigr)\Bigr)+\frac{2\pi i}{9}\Bigl(\phi\Bigl(8,\frac{1}{3}\Bigr)-\phi\Bigl(8,\frac{2}{3}\Bigr)\Bigr)\\
&\qquad-\frac{\pi^2}{27}\Bigl(\phi\Bigl(7,\frac{1}{3}\Bigr)+\phi\Bigl(7,\frac{2}{3}\Bigr)\Bigr)+\frac{4\pi^3 i}{243}\Bigl(\phi\Bigl(6,\frac{1}{3}\Bigr)-\phi\Bigl(6,\frac{2}{3}\Bigr)\Bigr),\\
&\zeta_2((1,5,3);PU(3))-\zeta_2((1,3,5);PU(3))-\zeta_2((5,3,1);PU(3))\\
&\quad=-2\zeta(9)+\frac{\pi^2}{9}\zeta(7)+\frac{\pi^4}{135}\zeta(5)-2\Bigl(\phi\Bigl(9,\frac{1}{3}\Bigr)+\phi\Bigl(9,\frac{2}{3}\Bigr)\Bigr)\\
&\qquad+\frac{2\pi i}{9}\Bigl(\phi\Bigl(8,\frac{1}{3}\Bigr)-\phi\Bigl(8,\frac{2}{3}\Bigr)\Bigr)-\frac{\pi^2}{27}\Bigl(\phi\Bigl(7,\frac{1}{3}\Bigr)+\phi\Bigl(7,\frac{2}{3}\Bigr)\Bigr)\\
&\qquad+\frac{4\pi^3 i}{243}\Bigl(\phi\Bigl(6,\frac{1}{3}\Bigr)-\phi\Bigl(6,\frac{2}{3}\Bigr)\Bigr)-\frac{13\pi^4}{3645}\Bigl(\phi\Bigl(5,\frac{1}{3}\Bigr)+\phi\Bigl(5,\frac{2}{3}\Bigr)\Bigr)\\
&\qquad+\frac{4\pi^5 i}{2187}\Bigl(\phi\Bigl(4,\frac{1}{3}\Bigr)-\phi\Bigl(4,\frac{2}{3}\Bigr)\Bigr).
\end{aligned}
$$

Combining these results and noting $\zeta_2((3,5,1);\ PU(3))=\zeta_2((5,3,1);\ PU(3))$, we have

$$
\begin{aligned}
&\zeta_2((3,5,1);PU(3))\\
&\quad=\frac{5}{3}\zeta(9)-\frac{\pi^2}{9}\zeta(7)-\frac{\pi^4}{270}\zeta(5)\\
&\qquad+\frac{5}{3}\Bigl(\phi\Bigl(9,\frac{1}{3}\Bigr)+\phi\Bigl(9,\frac{2}{3}\Bigr)\Bigr)-\frac{2\pi i}{9}\Bigl(\phi\Bigl(8,\frac{1}{3}\Bigr)-\phi\Bigl(8,\frac{2}{3}\Bigr)\Bigr)\\
&\qquad+\frac{\pi^2}{27}\Bigl(\phi\Bigl(7,\frac{1}{3}\Bigr)+\phi\Bigl(7,\frac{2}{3}\Bigr)\Bigr)-\frac{4\pi^3 i}{243}\Bigl(\phi\Bigl(6,\frac{1}{3}\Bigr)-\phi\Bigl(6,\frac{2}{3}\Bigr)\Bigr)\\
&\qquad+\frac{13\pi^4}{7290}\Bigl(\phi\Bigl(5,\frac{1}{3}\Bigr)+\phi\Bigl(5,\frac{2}{3}\Bigr)\Bigr)-\frac{2\pi^5 i}{2187}\Bigl(\phi\Bigl(4,\frac{1}{3}\Bigr)-\phi\Bigl(4,\frac{2}{3}\Bigr)\Bigr).
\end{aligned}
$$

Remark 8. Applying (5.7) and

$$\phi\Bigl(s,\frac{1}{3}\Bigr)+\phi\Bigl(s,\frac{2}{3}\Bigr)=(3^{1-s}-1)\zeta(s) \tag{6.1}$$

to the above expression, we find that $\zeta_2((3,5,1);PU(3))$ can actually be written in terms of Riemann-zeta values and values of the Dirichlet L-function attached to χ_3. In general, since from (5.7) and (6.1) we see that $\phi\bigl(s,\frac{1}{3}\bigr)$ and $\phi\bigl(s,\frac{2}{3}\bigr)$ can be written in terms of $L(s,\chi_3)$ and $\zeta(s)$, Theorem 6.1 can be reinterpreted that if $a+b+c$ is odd then $\zeta_2((a,b,c);PU(3))$ can be expressed as a polynomial in $\{\zeta(j),L(j,\chi_3)\,|\,j\in\mathbb{N}\}$ with $\mathbb{Q}[\pi,i]$-coefficients. This agrees with the results stated in Example 5.2.

Next consider the C_2 case. Since we have already known the parity result for $\zeta_2((a,b,c,d);Sp(2))$ ([30]), it should be not surprising to know that the following parity result for $\zeta_2((a,b,c,d);PSp(2))$ holds.

THEOREM 6.3. *Let $a,b,c,d \in \mathbb{N}$. If $a+b+c+d$ is odd then $\zeta_2((a,b,c,d);\ PSp(2))$ can be expressed as a polynomial in $\{\zeta(j+1)\,|\,j\in\mathbb{N}\}$ with $\mathbb{Q}$-coefficients.*

In this case, we cannot directly obtain the assertion from Theorem 5.6 unlike the case of $PU(3)$. In fact, even if we use (5.20), it seems unable to obtain an expression of $\zeta_2((1,2,2,2);PSp(2))$ because this value vanishes if we set $(p,s,q,r)=(1,2,2,2)$ or $(2,2,2,1)$ in (5.20). Hence, we use another method as follows. First we quote the following.

LEMMA 6.4 ([34, Theorem 4.1]). *Let*

$$\mathfrak{T}_{\tau,\mu}(k,l,d)=\sum_{\substack{l\geqslant 0\\ m\geqslant 0}}\frac{1}{(2l+\tau)^a(2m+\mu)^b(2l+2m+\tau+\mu)^c} \tag{6.2}$$

for $k,l,d\in\mathbb{N}$ and $\tau,\mu\in\{1,2\}$. Suppose $k+l+d$ is odd. Then $\mathfrak{T}_{\tau,\mu}(k,l,d)$ can be expressed as a polynomial in $\{\zeta(j+1)\,|\,j\in\mathbb{N}\}$ with $\mathbb{Q}$-coefficients.

It should be noted that the assertion in [34, Theorem 4.1] includes a condition $d\geqslant 2$. However, by examining its proof, we can remove this condition. More precisely, we know that [34, Theorem 4.1] can be derived from [34, Theorem 3.4] which includes a condition $d\geqslant 2$. We can easily check that [34, Theorem 3.4] holds for $d=1$ if we interpret the empty sum as 0 in its statement. Thus [34, Theorem 4.1] holds for $d=1$ which implies the above lemma. By this lemma we can prove Theorem 6.3 as follows.

Proof of Theorem 6.3. First we use the relation

$$\begin{aligned}\frac{(-1)^c}{X^c(X+Y)^d}&=\sum_{j=1}^{c}\binom{c+d-j-1}{c-j}(-1)^j\frac{1}{Y^{c+d-j}X^j}\\&\quad+\sum_{j=1}^{d}\binom{c+d-j-1}{d-j}\frac{1}{Y^{c+d-j}(X+Y)^j}\end{aligned} \tag{6.3}$$

for $c,d\in\mathbb{N}$, which can be elementarily proved by induction on $c+d$ by using the partial fraction decomposition repeatedly. Therefore, setting $(X,Y)=(2l+1+m,m)$ in (6.3), we see that

$$\begin{aligned}&(-1)^c\zeta_2(a,b,c,d;PSp(2))\\&\quad=\sum_{\substack{l\geqslant 0\\ m\geqslant 1}}\frac{1}{(2l+1)^a m^b(2l+1+m)^c(2l+1+2m)^d}\\&\quad=\sum_{j=1}^{c}\binom{c+d-j-1}{c-j}(-1)^j\sum_{\substack{l\geqslant 0\\ m\geqslant 1}}\frac{1}{(2l+1)^a m^{b+c+d-j}(2l+1+m)^j}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{d}\binom{c+d-j-1}{d-j}\sum_{\substack{l\geqslant 0\\ m\geqslant 1}}\frac{1}{(2l+1)^a m^{b+c+d-j}(2l+1+2m)^j}\\
&=\sum_{j=1}^{c}\binom{c+d-j-1}{c-j}\\
&\quad\times(-1)^j\left\{\mathfrak{T}_{1,1}(a,b+c+d-j,j)+\mathfrak{T}_{1,2}(a,b+c+d-j,j)\right\}\\
&\quad+\sum_{j=1}^{d}\binom{c+d-j-1}{d-j}2^{b+c+d-j}\mathfrak{T}_{1,2}(a,b+c+d-j,j). \qquad (6.4)
\end{aligned}
$$

Hence, by Lemma 6.4, we obtain the assertion.

Example 6.5. As we noted above, it seems impossible to obtain an expression of $\zeta_2((1,2,2,2);PSp(2))$ in terms of $\zeta(s)$, from (5.20). Hence we use (6.4). Then we have

$$
\begin{aligned}
&\zeta_2((1,2,2,2);PSp(2))\\
&\quad=-2\mathfrak{T}_{1,1}(1,5,1)+62\mathfrak{T}_{1,2}(1,5,1)+\mathfrak{T}_{1,1}(1,4,2)+17\mathfrak{T}_{1,2}(1,4,2). \qquad (6.5)
\end{aligned}
$$

By the method used in [34, Section 4], we can obtain

$$
\begin{aligned}
\mathfrak{T}_{1,1}(1,5,1)&=-\frac{105}{128}\zeta(3)\zeta(4)-\frac{93}{128}\zeta(5)\zeta(2)+\frac{381}{128}\zeta(7),\\
\mathfrak{T}_{1,2}(1,5,1)&=-\frac{7}{128}\zeta(3)\zeta(4)-\frac{31}{128}\zeta(5)\zeta(2)+\frac{127}{256}\zeta(7),\\
\mathfrak{T}_{1,1}(1,4,2)&=\frac{105}{128}\zeta(3)\zeta(4)+\frac{279}{128}\zeta(5)\zeta(2)-\frac{1143}{256}\zeta(7),\\
\mathfrak{T}_{1,1}(1,4,2)&=\frac{7}{128}\zeta(3)\zeta(4)+\frac{183}{128}\zeta(5)\zeta(2)-\frac{635}{256}\zeta(7).
\end{aligned}
$$

Substituting these results into (6.5), we obtain

$$
\zeta_2((1,2,2,2);PSp(2))=\frac{827}{64}\zeta(5)\zeta(2)-\frac{1397}{64}\zeta(7).
$$

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