Šiauliai Math. Semin.,
10 (18), 2015, 181–197

ŠIAULIAI
MATHEMATICAL
SEMINAR
http://siauliaims.su.lt

EXPONENTIAL DIVISOR FUNCTIONS

Andrew V. Lelechenko

Abstract. We consider the operator E on arithmetic functions such that Ef is the multiplicative arithmetic function defined by $(Ef)(p^a) = f(a)$ for every prime power p^a. We investigate the behaviour of $E^m \tau_k$, where τ_k is a k-dimensional divisor function and E^m stands for the m-fold iterate of E. We estimate the error terms of $\sum_{n \leqslant x} E^m \tau_k(n)$ for various combinations of m and k. We also study properties of $E^m f$ for arbitrary f and sufficiently large m. We improve special cases of the Dirichlet asymmetric divisor problem and several results on the exponential divisor and totient functions, also.

Key words and phrases: average order, exponential divisor function, generalized divisor function.

2010 Mathematics Subject Classification: 11N37, 11M06, 11A25.

Submitted: 15 August 2013

1. Introduction

Consider the set $\mathcal{A}$ of arithmetic functions, the set $\mathcal{M}_{PI}$ of multiplicative prime-independent functions and the operator $E\colon \mathcal{A} \to \mathcal{M}_{PI}$ such that

$$(Ef)(p^a) = f(a)$$

for every prime power p^a. The behaviour of Ef for various special cases of f has been widely studied, starting with the pioneering paper of Subbarao [18] on $E\tau$ and $E\mu$, where τ is the divisor function and μ is the Möbius function. The most notable of them are the paper of Wu [24] on $E\tau$, the paper of Pétermann and Wu [16] on the sum of E-divisors, the papers of Tóth [21, 22, 23] on certain E-functions, the paper of Pétermann [15] on $E\phi$, the paper of Cao and Zhai [1] on estimates of certain E-functions under Riemann hypothesis (RH).

The used by previous authors notation for Ef is $f^{(e)}$. We write $E^2 f(n)$, $E^3 f(n), \ldots$ meaning $(E^2 f)(n)$, $(E^3 f)(n), \ldots$.

The primary aim of the current paper is to investigate effects of essentially multiple applications of operator E on different functions, but we also obtain several important results in the case of the single application.

We improve previously known error terms for the special cases of Dirichlet asymmetric divisor problem (namely, for $\tau_{1,2^r}$, in Lemma 5 and, for $\tau_{1,2^r,2^r}$, in Lemma 7). Section 5 is devoted to the refinement of Tóth's theorem [23].

Mostly we consider divisor functions, but in the last section a generalization over all arithmetic functions is discussed.

2. Notations

In asymptotic relations, we use Landau symbols Ω, O and o, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument (usually x) tends to $+\infty$. p with or without indexes denotes a rational prime. As usual, $\zeta(s)$ is the Riemann zeta-function. For complex s, we denote $\sigma := \operatorname{Re} s$ and $t := \operatorname{Im} s$. γ denotes the Euler-Mascheroni constant, $\gamma \approx 0.577$. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

We write $f \star g$ for the Dirichlet convolution: $(f \star g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$.

Let τ be the divisor function, $\tau(n) = \sum_{d|n} 1$. Denote

$$\tau(a_1, \ldots, a_k; n) = \sum_{d_1^{a_1} \cdots d_k^{a_k} = n} 1$$

and $\tau_k = \tau(\underbrace{1, \ldots, 1}_{k \text{ times}}; \cdot)$. Then $\tau \equiv \tau_2 \equiv \tau(1, 1; \cdot)$.

Now let $\Delta(a_1, \ldots, a_k; x)$ be an error term in the asymptotic estimate of the sum $\sum_{n \leqslant x} \tau(a_1, \ldots, a_k; n)$ (see [10] for the form of the main term.) For brevity, denote $\Delta_k(x) = \Delta(\underbrace{1, \ldots, 1}_{k \text{ times}}; x)$. Finally, $\theta(a_1, \ldots, a_k)$ denotes throughout the paper, a real value such that

$$\Delta(a_1, \ldots, a_k; x) \ll x^{\theta(a_1, \ldots, a_k) + \varepsilon}.$$

We abbreviate θ_k for the exponent of x in $\Delta_k(x)$.

3. Elementary properties of E

One can easily check that

$$E(f + g) = Ef + Eg \quad \text{and} \quad E(f \cdot g) = Ef \cdot Eg.$$

Define (following Subbarao [18]) exponential convolution $\odot$ as

$$(f \odot g)(p^n) = \sum_{d|n} f(p^d) g(p^{n/d}).$$

Then $E(f \star g) = Ef \odot Eg$, since

$$E(f \star g)(p^n) \quad = \quad (f \star g)(n) = \sum_{d|n} f(d) g(n/d)$$

$$= \sum_{d|n} Ef(p^d)Eg(p^{n/d}) = (Ef \odot Eg)(p^n).$$

LEMMA 1. *The only solution of $Ef = f$ is $f(n) = 1$ for all n.*

Proof. Assume the contrary: there is n such that $f(n) \neq 1$. Choose the least of all such n. This number cannot have more than one prime divisor: if $n = \prod_i p_i^{a_i}$ and $f(n) \neq 1$, then, for some index i, we have $f(p_i^{a_i}) \neq 1$ and n is not the least. Now $n = p^a$. Since f is prime-independent $f(2^a) = f(n) \neq 1$, so in fact $n = 2^a$.

The requirement $Ef = f$ induces $f(n) = f(a)$. But $a < n$, and this is a contradiction.

LEMMA 2. *For every arithmetic function f such that $f(n) \ll n^a$ for some constant a, we have $Ef(n) \ll n^\varepsilon$.*

Proof. By result of Suryanarayana and Sita Rama Chandra Rao [19], we have

$$\limsup_{n\to\infty} \frac{\log Ef(n) \log\log n}{\log n} = \sup_n \frac{\log f(n)}{n}.$$

But as soon as $f(n) \ll n^a$, the right side is bounded, and so,

$$\frac{\log Ef(n)}{\log n} \to 0 \quad \text{as} \quad n \to \infty.$$

This means that $Ef(n) \ll n^\varepsilon$.

4. Multiexponential divisor functions

Consider the family of functions

$$\tau^{(e)}, E^2\tau, E^3\tau \ldots$$

Define $A_m := \{n \mid E^m\tau(n) \neq 1\}$. Then

$$A_0 = \mathbb{N} \setminus \{1\}, \quad A_1 = \{n \mid \mu(n) = 0\} = \{4, 8, 9, 12 \ldots\},$$
$$A_m = \bigcup_p \bigcup_{a\in A_{m-1}} p^a R_p, \quad \text{where} \quad R_p := \{n \mid \gcd(n, p) = 1\}$$

Which are the lowest elements of these infinite sets? Denote

$$(\mathfrak{m}) = \min A_m, \quad (\mathfrak{m})' = \min (A_m \setminus \{(\mathfrak{m})\}), \quad (\mathfrak{m})'' = \min (A_m \setminus \{(\mathfrak{m}), (\mathfrak{m})'\}).$$

Then $(\mathfrak{o}) = 2, (\mathfrak{1}) = 4, (\mathfrak{2}) = 16, (\mathfrak{3}) = 65536$ and generally $(\mathfrak{m}) = 2^{(\mathfrak{m}-\mathfrak{1})}$.

The set A_m contains not only $(\mathfrak{m})$, but also numbers of form $(2k + 1)(\mathfrak{m})$ for each integer k. Other elements of A_m must be divisible by $n = p^a$, where either $p > 2$, or $a > (\mathfrak{m}-\mathfrak{1})$. Let

$$B_m := A_m \setminus (2\mathbb{N} + 1)(\mathfrak{m}).$$

Lemma 3. *For integer $m > 1$, we have*

$$(\mathfrak{m})' = 3(\mathfrak{m}), \qquad (\mathfrak{m})'' = 5(\mathfrak{m}). \tag{1}$$

Proof. We prove only the first statement. The second can be proven in the same way.

It is enough to show that $\min B_m > 3(\mathfrak{m})$; let us prove this by induction. From the discussion above, we have

$$\min B_m \geqslant \min\{3^{(\mathfrak{m-1})}, 2^{(\mathfrak{m-1})'}\}.$$

For $m = 2$, one can evaluate

$$\min B_2 \geqslant \min\{3^4, 2^8\} = 81 > 3 \cdot (\mathfrak{2}) = 3 \cdot 16.$$

Now let $m > 1$, $(\mathfrak{m})' = 3(\mathfrak{m})$. Then

$$\min B_{m+1} \geqslant \min\{3^{(\mathfrak{m})}, 2^{3(\mathfrak{m})}\} = 3^{(\mathfrak{m})} > 3 \cdot 2^{(\mathfrak{m})} = 3(\mathfrak{m+1}).$$

The asymptotic behaviour of $\tau^{(e)}$ has been studied in details by Wu [24] and Smati [17]. Now we are going to study $E^m\tau$ for $m > 1$.

Lemma 4. *For a fixed integer $m > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{E^m\tau(n)}{n^s} = \frac{\zeta(s)\zeta((\mathfrak{m})s)\zeta\big((2(\mathfrak{m})+1)s\big)\zeta(3(\mathfrak{m})s)}{\zeta\big(((\mathfrak{m})+1)s\big)\zeta(2(\mathfrak{m})s)} H(s), \tag{2}$$

where $H(s)$ converges absolutely for $\sigma > \frac{1}{(3(\mathfrak{m})+1)}$.

Proof. Denote

$$S(x) := \sum_{n\geqslant 0} E^m\tau(p^n)x^n = \sum_{n\geqslant 0} x^n + x^{(\mathfrak{m})} + x^{3(\mathfrak{m})} + \mathrm{O}(x^{5(\mathfrak{m})}).$$

Taking into account (1) and $E^m\tau((\mathfrak{m})) = E^m\tau(3(\mathfrak{m})) = 2$, then

$$\begin{aligned}
&(1-x)S(x) = 1 + x^{(\mathfrak{m})} - x^{(\mathfrak{m})+1} + x^{3(\mathfrak{m})} + \mathrm{O}(x^{3(\mathfrak{m})+1}),\\
&(1-x)(1-x^m)S(x) = 1 - x^{(\mathfrak{m})+1} - x^{2(\mathfrak{m})} + x^{2(\mathfrak{m})+1} + x^{3(\mathfrak{m})} + \mathrm{O}(x^{3(\mathfrak{m})+1})\\
&\frac{(1-x)(1-x^{(\mathfrak{m})})}{1-x^{(\mathfrak{m})+1}} S(x) = 1 - x^{2(\mathfrak{m})} + x^{2(\mathfrak{m})+1} + x^{3(\mathfrak{m})} + \mathrm{O}(x^{3(\mathfrak{m})+1})\\
&\frac{(1-x)(1-x^{(\mathfrak{m})})}{(1-x^{(\mathfrak{m})+1})(1-x^{2(\mathfrak{m})})} S(x) = 1 + x^{2(\mathfrak{m})+1} + x^{3(\mathfrak{m})} + \mathrm{O}(x^{3(\mathfrak{m})+1}).
\end{aligned}$$

So,

$$\frac{(1-x)(1-x^{(\mathfrak{m})})(1-x^{2(\mathfrak{m})+1})(1-x^{3(\mathfrak{m})})}{(1-x^{(\mathfrak{m})+1})(1-x^{2(\mathfrak{m})})} S(x) = 1 + \mathrm{O}(x^{3(\mathfrak{m})+1}),$$

and this implies (2).

THEOREM 1. *For a fixed integer $m > 1$, we have*

$$\sum_{n \leqslant x} E^m \tau(n) = A_m x + B_m x^{1/(\mathfrak{m})} + \mathrm{O}(x^{\alpha_m}), \quad \alpha_m = \frac{1}{(\mathfrak{m}) + 1}, \tag{3}$$

where A_m and B_m are computable constants.

Proof. We use (2), a classic estimate from the book of Krätzel [10, Theorem 5.1]

$$\begin{aligned} \sum_{n \leqslant x} \tau(1, (\mathfrak{m}); n) &= \zeta((\mathfrak{m}))x + \zeta\left(\frac{1}{(\mathfrak{m})}\right) x^{1/(\mathfrak{m})} + \mathrm{O}(x^{\theta(1,(\mathfrak{m}))}), \\ \theta(1, (\mathfrak{m})) &= \frac{1}{(\mathfrak{m}) + 1}, \end{aligned} \tag{4}$$

and the convolution method.

We can improve the error term in (3) under RH. First of all we should estimate (4) more precisely. Using an exponent pair $A^{(\mathfrak{m}-1)-1}B(0,1)$, where A and B are van der Corput's processes, one can obtain $\theta(1, (\mathfrak{m})) = \frac{1}{(\mathfrak{m}) + (\mathfrak{m}-1)}$. The following lemma provides even better result by sophisticated selection of the exponent pair.

LEMMA 5. *For a fixed integer $r \geqslant 5$, we have*

$$\theta(1, 2^r) = \frac{2^r - 2r}{2^{2r} - r2^r - 2r^2 + 2r - 4} < \frac{1}{2^r + r}.$$

Proof. Consider an exponent pair

$$(k_r, l_r) := A^{r-1} B A^{r-3} BAB(0, 1).$$

To evaluate k_r and l_r, we map exponent pairs into the real projective space (the concept of such mapping traces back to Graham [4]):

$$\mathbb{R}^2 \ni (k, l) \mapsto (k : l : 1) \in \mathbb{R}^3/(\mathbb{R} \setminus \{0\}).$$

We have

$$A(k, l) \mapsto \mathcal{A}(k : l : 1), \quad \mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} = \mathcal{S} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathcal{S}^{-1},$$

where $\mathcal{S} = \left(\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{smallmatrix}\right)$, and

$$B(k, l) \mapsto \mathcal{B}(k : l : 1), \quad \mathcal{B} = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Thus, $\mathcal{A}^n = \mathcal{S} \left(\begin{smallmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{smallmatrix} \right) \mathcal{S}^{-1}$ and

$$\mathcal{A}^{r-1}\mathcal{B}\mathcal{A}^{r-3}\mathcal{B}\mathcal{A}\mathcal{B}(0:1:1) = \begin{pmatrix} 4\cdot 2^r - 8r \\ 8\cdot 2^{2r} - (12r+16)\cdot 2^r + 8r^2 + 8r + 16 \\ 8\cdot 2^{2r} - (8r+16)\cdot 2^r + 16r \end{pmatrix}.$$

Returning to $\mathbb{R}^2$, we get

$$k_r = \frac{2^r - 2r}{2\cdot 2^{2r} - (2r+4)\cdot 2^r + 4r}, \quad l_r = 1 - \frac{r\cdot 2^r - 2r^2 + 2r - 4}{2\cdot 2^{2r} - (2r+4)\cdot 2^r + 4r}.$$

Now, for $r \geqslant 5$,

$$2l_r - 22^r k_r - 1 = \frac{2r^2 + 4 - 2\cdot 2^r}{2^{2r} - (r+2)\cdot 2^r + 2r} < 0.$$

This proves that (k_r, l_r) satisfies the second case of [10, Theorem 5.11], and finally,

$$\theta(1, 2^r) = \frac{k_r}{2^r k_r - l_r + 1}.$$

In special cases, the value of $\theta(1, (\mathfrak{m}))$ in (3) can be estimated even more precisely. The case of $m = 0$ is classic, the best modern result is given by Graham and Kolesnik [5]. For bigger m we calculated estimates using [10, Theorems 5.11 and 5.12], selecting appropriate exponent pair using the method described in [12] (see Table 1).

m	$(\mathfrak{m})$	$\theta(1,(\mathfrak{m}))$	Exponent pair or ref.
0	2	$\frac{1057}{4785}+\varepsilon\approx 0.220899$	Graham and Kolesnik [5]
1	4	$\frac{1448}{10331}+\varepsilon\approx 0.140161$	AH_{05}
2	16	$\frac{15}{307}\approx 0.048860$	$A^3BA^2BA^4BI$

Table 1: Special values of $\theta(1,\cdot)$. Exponent pairs are written in terms of A- and B-processes. Here $I = (0,1)$ and $H_{05} = (\frac{32}{205}+\varepsilon, \frac{269}{410}+\varepsilon)$ is an exponent pair from [6].

Theorem 2. *Under RH, for a fixed $m > 1$,*

$$\alpha_m = \frac{1 - \theta(1,(\mathfrak{m}))}{(\mathfrak{m}) + 2 - 2((\mathfrak{m})+1)\theta(1,(\mathfrak{m}))}, \tag{5}$$

where α_m is an exponent from (3). *Note that if $\theta(1,(\mathfrak{m})) < \frac{1}{(\mathfrak{m})+1}$, then $\alpha_m < \frac{1}{(\mathfrak{m})+1}$ too.*

Proof. By (2), we get

$$E^m\tau = \tau(1,(\mathfrak{m});\cdot) \star \mu_{(\mathfrak{m})+1} \star h,$$

where $\mu_k(n^k) = \mu(n)$ and $\mu_k(n) = 0$ in other cases, and $h(n)$ is the nth coefficient in Dirichlet series

$$\frac{\zeta\big((2(\mathfrak{m})+1)s\big)\zeta(3(\mathfrak{m})s)}{\zeta(2(\mathfrak{m})s)}H(s).$$

We see that $\sum_{n\leqslant x} h(n)n^{-s}$ converges absolutely for $\sigma > \frac{1}{2(\mathfrak{m})}$.

Nowak (see [14, Theorem 2] or Cao and Zhai [2, Theorem 1, Corolarry 1.1]) proved that if $a \leqslant b < c < 2(a+b)$ and $\theta(a,b) < \frac{1}{c}$, then, under RH for $f = \tau(a,b;\cdot)\star\mu_c$, one has

$$\sum_{n\leqslant x} f(n) = \frac{\zeta(b/a)}{\zeta(c/a)}x^{1/a} + \frac{\zeta(a/b)}{\zeta(c/b)}x^{1/b} + \mathrm{O}(x^{\beta+\varepsilon}), \tag{6}$$

where

$$\beta = \frac{1-a\theta(a,b)}{a+c-2ac\theta(a,b)}.$$

Substitution $a = 1$, $b = (\mathfrak{m})$, $c = (\mathfrak{m})+1$ gives us (5).

The rest of the theorem can be proven by convolution method.

We can also estimate the error term from the bottom.

Theorem 3. *For a fixed integer $m > 1$, we have*

$$\sum_{n\leqslant x} E^m\tau(n) = A_m x + B_m x^{1/(\mathfrak{m})} + \Omega(x^{1/2((\mathfrak{m})+1)}).$$

Proof. By (2), we have that the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{E^m\tau(n)}{n^s}$$

has $\zeta\big(((\mathfrak{m})+1)s\big)$ in the denominator, and thus, has infinitely many poles on line $\sigma = \frac{1}{2((\mathfrak{m})+1)}$.

5. Exponential multidimensional divisor functions

Consider a family of functions

$$\tau^{(e)}, \tau_3^{(e)}, \tau_4^{(e)} \ldots$$

In [23], Tóth has proved the following general result.

Theorem A. *Let f be a complex valued multiplicative arithmetic function such that*

a) $f(p) = f(p^2) = \ldots = f(p^{\ell-1}) = 1$, $f(p^\ell) = f(p^{\ell+1}) = k$ *for every prime p, where $\ell, k \geqslant 2$ are fixed integers;*

b) *there exist constants* $C, m > 0$ *such that* $|f(p^a)| \leqslant Ca^m$ *for every prime* p *and every* $a \geqslant \ell + 2$.

Then, for $s \in \mathbb{C}$,

(i)

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s)\zeta^{k-1}(\ell s)V(s), \quad \operatorname{Re} s > 1,$$

where the Dirichlet series $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ *is absolutely convergent for* $\operatorname{Re} s > \frac{1}{\ell+2}$;

(ii)

$$\sum_{n \leqslant x} f(n) = C_f x + x^{1/\ell} P_{f,k-2}(\log x) + \mathrm{O}(x^{u_{k,\ell}+\varepsilon}),$$

for every $\varepsilon > 0$, *where* $P_{f,k-2}$ *is a polynomial of degree* $k-2$, $u_{k,\ell} := \frac{2k-1}{3+(2k-1)\ell}$, *and*

$$C_f := \prod_p \left(1 + \sum_{a=\ell}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right);$$

(iii) *the error term can be improved for certain values of* k *and* ℓ. *For example, in the case* $k = 3$, $\ell = 2$, *it is* $\mathrm{O}(x^{8/25} \log^3 x)$.

Tóth considered $\tau_k^{(e)}$ and showed that

$$\sum_{n=1}^{\infty} \frac{\tau_k^{(e)}(n)}{n^s} = \zeta(s)\zeta^{k-1}(2s)H(s), \tag{7}$$

where $H(s)$ converges absolutely for $\sigma > \frac{1}{5}$, and thus, he obtained that

$$\sum_{n \leqslant x} \tau_k^{(e)}(n) = C_k x + x^{1/2} S_{k-2}(\log x) + \mathrm{O}(x^{w_k+\varepsilon}), \tag{8}$$

where $w_k = \frac{2k-1}{4k+1}$.

We are going to improve this estimate by improving Theorem A.

THEOREM 4. *Under hypotheses of* Theorem A, *we can take*

$$u_{k,\ell} = \frac{1}{\ell + 1 - \theta_{k-1}}. \tag{9}$$

In particular, for $k \geqslant 5$, *we have*

$$u_{k,\ell} \leqslant \frac{k+1}{3+(k+1)\ell} < \frac{2k-1}{3+(2k-1)\ell}, \tag{10}$$

and, for $k \geqslant 1000$*, the inequality*

$$u_{k,\ell} \leqslant \frac{1}{l + ck^{-2/3}}, \quad c \leqslant \left(\frac{40}{267}\right)^{2/3} \tag{11}$$

holds.

Proof. The analysis of the proof of Theorem A shows that the error term in it is caused by the estimate of

$$\Delta(1, \underbrace{\ell, \ldots, \ell}_{k-1 \text{ times}}).$$

Let us show how this estimate can be improved.

Taking into account the equality

$$\sum_{n \leqslant x} \tau(\underbrace{\ell, \ldots, \ell}_{k \text{ times}}; n) = \sum_{n \leqslant x^{1/\ell}} \tau_k(n),$$

we obtain that

$$\Delta(\underbrace{\ell, \ldots, \ell}_{k \text{ times}}; x) \ll x^{\theta_k/\ell + \varepsilon} \quad \text{and} \quad \Delta(\underbrace{\ell, \ldots, \ell}_{k-1 \text{ times}}; x) \ll x^{\theta_{k-1}/\ell + \varepsilon}.$$

Substituting $p = k, q = 1$, $(a_1, \ldots, a_p) = (1, \ell, \ldots, \ell)$ into [10, Theorem 6.8], we get

$$\Delta(1, \underbrace{\ell, \ldots, \ell}_{k-1 \text{ times}}; x) \ll x^{u_{k,l} + \varepsilon},$$

where $u_{k,l}$ can be chosen as in (9).

Values of θ_k have been widely studied: Huxley [6] proved that $\theta_2 \leqslant \frac{131}{416}$; Kolesnik [8] proved $\theta_3 \leqslant \frac{43}{96}$; many other special cases can be found in the book of Titchmarsh [20, Chapter 12] and in the paper of Ivič and Ouellet [7]. Namely, [20, Theorem 12.3] gives an estimate

$$\theta_k \leqslant \frac{k-1}{k+2}, \quad k \geqslant 4,$$

which implies inequality (10), and [7, Theorem 1] together with Ford's estimate [3] and Kolpakova's estimate [9] implies (11).

Substitution $l = 2$ and $k = k$ into Theorem 4 allows to decrease w_k in (8) to

$$w_k = \frac{1}{3 - \theta_{k-1}}.$$

Note that the application of Theorem 4 instead of Theorem A also improves all other results of [23]. Namely,

$$\sum_{n \leqslant x} \left(\tau^{(e)}(n)\right)^r = A_r x + x^{1/2} P_{2^r - 2}(\log x) + \mathrm{O}(x^{u_r + \varepsilon}), \quad u_r = \frac{2^r + 1}{2^{r+1} + 5},$$

$$\sum_{n \leqslant x} \left(\phi^{(e)}(n)\right)^r = B_r x + x^{1/3} R_{2^r - 2}(\log x) + \mathrm{O}(x^{t_r + \varepsilon}), \quad t_r = \frac{2^r + 1}{3(2^r + 2)}.$$

THEOREM 5. *For a fixed integer $k > 1$, we have*

$$\sum_{n \leqslant x} \tau_k^{(e)}(n) = C_k x + x^{1/2} S_{k-2}(\log x) + \Omega(x^{1/(4-2/k)}).$$

Proof. Application of Kühleitner and Nowak [11, Theorem 2] with $f_1 = \ldots = f_k = \zeta$, $m_1 = 1$, $m_2 = \ldots = m_k = 2$ gives us

$$\alpha = \frac{k}{2\big(1 + 2(k-1)\big)} = \frac{1}{4 - 2/k}.$$

6. Multiexponential multidimensional divisor functions

Consider the two-parametric family of functions

$$\{E^m \tau_k\}, \quad k > 2, \quad m > 1.$$

One can check that, for $m > 1$, the least arguments on which $E^m \tau_k(n) \neq 1$ are (as in the case of $k = 2$) $(\mathfrak{m}), 3(\mathfrak{m})$ and $5(\mathfrak{m})$.

LEMMA 6. *For fixed integers $k > 2$ and $m > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{E^m \tau_k(n)}{n^s} = \frac{\zeta(s) \zeta^{k-1}((\mathfrak{m})s)}{\zeta^{k-1}\big(((\mathfrak{m})+1)s\big) \zeta^{k(k-1)/2}(2(\mathfrak{m})s)} H(s), \tag{12}$$

where $H(s)$ converges absolutely for $\sigma > \frac{1}{2(\mathfrak{m})+1}$.

Proof. As soon as $E^m \tau_k((\mathfrak{m})) = \tau_k(2) = k$, for

$$S(x) := \sum_{n \geqslant 0} E^m \tau_k(p^n) = \sum_{n \geqslant 0} x^n + (k-1) x^{(\mathfrak{m})} + \mathrm{O}(x^{2(\mathfrak{m})+1}),$$

we get

$$(1-x) S(x) = 1 + (k-1) x^{(\mathfrak{m})} - (k-1) x^{(\mathfrak{m})+1} + \mathrm{O}(x^{2(\mathfrak{m})+1}),$$

$$(1-x)(1 - x^{(\mathfrak{m})})^{k-1} S(x) = 1 - (k-1) x^{(\mathfrak{m})+1} - \frac{k(k-1)}{2} x^{2(\mathfrak{m})} + \mathrm{O}(x^{2(\mathfrak{m})+1}),$$

which implies the statement of the lemma.

THEOREM 6. *For fixed integers $k > 2$ and $m > 1$, we have*

$$\sum_{n \leqslant x} E^m \tau_k(n) = K_{m,k} x + x^{1/(\mathfrak{m})} R_{m,k-2}(\log x) + \mathrm{O}(x^{\alpha_{m,k} + \varepsilon}), \tag{13}$$

where $K_{m,k}$ is a computable constant, $R_{m,k-2}$ is a polynomial of degree $k-2$ and

$$\alpha_{m,k} = \frac{1}{(\mathfrak{m}) + 1 - \theta_{k-1}}.$$

The proof follows from (12) and Theorem 4.

THEOREM 7. *For fixed integers $k > 2$ and $m > 1$, we have*

$$\sum_{n \leqslant x} E^m \tau_k(n) = K_{m,k} x + x^{1/(\mathfrak{m})} R_{m,k-2}(\log x) + \Omega(x^\alpha), \quad \alpha = \frac{1}{2\big((\mathfrak{m}) + 1/(k-1)\big)}.$$

Proof. By substitution

$$\begin{aligned} &f_1 = \ldots = f_k = g_1 = \cdots = g_{(k+2)(k-1)/2} = \zeta, \\ &m_1 = 1, \quad m_2 = \cdots = m_k = (\mathfrak{m}), \\ &n_1 = \cdots = n_{k-1} = (\mathfrak{m}) + 1, \quad n_k = \cdots = n_{(k+2)(k-1)/2} = 2(\mathfrak{m}) \end{aligned}$$

into [11, Theorem 2], we obtain

$$\alpha = \frac{k-1}{2(1 + (\mathfrak{m})(k-1))}.$$

7. Multiexponential three-dimensional divisor function

In the case of τ_3, the statement of Theorem 6 can be improved using stronger estimates for $\theta(1, (\mathfrak{m}), (\mathfrak{m}))$. Exponent pairs of the form

$$A^{(\mathfrak{m}-1)-1} B A^{(\mathfrak{m}-1)-2} BABA^2 (k_0, l_0)$$

seem to be especially useful for this task. We shall prove only the simplest result of this kind.

LEMMA 7. *For a fixed integer $r \geqslant 10$, we have*

$$\begin{aligned} \theta(1, 2^r, 2^r) &= \frac{26 \cdot 2^{2r} - (29r + 41)2^r + 16r^2 + 12r + 32}{26 \cdot 2^{3r} - (16r + 41)2^{2r} + (24r - 3)2^r + 16r + 12} \\ &< \frac{1}{2^r + 1}. \end{aligned} \tag{14}$$

Proof. We apply [10, Theorem 6.2] with

$$(k, l) = A^{r-1} B A^{r-2} BABA^2 \cdot B(0, 1).$$

Here (k, l) can be evaluated similarly to Lemma 5, i.e.,

$$k = \frac{13 \cdot 2^r - 16r - 12}{26 \cdot 2^{2r} - (16r + 54)2^r + 32r + 24},$$

$$l = 1 - \frac{13r \cdot 2^r - 16r^2 + 4r - 20}{26 \cdot 2^{2r} - (16r + 54)2^r + 32r + 24}.$$

For larger r, more complicated exponent pairs can be used. They provide slightly lesser values of $\theta(1, 2^r, 2^r)$.

For small m, good estimates of $\theta(1, (\mathfrak{m}), (\mathfrak{m}))$ may be obtained by the careful manual selection of exponent pairs, appropriate for the substitution into [10, Theorems 6.2 and 6.3]. We have calculated several first estimates, see Table 2.

m	$(\mathfrak{m})$	$\theta(1,(\mathfrak{m}),(\mathfrak{m}))$	Exponent pair or ref.
0	2	$\frac{8}{25}$=0.320000	[10, Theorem 6.4] with $k=3$
1	4	$\frac{7+\sqrt{809}}{190}\approx$0.186542	$A(BA)^4(A^2BAA)^\infty I$
2	16	$\frac{93607}{1698654}\approx$0.055107	$A^3BA(A(BA)^2A)^3BAA^3(BA)^2ABAI$

Table 2: Special values of $\theta(1,\cdot,\cdot)$. Exponent pairs are written in terms of A- and B-processes. Here $I=(0,1)$.

THEOREM 8. *For a fixed integer $m>1$, we have*

$$\sum_{n\leqslant x} E^m\tau_3(n) = K_{m,3}x + (r_1\log x + r_0)x^{1/(\mathfrak{m})} + O(x^{1/((\mathfrak{m})+1)}),$$

where $K_{m,3}, r_1$ and r_0 are computable constants.

Proof. The statement follows from (12), (14) and the convolution method.

Now we are going to refine the last estimate under RH. We need the following lemma, which generalizes [14, Theorem 2].

LEMMA 8. *Consider a multiplicative function f such that*

$$\sum_{n=1}^\infty \frac{f(n)}{n^s} = \frac{\zeta(as)\zeta^r(bs)}{\zeta^k(cs)} := F(s),$$

where $2a\leqslant b<c<2(a+b)$. Let $\Delta(x)$ be defined implicitly by the equation

$$S(x) := \sum_{n\leqslant x} f(n) = \left(\operatorname*{Res}_{s=1/a} + \operatorname*{Res}_{s=1/b}\right) F(s)x^s s^{-1} + \Delta(x).$$

Denote $\underbrace{b,\ldots,b}_{r\ times}$ by $b\times r$. Then, under RH, for any $1\leqslant y\leqslant x^{1/c}$,

$$\Delta(x) = \sum_{l\leqslant y} \mu_k(l)\Delta(a, b\times r; x/l^c) + O(x^{1/2a+\varepsilon}y^{1/2-c/2a} + x^\varepsilon),$$

where μ_k is a multiplicative function such that

$$\sum_{n=1}^\infty \frac{\mu_k(n)}{n^s} = \zeta^{-k}(s).$$

Proof. For a fixed y, let us split $f(n)$ into two parts: $f(n)=f_1(n)+f_2(n)$, where

$$f_1(n) = \sum_{\substack{l^c m=n \\ l\leqslant y}} \mu_k(l)\tau(a,b\times r;m), \quad f_2(n) = \sum_{\substack{l^c m=n \\ l>y}} \mu_k(l)\tau(a,b\times r;m).$$

This split naturally implies a split of $S(x)$ into $S_1(x) := \sum_{n\leqslant x} f_1(n)$ and $S_2(x) := \sum_{n\leqslant x} f_2(n)$. For S_1, we obtain

$$\begin{aligned} S_1(x) &= \sum_{l\leqslant y} \mu_k(l) \sum_{m\leqslant x/l^c} \tau(a, b\times r; m) \\ &= \Big(\operatorname*{Res}_{s=1/a} + \operatorname*{Res}_{s=1/b}\Big) F_1(s)x^s s^{-1} + \sum_{l\leqslant y} \mu_k(l)\Delta\Big(a, b\times r; \frac{x}{l^c}\Big), \end{aligned} \tag{15}$$

where

$$F_1(s) := \zeta(as)\zeta^r(bs)\sum_{l\leqslant y}\frac{\mu_k(l)}{l^{cs}}.$$

Here

$$\operatorname*{Res}_{s=1/b} F_1(s)x^s s^{-1} = x^{1/b}\sum_{l\leqslant y}\frac{\mu_k(l)}{l^{c/b}}P(\log x) - cx^{1/b}\sum_{l\leqslant y}\frac{\mu_k(l)\log l}{l^{c/b}} \cdot \frac{1}{b}\zeta\Big(\frac{a}{b}\Big),$$

where P is a polynomial of $\deg P = r-1$. As soon as $\sum_{l\leqslant x}\mu_k(l) \ll x^{1/2+\varepsilon}$, under RH, we get

$$x^{1/b+\varepsilon}\sum_{l>y}\mu_k(l)l^{-c/b+\varepsilon} \ll x^{1/b+\varepsilon}y^{1/2-c/b+\varepsilon} \ll x^{1/2a+\varepsilon}y^{1/2-c/2a}.$$

So,

$$\operatorname*{Res}_{s=1/b} F_1(s)x^s s^{-1} = \operatorname*{Res}_{s=1/b}\frac{\zeta(as)\zeta^r(bs)}{\zeta^k(cs)} + \mathrm{O}(x^{1/2a+\varepsilon}y^{1/2-c/2a}). \tag{16}$$

Now consider S_2. Define

$$g(s) := \sum_{l>y}\frac{\mu_k(l)}{l^s} \quad \text{for} \quad \sigma > 1.$$

Then, under RH, function g can be continued analytically to the half-plane $\sigma > \frac{1}{2}+\varepsilon$, and we have uniformly for such σ that

$$g(s) \ll y^{1/2-\sigma+\varepsilon}\big(|t|+1\big)^{\varepsilon}$$

(see [14, Lemma 3] for the proof). Thus, by the Perron formula with $c = \frac{1}{a}+\varepsilon$ and $T = x^2$, we get

$$S_2(x) = \int_{c-iT}^{c+iT} F_2(s)x^s s^{-1}\mathrm{d}s + \mathrm{O}(x^{\varepsilon}),$$

where $F_2(s) := \zeta(as)\zeta^r(bs)g(cs)$. Moving the line of integration to $\sigma = d = \frac{1}{2}a+\varepsilon$, we obtain

$$S_2(x) - \operatorname*{Res}_{s=1/a} F_2(s)x^s s^{-1} \ll \left(\int_{d+iT}^{c+iT} + \int_{d-iT}^{d+iT} + \int_{d-iT}^{d+iT}\right) F_2(s)x^s s^{-1}\mathrm{d}s$$

$$=: I_1 + I_2 + I_3. \tag{17}$$

But, for $\sigma > \frac{1}{2}a$,

$$g(cs) \ll y^{1/2-c/2a+\varepsilon} \bigl(|t|+1\bigr)^{\varepsilon},$$

and RH implies the Lindelöf hypothesis, so $\zeta(s) \ll \bigl(|t|+1\bigr)^{\varepsilon}$ for $\sigma > \frac{1}{2}$. Thus,

$$I_1 + I_2 + I_3 \ll y^{1/2-c/2a+\varepsilon} x^{\varepsilon}. \tag{18}$$

Combining of (15), (16), (17) and (18) completes the proof.

Theorem 9. *Under RH, for a fixed $m > 1$, we have*

$$\alpha_{m,3} = \frac{1-\theta(1,(\mathfrak{m}),(\mathfrak{m}))}{(\mathfrak{m})+2-2((\mathfrak{m})+1)\theta(1,(\mathfrak{m}),(\mathfrak{m}))},$$

where $\alpha_{m,3}$ is an exponent in (13).

Proof. Similarly to Theorem 2, we use Lemma 8 with $r=2$ instead of Nowak's result. The equation (6) transforms into

$$\sum_{n\leqslant x} (\tau(1,b,b;\cdot)\star\mu_c)(n) = C_1 x^{1/a} + (C_2 \log x + C_3) x^{1/b} + \mathrm{O}(x^{\beta+\varepsilon}),$$

with

$$\beta = \frac{1-a\theta(a,b,b)}{a+c-2ac\theta(a,b,b)}. \tag{19}$$

We give one more application of Lemma 8 on somewhat off-topic function. Define $\mathfrak{t}\colon \mathbb{Z}[i] \to \mathbb{Z}$, where $\mathfrak{t}(n)$ is a number of $d \in \mathbb{Z}[i]$ such that $d \mid n$ over $\mathbb{Z}[i]$. Consider $\tau_*^{(e)} := E(\mathfrak{t}|_{\mathbb{Z}})\colon \mathbb{Z} \to \mathbb{Z}$. We proved in [13, (10)] that

$$\sum_{n=1}^{\infty} \frac{\tau_*^{(e)}(n)}{n^s} = \frac{\zeta(s)\zeta^2(2s)}{\zeta(3s)} H(s),$$

where $H(s)$ converges absolutely for $\sigma > \frac{1}{5}$, and, due to [13, Theorem 3],

$$\sum_{n\leqslant x} \tau_*^{(e)}(n) = A_1 x + (A_2 \log x + A_3) x^{1/2} + \mathrm{O}(x^{1/3+\varepsilon}).$$

Lemma 8 shows that the last estimate can be improved under RH. Taking into account (19) with $(a,b,c) = (1,2,3)$, we obtain, for $\theta(1,2,2) = \frac{8}{25}$, that

$$\beta = \frac{1-\theta(1,2,2)}{4-6\theta(1,2,2)} = \frac{17}{52} < \frac{1}{3}.$$

8. Generalization

Consider an arbitrary multiplicative function f. Define

$$A(f) := \{n \mid f(n) \neq 1\}, \quad n(f) := \min A(f),$$
$$B(f) := A(f) \setminus (2\mathbb{N}+1)n(f), \quad n'(f) := \min\left(A(f) \setminus \{n(f)\}\right).$$

Then $n(E^m f) = 2^{n(E^{m-1}f)}$.

LEMMA 9. *Let f be an arbitrary arithmetic function. For every $k \in \mathbb{N}$, there exists sufficiently large, constructively defined $m_0 = m_0(k, f)$ such that, for $m \geqslant m_0$ integers,*

$$n(E^m f), 3n(E^m f), \ldots, (2k-1)n(E^m f)$$

are k lowest elements of $A(E^m f)$.

Proof. Without loss of generality, we can suppose that f is multiplicative prime-independent and $n(f)$ is a power of 2. On the contrary, we take Ef instead of f.

We can also suppose without loss of generality that $n'(f) \geqslant 2n(f)$. On the contrary, we can take Ef instead of f once again, because $n'(f) \geqslant n(f)+1$ and

$$n'(Ef) \geqslant \min\left\{3n(Ef), \min B(Ef)\right\},$$

where

$$\min B(Ef) \geqslant \min\left\{3^{n(f)}, 2^{n'(f)}\right\} \geqslant \min\left\{\left(\frac{3}{2}\right)^2 \cdot 2^{n(f)}, 2^{n(f)+1}\right\} = 2n(Ef).$$

If $n'(f) \geqslant 2n(f)$, then, for every m, we have $n'(E^m f) \geqslant 2n(E^m f)$, too.

Now, to prove the statement of the lemma it is enough to show that

$$\min B(E^m f) \geqslant 2kn(E^m f).$$

As soon as sequence $n(f), n(Ef), n(E^2 f), \ldots$ is tending to $+\infty$, we can choose m such that $n(E^m f) \geqslant 2k$ and $\left(\frac{3}{2}\right)^{n(E^{m-1}f)} \geqslant 2k$. Then

$$\begin{aligned}\min B(E^m f) &\geqslant \min\left\{3^{n(E^{m-1}f)}, 2^{n'(E^{m-1}f)}\right\} \\ &\geqslant \min\left\{2kn(E^m f), \left(n(E^m f)\right)^2\right\} \geqslant 2kn(E^m f).\end{aligned}$$

LEMMA 10. *Let f be an arbitrary arithmetic function, and let $m_0 = m_0(2, f)$ be as in* Lemma 9. *For a fixed integer $m \geqslant m_0$, we have*

$$\sum_{n=1}^{\infty} \frac{E^m f(n)}{n^s} = \zeta(s) \left(\frac{\zeta\left(n(E^m f)s\right)}{\zeta\left(\left(n(E^m f)+1\right)s\right)} \right)^{f(n(f))-1} H(s),$$

where $H(s)$ converges absolutely for $\sigma > \frac{1}{2n(E^m f)}$.

Proof. By Lemma 9, all values $E^m f(1), \ldots, E^m f\bigl(3n(E^m f)-1\bigr)$ equal to 1, except for $E^m f\bigl(n(E^m f)\bigr) = f\bigl(n(f)\bigr)$. Thus, for

$$S(x) := \sum_{n\geqslant 0} E^m f(p^n) x^n = \frac{1}{1-x} + \Bigl(f\bigl(n(f)\bigr)-1\Bigr) x^{n(E^m f)} + \mathrm{O}\left(x^{3n(E^m f)}\right),$$

we have

$$(1-x)\left(\frac{1-x^{n(E^m f)}}{1-x^{n(E^m f)+1}}\right)^{f(n(f))-1} S(x) = 1 + \mathrm{O}\left(x^{2n(E^m f)}\right).$$

This asymptotic identity implies the lemma.

Now, if $f\bigl(n(f)\bigr) \in \mathbb{N}$, then Theorem 4 with $k = f\bigl(n(f)\bigr)$, $l = n(E^m f)$ can be applied on $E^m f$.

References

[1] X. Cao, W. Zhai, Some arithmetic functions involving exponential divisors, *J. Integer Seq.,* **13**(3), 10.3.7 (2010).

[2] X. Cao, W. Zhai, Conditional results for a class of arithmetic functions: a variant of H.L. Montgomery and R.C. Vaughan's method, *ArXiv e-prints,* 1301.4530v1 (2013).

[3] K. Ford, Vinogradov's integral and bounds for the Riemann zeta function, *Proc. Lond. Math. Soc.,* **85**(3), 565–633 (2002).

[4] S.W. Graham, An algorithm for computing optimal exponent pair, *J. Lond. Math. Soc.,* **33**(2), 203–218 (1986).

[5] S.W. Graham, G. Kolesnik, On the difference between consecutive squarefree integers, *Acta Arith.,* **49**(5), 435–447 (1988).

[6] M.N. Huxley, Exponential sums and the Riemann zeta function. V, *Proc. Lond. Math. Soc.,* **90** (1), 1–41 (2005).

[7] A. Ivič, M. Ouellet, Some new estimates in the Dirichlet divisor problem, *Acta Arith.,* **52**(3), 241–253 (1989).

[8] G. Kolesnik, On the estimation of multiple exponential sums, in: *Recent Progress in Analytic Number Theory,* Academic Press, London, 231–246, 1981.

[9] O.V. Kolpakova, New estimates of the remainder in an asymptotic formula in the multidimensional Dirichlet divisor problem, *Math. Notes,* **89**(3–4), 504–518 (2011).

[10] E. Krätzel, *Lattice Points,* Kluwer, Dordrecht, 1988.

[11] M. Kühleitner, W.G. Nowak, An omega theorem for a class of arithmetic functions, *Math. Nachr.,* **165**(1), 79–98 (1994).

[12] A.V. Lelechenko, Linear programming over exponent pairs, *Acta Univ. Sapientiae, Inform.,* **5**(2), 271–287 (2013).

[13] A.V. Lelechenko, An exponential divisor function over Gaussian integers, *Mat. Visn. Nauk. Tov. Im. Shevchenka,* **10**, 65–74 (2013).

[14] W.G. Nowak, On the average order of arithmetic functions connected with the Riemann zeta-function, in: *Analysis, Geometry and Groups: A Riemann Legacy Volume,* H.M. Srivastava et al. (Eds.), Hadronic Press, Palm Harbor, Florida, 449–470, 1993.

[15] Y.-F.S. Pétermann, Arithmetical functions involving exponential divisors: note on two papers by L. Tóth, *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput.,* **32**, 143–149 (2010).

[16] Y.-F.S. Pétermann, J. Wu, On the sum of exponential divisors of an integer, *Acta Math. Hung.,* **77**(1–2), 159–175 (1997).

[17] A. Smati, J. Wu, On the exponential divisor function, *Publ. Inst. Math., Nouv. Sér.,* **61**(75), 21–32 (1997).

[18] M.V. Subbarao, On some arithmetic convolutions, in: *The Theory of Arithmetical Functions, Proc. of the Conf. at Western Michigan University, April 29 – May 1, 1971,* Springer Verlag, Berlin, 247–271, 1972.

[19] D. Suryanarayana, R. Sita Rama Chandra Rao, On the true maximum order of a class of arithmetic functions, *Math. J. Okayama Univ.,* **17**, 95–101 (1975).

[20] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function,* Oxford University Press, New York, 1986.

[21] L. Tóth, On certain arithmetic functions involving exponential divisors, *Ann. Univ. Sci. Budap. Rolando Eótvös, Sect. Comput.,* **24**, 285–294 (2004).

[22] L. Tóth, On certain arithmetic functions involving exponential divisors. II, *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput.,* **27**, 155–166 (2007).

[23] L. Tóth, An order result for the exponential divisor function, *Publ. Math. Debrecen,* **71**(1–2), 165–171 (2007).

[24] J. Wu, Probléme de diviseurs exponentiels et entiers exponentiellement sans facteur carré, *J. Théor. Nombres Bord.,* **7**(1), 133–141 (1995).

Andrew V. Lelechenko
Department Of Computer Algebra and Discrete Mathematics,
I.I. Mechnikov Odessa National University,
Dvoryanskaya Str. 2, 65026 Odessa, Ukraine;
e-mail: 1@dxdy.ru