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GLOBAL EXISTENCE OF SOLUTIONS OF IVP'S FOR DIFFERENTIAL SYSTEMS WITH SEQUENTIAL FRACTIONAL DERIVATIVE OPERATORS

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Abstract. In this article, we establish results for global existence of solutions of an initial value problem of nonlinear fractional differential system on half-line involving sequential Riemann-Liouville fractional operators. Our analysis rely on the well-known fixed point theorem of Schauder. An example is presented to illustrate the main theorem. As far as the author knows, the present work is perhaps the first one that deals with such kind of initial value problems for fractional differential systems on half-line.

Key words and phrases: initial value problem, fixed point theorem, sequential fractional differential system.

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1. Introduction

Fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They have been used in modeling of many physical and chemical processes and in engineering, see the text books [13, 20, 24]. For more details on the geometric and physical interpretation for derivatives, see [11, 21, 30].

The existence and uniqueness of solutions of initial value problems or boundary value problems for fractional order differential systems on finite intervals are a fascinating subject. See papers [2, 6, 9, 22, 23, 25, 31, 32] and the references therein. The solvability of initial value problems or boundary value problems for fractional differential equations on the half-line have been addressed, for example, Arara, Benchohra, Hamidi and Nieto [3], Agarwal, Benchohra, Hamidi and Pinelas

[1], Chen and Tang [5], Kou etc [12], Liu and Jia [17], Liang and Zhang [15], Su and Zhang [26] and Trif [28], Wei, Li and Che [29], Zhao and Ge [33] studied the existence of solutions for some in initial value problems or boundary value problems for fractional order differential equations on the half-lines.

Applications of fractional order differential systems are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others, see [4]. Diethehm [8] proposed the model of the type (which is called a multi-order fractional differential system)

$$D_{0^+}^{n_i} y_i(t) = f_i(t, y_1(t), \ldots, y_n(t)), \quad i = 1, 2, \ldots, n$$

subjected to the initial conditions

$$y_j(0) = y_{j,0}, \quad j = 1, 2, \ldots, n.$$

These systems contain many models as special cases, see Chen's fractional order system [28] with a double scroll attractor, Genesio-Tesi fractional-order system [10], Lu's fractional order system [7], Volta's fractional-order system [19, 18], Rossler's fractional-order system [14], and so on.

One knows that $x \to D_{0^+}^{p_1} \ldots D_{0^+}^{p_m} x$ is called a sequential fractional derivative operator. Sequential fractional derivative operators appear in the formulation of various applied problems in physics and applied science. Indeed, differential equations modelling processes or objects arise usually as a result of a substitution of one relationship involving derivatives into another one. If the derivatives in both relationships are fractional derivatives, then the resulting expression (equation) will contain, in general case, sequential fractional derivative operators ([20], p. 88). Therefore the consideration of existence and uniqueness of solutions of a sequential fractional differential equation is of interest [20].

There has been no paper concerned with the existence of solutions of initial value problems on half-lines for fractional differential systems with sequential fractional derivative operators and the nonlinearities depending on the lower order derivatives [20], where initial value problems on the half-line have been applied (unsteady flow of gas through a semi-infinite porous medium, the theory of drain flows etc.)

In this paper, to fill this gap, we discuss the global existence of solutions of the initial value problem of the nonlinear fractional differential system on the half-line with sequential fractional derivative operators

$$\begin{cases} D_{0^+}^{\sigma_n} x(t) + \phi(t) f(t, y(t), D_{0^+}^{p} y(t)) = 0, \quad t \in (0, +\infty), \\ D_{0^+}^{\tau_m} y(t) + \psi(t) g(t, x(t), D_{0^+}^{q} x(t)) = 0, \quad t \in (0, +\infty), \\ \lim\limits_{t \to 0} t^{1-\alpha_i} D^{\sigma_{i-1}} x(t) = x_{i-1}, \quad i \in N_{1,n}, \\ \lim\limits_{t \to 0} t^{1-\beta_i} D^{\tau_{i-1}} y(t) = y_{i-1}, \quad i \in N_{1,n}, \end{cases} \tag{1}$$

where

(a) $N_{a,b} = \{a, a+1, a+2, \ldots, b\}$ with $a \leqslant b$,

(b) $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $* > 0$,

(c) $\alpha_i \in (0,1)$, $i \in N_{1,n}$,, $\sigma_j = \alpha_1 + \ldots + \alpha_j$, $j \in N_{1,n}$, $q \in (0,1)$ with $q < \sigma_n$, $D^{\sigma_j}x = D_{0^+}^{\alpha_j} \ldots D_{0^+}^{\alpha_2} D_{0^+}^{\alpha_1} x$, $j \in N_{1,n}$, is a sequential fractional derivative operator, $D^{\sigma_0}x =: x$,

(d) $\beta_i \in (0,1)$, $i \in N_{1,m}$, $\tau_j = \beta_1 + \ldots + \beta_j$, $j \in N_{1,m}$, $p \in (0,1)$ with $p < \tau_m$, $D^{\tau_j}y = D_{0^+}^{\beta_j} \ldots D_{0^+}^{\beta_2} D_{0^+}^{\beta_1} y$, $j \in N_{1,m}$, is a sequential fractional derivative operator, $D^{\beta_0}y =: y$,

(e) $x_i \in \mathbb{R}$, $i \in N_{0,n-1}$, $y_i \in \mathbb{R}$, $i \in N_{0,m-1}$, are initial data,

(f) $\phi, \psi : (0,+\infty) \to \mathbb{R}$ satisfy that there exist constants $k_i > -1$, $i = 1,2$, such that
$$|\phi(t)| \leqslant t^{k_1}, \quad |\psi(t)| \leqslant t^{k_2}, \quad t \in (0,\infty),$$

(g) $f, g : (0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$, f is a τ-Caratheodory function, and g is a σ-Caratheodory function (see Definitions 2.3 and 2.4).

We establish sufficient conditions for the global existence of solutions of IVP (1). The methods used in this paper are based upon the Schauder fixed point theorem. The novelty of this paper is that IVP (1) is defined on a half-line, and f, g are allowed to be sub-linear, linear or supper linear functions. An example is presented to illustrate the main theorem.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3. In Section 4, an example is given to show the application of the main theorem.

2. Preliminary results

For the convenience of the reader, we present here the necessary definitions from fixed point theory and fractional calculus theory. These definitions and properties can be found in the literatures [13, 20, 24]. Denote the Gamma function and Beta function by
$$\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} \mathrm{e}^{-s} \mathrm{d}s, \quad \mathbf{B}(\alpha,\beta) = \int_0^1 (1-x)^{\alpha-1} x^{\beta-1} \mathrm{d}x.$$

Definition 2.1 ([20]). *Let $c \in \mathbb{R}$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (c,+\infty) \to \mathbb{R}$ is given by*
$$I_{c^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$
provided that the right-hand side exists.

Definition 2.2 ([20]). *Let $c \in \mathbb{R}$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ with $n-1 \leqslant \alpha < n$ of a function $f : (c, +\infty) \to \mathbb{R}$ is given by*

$$D_{c+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_c^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

provided that the right-hand side exists.

Let $\sigma_n, \tau_m, k_1, k_2$ be defined by (c), (d) and (f), $\sigma > \sigma_n + k_1 + 1$ and $\tau > \tau_m + k_2 + 1$. Denote $\rho(t) = \frac{t^{1-\alpha_1}}{1+t^{\sigma}}$ and $\varrho(t) = \frac{t^{1-\beta_1}}{1+t^{\tau}}$.

Definition 2.3. *Let τ_m, k_2 be defined in* (d) *and* (f) *in* Section 1. *Suppose that $\tau > \tau_m + k_2 + 1$. $f : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is called a τ-Caratheodory function if it satisfies the following assumptions:*

(i) $t \to f\left(t, \frac{x}{\varrho(t)}, \frac{y}{t^p \varrho(t)}\right)$ *is measurable on $(0, +\infty)$ for each $(x, y) \in \mathbb{R}^2$;*

(ii) $(x, y) \to f\left(t, \frac{x}{\varrho(t)}, \frac{y}{t^p \varrho(t)}\right)$ *is continuous on $\mathbb{R}^2$ for each $t \in (0, +\infty)$;*

(iii) *for each $r > 0$, there exists a constant $M_r \geqslant 0$ such that $|x|, |y| \leqslant r$ imply*

$$\left| f\left(t, \frac{x}{\varrho(t)}, \frac{y}{t^p \varrho(t)}\right) \right| \leqslant M_r, \quad t \in (0, +\infty).$$

Definition 2.4. *Let σ_n, k_1 be defined in* (c) *and* (f) *in* Section 1. *Suppose that $\sigma > \sigma_n + k_1 + 1$. $g : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is called a σ-Caratheodory function if it satisfies the following assumptions:*

(i) $t \to g\left(t, \frac{x}{\rho(t)}, \frac{y}{t^q \rho(t)}\right)$ *is measurable on $(0, +\infty)$ for each $(x, y) \in \mathbb{R}^2$;*

(ii) $(x, y) \to g\left(t, \frac{x}{\rho(t)}, \frac{y}{t^q \rho(t)}\right)$ *is continuous on $\mathbb{R}^2$ for each $t \in (0, +\infty)$;*

(iii) *for each $r > 0$, there exists a constant $M_r > 0$ such that $|x|, |y| \leqslant r$ imply*

$$\left| g\left(t, \frac{x}{\rho(t)}, \frac{y}{t^q \rho(t)}\right) \right| \leqslant M_r, \quad t \in (0, +\infty).$$

Definition 2.5. *Let Z_1 and Z_2 be Banach spaces and $T : Z_1 \to Z_2$. T is called completely continuous if T is continuous and maps bounded sets into relatively compact sets.*

For $\alpha > 0$ and $\mu > -1$, it holds that

$$I_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad D_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

Let $A > B > 0$. It is easy to show that

$$\sup_{t \in (0,+\infty)} \frac{t^B}{1+t^A} = \frac{A-B}{A} \left(\frac{B}{A-B}\right)^{B/A} =: M_{A,B}.$$

Let $C(0,+\infty)$ denote the set of all continuous functions on $(0,+\infty)$. Choose

$$X=\left\{x:\begin{array}{c} x, D_{0^+}^q x\in C(0,+\infty), \text{ and the following limits exist}\\ \lim\limits_{t\to 0}\rho(t)x(t),\quad \lim\limits_{t\to 0}t^q\rho(t)D_{0^+}^q x(t),\\ \lim\limits_{t\to +\infty}\rho(t)x(t),\quad \lim\limits_{t\to +\infty}t^q\rho(t)D_{0^+}^q x(t)\end{array}\right\}$$

and

$$Y=\left\{y:\begin{array}{c} y, D_{0^+}^p y\in C(0,+\infty) \text{ and the following limits exist}\\ \lim\limits_{t\to 0}\varrho(t)y(t),\quad \lim\limits_{t\to 0}t^p\varrho(t)D_{0^+}^p y(t),\\ \lim\limits_{t\to +\infty}\varrho(t)y(t),\quad \lim\limits_{t\to +\infty}t^p\varrho(t)D_{0^+}^p y(t)\end{array}\right\}.$$

For $x\in X$, define

$$||x||_X=\max\left\{\sup_{t\in(0,+\infty)}\rho(t)|x(t)|,\ \sup_{t\in(0,+\infty)}t^q\rho(t)|D_{0^+}^q x(t)|\right\}.$$

For $y\in Y$, define

$$||y||_Y=\max\left\{\sup_{t\in(0,+\infty)}\varrho(t)|y(t)|,\quad \sup_{t\in(0,+\infty)}t^p\varrho(t)D_{0^+}^p y(t)|\right\}.$$

LEMMA 2.1. *X is a Banach space with the norm $||\cdot||_X$, Y is a Banach space with the norm $||\cdot||_Y$ and $X\times Y$ is a Banach space with the norm $||(\cdot,\cdot)||=\max\{||\cdot||_X,||\cdot||_Y\}$.*

Proof. We prove that X is a Banach space. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $||x_u-x_v||\to 0,\ u,v\to +\infty$. It follows that there exist

$$\begin{gathered}\lim_{t\to 0}\rho(t)x_u(t),\ \lim_{t\to+\infty}\rho(t)x_u(t),\ \lim_{t\to 0}t^q\rho(t)D_{0^+}^q x_u(t),\ \lim_{t\to+\infty}t^q\rho(t)D_{0^+}^q x_u(t),\\ \sup_{t\in(0,+\infty)}\rho(t)|x_u(t)-x_v(t)|\to 0,\quad \sup_{t\in(0,+\infty)}t^q\rho(t)|D_{0^+}^q x_u(t)-D_{0^+}^q x_v(t)|\to 0\end{gathered}\tag{2}$$

as $u,v\to+\infty$. Thus there exists two functions x_0,y_0 defined on $(0,+\infty)$ such that

$$\lim_{u\to+\infty}\rho(t)x_u(t)=x_0(t),\quad \lim_{u\to+\infty}t^q\rho(t)D_{0^+}^q x_u(t)=y_0(t).$$

It follows that

$$\begin{gathered}\sup_{t\in(0,+\infty)}|\rho(t)x_u(t)-x_0(t)|\to 0,\\ \sup_{t\in(0,+\infty)}\left|t^q\rho(t)D_{0^+}^q x_u(t)-y_0(t)\right|\to 0\end{gathered}\tag{3}$$

as $u\to+\infty$.

Step 1. Prove that $x_0,y_0\in C(0,+\infty)$.

We have $t_0 \in (0,+\infty)$ that

$$\begin{aligned}
&|x_0(t)-x_0(t_0)| \\
&\quad \leqslant |x_0(t)-\rho(t)x_N(t)| + |\rho(t)x_N(t)-\rho(t)x_N(t_0)| + |\rho(t)x_N(t_0)-x_0(t_0)| \\
&\quad \leqslant 2\sup_{t\in(0,+\infty)} |\rho(t)x_N(t)-x_0(t)| + |\rho(t)x_N(t)-\rho(t)x_N(t_0)|\,.
\end{aligned}$$

Since $\sup_{t\in(0,+\infty)}|\rho(t)x_u(t)-x_0(t)|\to 0$, $u\to+\infty$ and $\rho(t)x_u(t)$ is continuous on $(0,+\infty)$, then, for any $\varepsilon>0$, we can choose N and $\delta>0$ such that $\sup_{t\in(0,+\infty)}|\rho(t)x_N(t)-x_0(t)|<\varepsilon$ and $|\rho(t)x_N(t)-\rho(t)x_N(t_0)|<\varepsilon$ for all $|t-t_0|<\delta$. Thus, $|x_0(t)-x_0(t_0)|<3\varepsilon$ for all $|t-t_0|<\delta$. So, $x_0\in C(0,+\infty)$. Similarly, we can prove that $y_0\in C(0,+\infty)$.

Step 2. Prove that the limits $\lim_{t\to 0}x_0(t)$, $\lim_{t\to+\infty}x_0(t)$, $\lim_{t\to 0}y_0(t)$, $\lim_{t\to+\infty}y_0(t)$ exist.

Suppose that $\lim_{t\to 0}\rho(t)x_u(t)=A_u$. By $\sup_{t\in(0,+\infty)}\rho(t)|x_u(t)-x_v(t)|\to 0$, $u,v\to+\infty$, we know that A_u is a Cauchy sequence. Then $\lim_{u\to+\infty}A_u$ exists. By $\sup_{t\in(0,+\infty)}|\rho(t)x_u(t)-x_0(t)|\to 0$, $u\to+\infty$, we get that

$$\lim_{t\to 0}x_0(t)=\lim_{t\to 0}\lim_{u\to+\infty}\rho(t)x_u(t)=\lim_{u\to+\infty}\lim_{t\to 0}\rho(t)x_u(t)=\lim_{u\to+\infty}A_u.$$

Hence, $\lim_{t\to 0}x_0(t)$ exists. Similarly, we can prove that $\lim_{t\to+\infty}x_0(t)$, $\lim_{t\to 0}y_0(t)$, $\lim_{t\to+\infty}y_0(t)$ exist.

Step 3. Prove that $\frac{y_0(t)}{t^q\rho(t)}=D^q_{0^+}\left(\frac{x_0(t)}{\rho(t)}\right)$.

For some $c_u\in\mathbb{R}$, we have that

$$\begin{aligned}
&\left|x_u(t)+c_ut^{q-1}-I^q_{0^+}\left(\frac{y_0(t)}{t^q\rho(t)}\right)\right| \\
&\quad=\left|I^q_{0^+}D^q_{0^+}x_u(t)-I^q_{0^+}\left(\frac{y_0(t)}{t^q\rho(t)}\right)\right| \\
&\quad=\left|\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left(D^q_{0^+}x_u(s)-\frac{y_0(s)}{t^q\rho(s)}\right)\mathrm{d}s\right| \\
&\quad\leqslant\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}s^q\rho(s)\mathrm{d}s\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D^q_{0^+}x_u(t)-y_0(t)\right| \\
&\quad\leqslant\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}s^qs^{1-\alpha_1}\mathrm{d}s\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D^q_{0^+}x_u(t)-y_0(t)\right| \\
&\quad=t^{1+2q-\alpha_1}\frac{\mathbf{B}(q,2+q-\alpha_1)}{\Gamma(q)}\sup_{t\in(0,+\infty)}\left|t^q\rho(t)D^q_{0^+}x_u(t)-y_0(t)\right|\to 0
\end{aligned}$$

as $u\to+\infty$. So,

$$\lim_{u\to+\infty}\left(x_u(t)+c_ut^{q-1}\right)=I^q_{0^+}\left(\frac{y_0(t)}{t^q\rho(t)}\right).$$

Then

$$\frac{x_0(t)}{\rho(t)}+c_0t^{q-1}=I^q_{0^+}\left(\frac{y_0(t)}{t^q\rho(t)}\right).$$

It follows that

$$\frac{y_0(t)}{t^q\rho(t)} = D^q_{0+}\left(\frac{x_0(t)}{\rho(t)}\right).$$

So, $t \to \frac{x_0(t)}{\rho(t)}$ is a element in X with $x_u \to \frac{x_0}{\rho(t)}$ as $u \to +\infty$ in X. It follows that X is a Banach space. Similarly, we can prove that Y is a Banach space. So, $X \times Y$ is a Banach space with the norm $||(\cdot,\cdot)|| = \max\{||\cdot||_X, ||\cdot||_Y\}$. The proof is complete.

LEMMA 2.2. *Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:*

(i) *both $\{\rho(t)x : x \in M\}$ and $\{t^q\rho(t)D^q_{0+}x : x \in M\}$ are uniformly bounded;*

(ii) *both $\{\rho(t)x : x \in M\}$ and $\{t^q\rho(t)D^q_{0+}x : x \in M\}$ are equicontinuous in any subinterval $[a,b]$ in $[0,+\infty)$;*

(iii) *both $\{\rho(t)x : x \in M\}$ and $\{t^q\rho(t)D^q_{0+}x : x \in M\}$ are equiconverges as $t \to +\infty$.*

Proof. From Lemma 2.1, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is, for all $\varepsilon > 0$, M has a finite ε-net.

For any given $\varepsilon > 0$, by (ii)–(iii), there exist constants $\delta > 0$, $T > 0$, and we have, for $x \in M$,

$$\begin{aligned}
&|\rho(t_1)x(t_1) - \rho(t_2)x(t_2)| \leqslant \frac{\varepsilon}{3}, \quad t_1, t_2 \geqslant T,\\
&|\rho(t_1)x(t_1) - \rho(t_2)x(t_2)| \leqslant \frac{\varepsilon}{3}, \quad t_1, t_2 \in [0,T], \quad |t_1 - t_2| < \delta,\\
&|t_1^q\rho(t_1)D^q_{0+}x(t_1) - t_1^q\rho(t_1)D^q_{0+}x(t_1)| < \frac{\varepsilon}{3}, \quad t_1, t_2 \geqslant T,\\
&|t_1^q\rho(t_1)D^q_{0+}x(t_1) - t_1^q\rho(t_1)D^q_{0+}x(t_1)| < \frac{\varepsilon}{3}, \quad t_1, t_2 \in [0,T], \quad |t_1 - t_2| < \delta.
\end{aligned}$$

For $T > 0$, define

$$X|_{(0,T]} = \left\{x : \begin{array}{c} x, D^q_{0+}x \in C(0,T] \text{ and the following limits exist} \\ \lim\limits_{t\to 0}\rho(t)x(t), \lim\limits_{t\to 0}t^q\rho(t)D^q_{0+}x(t) \end{array}\right\}.$$

For $x \in X|_{(0,T]}$, define

$$||x||_T = \max\left\{\sup_{t\in(0,T]}\rho(t)|x(t)|, \ \sup_{t\in(0,T]}t^q\rho(t)|D^q_{0+}x(t)|\right\}.$$

Similarly to Lemma 2.1, we can prove that $X_{(0,T]}$ is a Banach space. Let $M|_{(0,T]} = \{t \to x(t), t \in (0,T] : x \in M\}$. Then $M|_{(0,T]}$ is a subset of $X|_{(0,T]}$. By (i) and (ii), and Ascoli-Arzela theorem, we can know that $M|_{(0,T]}$ is relatively compact.

Thus, there exist $x_1, x_2, \dots, x_k \in M$ such that, for any $x \in M|_{(0,T]}$, we have that there exists some $i = 1, 2, \dots, k$ such that

$$||x - x_i||_T = \max\left\{\sup_{t\in(0,T]} \rho(t)|x(t)-x_i(t)|,\ \sup_{t\in(0,T]} t^q\rho(t)|D^q_{0^+}x(t)-D^q_{0^+}x_i(t)|\right\} \leqslant \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned}&\sup_{t\geqslant T}\rho(t)|x(t)-x_i(t)|\\&\quad\leqslant \sup_{t\geqslant T}|\rho(t)x(t)-\rho(T)x(T)| + |\rho(T)x(T)-\rho(T)x_i(T)|\\&\qquad+\sup_{t\geqslant T}|\rho(T)x_i(T)-\rho(t)x_i(t)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Similarly, we have

$$\sup_{t\in(0,T]} t^q\rho(t)|D^q_{0^+}x(t)-D^q_{0^+}x_i(t)| < \varepsilon.$$

Then

$$\begin{aligned}&||x-x_i||_X\\&\quad=\max\left\{\sup_{t\in(0,T]}\rho(t)|x(t)-x_i(t)|,\ \sup_{t\in(0,T]} t^q\rho(t)|D^q_{0^+}x(t)-D^q_{0^+}x_i(t)|,\right.\\&\qquad\left.\sup_{t\geqslant T}\rho(t)|x(t)-x_i(t)|, \sup_{t\geqslant T} t^q\rho(t)|D^q_{0^+}x(t)-D^q_{0^+}x_i(t)|\right\} < \varepsilon.\end{aligned}$$

So, for any $\varepsilon > 0$, M has a finite ε-net $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

Now we assume that M is relatively compact, then for any $\varepsilon > 0$, there exists a finite ε-net of M. Let the finite ε-net be $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$ with $U_{x_i} \subset M$. Then, for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and

$$||x|| \leqslant ||x - x_i|| + ||x_i|| \leqslant \varepsilon + \max\{||x_i|| : i = 1, 2, \dots, k\}.$$

It is easy to know (i) is satisfied.

We define, for $x \in X$, that

$$\rho(t)x(t)|_{t=0} = \lim_{t\to 0}\rho(t)x(t)$$

and

$$t^q\rho(t)D_{0^+}x(t)|_{t=0} = \lim_{t\to 0} t^q\rho(t)D_{0^+}x(t).$$

Then both $t \to \rho(t)x(t)$ and $t \to t^q\rho(t)D^q_{0^+}x(t)$ are continuous on $[0, +\infty)$. Let $[a, b] \subset [0, +\infty)$. On the other hand, we know, for each $\varepsilon > 0$, that there exists $\delta > 0$ such that

$$|\rho(t_1)x_i(t_1) - \rho(t_2)x_i(t_2)| \leqslant \varepsilon,$$

$$|t_1^q\rho(t_1)D_{0^+}^q x_i(t_1)-t_1^q\rho(t_1)D_{0^+}^q x_i(t_1)|<\varepsilon$$

for $t_1,t_2\in[a,b]$, $|t_1-t_2|<\delta$, $i=1,2,\ldots,k$. We know that

$$\begin{aligned}&|\rho(t_1)x(t_1)-\rho(t_2)x(t_2)|\\&\quad\leqslant|\rho(t_1)x(t_1)-\rho(t_1)x_i(t_1)|+|\rho(t_1)x_i(t_1)-\rho(t_2)x_i(t_2)|\\&\qquad+|\rho(t_2)x_i(t_2)-\rho(t_2)x(t_2)|<3\varepsilon.\end{aligned}$$

Similarly, we get

$$|t_1^q\rho(t_1)D_{0^+}^q x(t_1)-t_1^q\rho(t_1)D_{0^+}^q x(t_1)|<3\varepsilon$$

for $t_1,t_2\in[a,b]$, $|t_1-t_2|<\delta$. So, (ii) is valid.

Finally, we know, for each $\varepsilon>0$, that there exists $\delta>0$ such that

$$\begin{aligned}&|\rho(t_1)x_i(t_1)-\rho(t_2)x_i(t_2)|\leqslant\varepsilon,\\&|t_1^q\rho(t_1)D_{0^+}^q x_i(t_1)-t_1^q\rho(t_1)D_{0^+}^q x_i(t_1)|<\varepsilon,\end{aligned}$$

$t_1,t_2\geqslant T$, $i=1,2,\ldots,k$. For $x\in M$, we know that

$$\begin{aligned}&|\rho(t_1)x(t_1)-\rho(t_2)x(t_2)|\\&\quad\leqslant|\rho(t_1)x(t_1)-\rho(t_1)x_i(t_1)|+|\rho(t_1)x_i(t_1)-\rho(t_2)x_i(t_2)|\\&\qquad+|\rho(t_2)x_i(t_2)-\rho(t_2)x(t_2)|<3\varepsilon,\quad t_1,t_2\geqslant T.\end{aligned}$$

Similarly, we get $|t_1^q\rho(t_1)D_{0^+}^q x(t_1)-t_1^q\rho(t_1)D_{0^+}^q x(t_1)|<3\varepsilon$ for $t_1,t_2\geqslant T$. So, (iii) is valid.

From above discussion, the proof is complete.

LEMMA 2.3. *Suppose that $h:(0,+\infty)\to\mathbb{R}$ satisfies that there exists a constant $k_1>-1$ such that $|h(t)|\leqslant t^{k_1}$ for all $t\in(0,+\infty)$. Then $u\in X$ is a solution of the system*

$$\begin{cases}D^{\sigma_n}u(t)+h(t)=0,\quad t\in(0,\infty),\\ \lim\limits_{t\to0}t^{1-\alpha_i}D^{\sigma_{i-1}}u(t)=x_{i-1},\quad i\in N_{1,n}\end{cases}\tag{4}$$

if and only if $u\in X$ satisfies

$$u(t)=-\int_0^t\frac{(t-s)^{\sigma_n-1}}{\Gamma(\sigma_n)}h(s)\mathrm{d}s+\sum_{j=0}^{n-1}\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}x_jt^{\sigma_{j+1}-1}.\tag{5}$$

Proof. Suppose that $x\in X$ is a solution of (4). We have that there exists a constant c_1 such that

$$D_{0^+}^{\alpha_{n-1}}\cdots D_{0^+}^{\alpha_1}u(t)=-\frac{1}{\Gamma(\alpha_n)}\int_0^t(t-s)^{\alpha_n-1}h(s)\mathrm{d}s+c_1t^{\alpha_n-1}.$$

By $\lim_{t\to0}t^{1-\alpha_n}D_{0^+}^{\alpha_{n-1}}\ldots D_{0^+}^{\alpha_1}u(t)=x_{n-1}$, we get that

$$D_{0^+}^{\alpha_{n-1}}\ldots D_{0^+}^{\alpha_1}u(t)=-\frac{1}{\Gamma(\alpha_n)}\int_0^t(t-s)^{\alpha_n-1}h(s)\mathrm{d}s+x_{n-1}t^{\alpha_n-1}.$$

Using similar methods, by the other boundary conditions, we get

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha_1+\cdots+\alpha_n)}\int_0^t (t-s)^{\alpha_1+\cdots+\alpha_n-1}h(s)\mathrm{d}s \\ &\quad+\sum_{j=0}^{n-1} x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\alpha_1+\cdots+\alpha_{j+1})}t^{\alpha_1+\cdots+\alpha_{j+1}-1}. \end{aligned}$$

Since

$$\begin{aligned} \frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}|h(s)|\mathrm{d}s &\leqslant \frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}s^{k_1}\mathrm{d}s \\ &\leqslant \frac{t^{1-\alpha_1}}{1+t^\sigma}t^{\sigma_n+k}\int_0^1 (t-w)^{\sigma_n-1}w^{k_1}\mathrm{d}w \\ &= \frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\mathbf{B}(\sigma_n,k_1+1), \end{aligned}$$

we find that $u\in C(0,\infty)$ and $\lim_{t\to 0}\rho(t)u(t)$ exists. From $\sigma>\sigma_n+k_1+1$, we find that $\lim_{t\to+\infty}\rho(t)u(t)$ exists. One sees that

$$\begin{aligned} \frac{t^{1+q-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-q-1}|h(s)|\mathrm{d}s &\leqslant \frac{t^{1+q-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-q-1}s^{k_1}\mathrm{d}s \\ &\leqslant \frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\mathbf{B}(\sigma_n-q,k_1+1). \end{aligned}$$

Similarly, we can show that $D_{0+}^q u\in C(0,\infty)$, and both $\lim_{t\to 0}t^q\rho(t)D_{0+}^q u(t)$ and $\lim_{t\to+\infty}t^q\rho(t)D_{0+}^q u(t)$ exist. Then $u\in X$ satisfies (5).

On the other hand, if $u\in X$ satisfies (5), then we can prove that $u\in X$ satisfies (4) easily. The proof is completed.

LEMMA 2.4. *Suppose that $h:(0,\infty)\to[0,\infty)$ satisfies that there exists a constant $k_2>-1$ such that $h(t)\leqslant t^{k_2}$ for all $t\in(0,\infty)$. Then $v\in Y$ is a solution of the system*

$$\begin{cases} D^{\tau_m}v(t)+h(t)=0, \quad t\in(0,\infty), \\ \lim\limits_{t\to 0}t^{1-\beta_i}D^{\tau_{i-1}}v(t)=y_{i-1}, \quad i\in N_{1,m} \end{cases} \tag{6}$$

if and only if $v\in Y$ satisfies

$$v(t)=-\frac{1}{\Gamma(\tau_m)}\int_0^t (t-s)^{\tau_m-1}h(s)\mathrm{d}s+\sum_{j=0}^{m-1} y_j\frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})}t^{\tau_{j+1}-1}. \tag{7}$$

Proof. The proof is similar to that of the proof of Lemma 2.3 and is omitted.

For $(x,y)\in X\times Y$, let us define T by $T(x,y)(t)=((T_1y)(t),(T_2x)(t))$ with

$$(T_1y)(t) = -\int_0^t \frac{(t-s)^{\sigma_n-1}}{\Gamma(\sigma_n)}\phi(s)f(s,y(s),D_{0+}^p y(s))\mathrm{d}s$$

$$
\begin{aligned}
& +\sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})} t^{\sigma_{j+1}-1}, \\
(T_2 x)(t) \quad = \quad & -\int_0^t \frac{(t-s)^{\tau_m-1}}{\Gamma(\tau_m)} \psi(s) g(s, x(s), D_{0+}^q x(s)) \mathrm{d}s \\
& +\sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})} t^{\tau_{j+1}-1}.
\end{aligned} \tag{8}
$$

It is easy to show that

$$
\begin{aligned}
D_{0+}^q (T_1 y)(t) \quad = \quad & -\int_0^t \frac{(t-s)^{\sigma_n-q-1}}{\Gamma(\sigma_n-q)} \phi(s) f(s, y(s), D_{0+}^p y(s)) \mathrm{d}s \\
& +\sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)} t^{\sigma_{j+1}-q-1}, \\
D_{0+}^p (T_2 x)(t) \quad = \quad & -\int_0^t \frac{(t-s)^{\tau_m-p-1}}{\Gamma(\tau_m-p)} \psi(s) g(s, x(s), D_{0+}^q x(s)) \mathrm{d}s \\
& +\sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1}-p)} t^{\tau_{j+1}-p-1}.
\end{aligned} \tag{9}
$$

LEMMA 2.5. *Suppose that f is a $\tau-$Caratheodory function, and g is a σ-Caratheodory function. Then*

(i) *both $T_1 : Y \to X$ and $T_2 : X \to Y$ are well defined, and so $T : X \to X$ is well defined too;*

(ii) *the fixed point of the operator T coincides with the solution of IVP* (1)*;*

(iii) *both $T_1 : Y \to X$ and $T_2 : X \to Y$ are completely continuous and so $T : X \to X$ is completely continuous.*

Proof. The proof is similar to that of the proof of Lemma 2.2 in [16] and is omitted.

3. Main results

We are in the position to prove the main results of the paper. We present the main assumptions:

(H). f is a τ-Caratheodory function and g is a σ-Caratheodory function satisfying the following assumptions: there exist non-zero functions $\Phi, \Psi : (0, \infty) \to \mathbb{R}$ measurable on each subinterval $(0, t]$ on $(0, \infty)$ and numbers

$$
\begin{aligned}
& A_i, B_i, \quad i = 1, 2, \ldots, s, \quad C_i, D_i \geqslant 0, \quad i = 1, 2, \ldots, r, \\
& \mu_s > \mu_{s-1} > \ldots > \mu_1 > 0, \quad \delta_r > \delta_{r-1} > \ldots > \delta_1 > 0
\end{aligned}
$$

such that

$$\begin{aligned}\left|f\left(t,\frac{u}{\varrho(t)},\frac{v}{t^p\varrho(t)}\right)-\Phi(t)\right| &\leqslant \sum_{j=1}^{s}A_j|u|^{\mu_j}+\sum_{j=1}^{s}B_j|v|^{\mu_j},\\ \left|g\left(t,\frac{u}{\rho(t)},\frac{v}{t^q\rho(t)}\right)-\Psi(t)\right| &\leqslant \sum_{j=1}^{r}C_j|u|^{\delta_j}+\sum_{j=1}^{r}D_j|v|^{\delta_j}\end{aligned}$$

hold for all $t\in(0,\infty)$, $u,v\in\mathbb{R}$.

For $A>B>0$, let $M_{A,B}$ be defined in Section 2. Let

$$\begin{aligned}\Phi_0(t) &= -\int_0^t\frac{(t-s)^{\sigma_n-1}}{\Gamma(\sigma_n)}\phi(s)\Phi(s)\mathrm{d}s+\sum_{j=0}^{n-1}x_j\frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})}t^{\sigma_{j+1}-1},\\ \Psi_0(t) &= -\int_0^t\frac{(t-s)^{\tau_m-1}}{\Gamma(\tau_m)}\psi(s)\Psi(s)\mathrm{d}s+\sum_{j=0}^{m-1}y_j\frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})}t^{\tau_{j+1}-1}. \qquad (10)\end{aligned}$$

Denote

$$\begin{aligned}M_0 &= \max\left\{\frac{\mathbf{B}(\sigma_n-q,k_1+1)}{\Gamma(\sigma_n-q)}M_{\sigma,\sigma_n+k_1+1-\alpha_1},\ \frac{\mathbf{B}(\sigma_n,k_1+1)}{\Gamma(\sigma_n)}M_{\sigma,\sigma_n+k_1+1-\alpha_1}\right\},\\ N_0 &= \max\left\{\frac{\mathbf{B}(\tau_n-p,k_2+1)}{\Gamma(\tau_n-p)}M_{\tau,\tau_n+k_2+1-\beta_1},\ \frac{\mathbf{B}(\tau_n,k_2+1)}{\Gamma(\tau_n)}M_{\tau,\tau_n+k_2+1-\beta_1}\right\},\\ a &= M_0\left[\sum_{j=1}^{s-1}[A_j+B_j]||\Psi_0||^{\mu_j-\mu_s}+[A_s+B_s]\right],\\ b &= N_0\left[\sum_{j=1}^{r-1}[C_j+D_j]||\Phi_0||^{\delta_j-\delta_r}+[C_r+D_r]\right].\end{aligned}$$

THEOREM 3.1. *Suppose that* (**H**) *holds. Then IVP* (1) *has at least one solution* $(x,y)\in Z$ *if*

(i) $\mu_s\delta_r>1$ *with*

$$\begin{aligned}\frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s-1)^{\delta_r\mu_s-1}}\left[||\Phi_0||+\left(\frac{||\Psi_0||}{b}\right)^{\frac{1}{\delta_r}}\right]^{\delta_r\mu_s-1} &\leqslant \frac{1}{ab^{\mu_s}}\quad \textit{for}\quad \delta_r>1,\\ \frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s-1)^{\delta_r\mu_s-1}}\left[||\Psi_0||+\left(\frac{||\Phi_0||}{a}\right)^{\frac{1}{\mu_s}}\right]^{\delta_r\mu_s-1} &\leqslant \frac{1}{ba^{\delta_r}}\quad \textit{for}\quad \mu_s>1; \qquad (11)\end{aligned}$$

(ii) $\mu_s\delta_r=1$ *with either* $a<\left(\frac{1}{b}\right)^{1/\delta_r}$ *or* $b<\left(\frac{1}{a}\right)^{1/\mu_s}$*;*

(iii) $\mu_s\delta_r<1$.

Proof. Let the Banach spaces X, Y and Z with their norms be defined in Section 2. Let $T : Z \to Z$ be defined by (10).

By Lemma 2.5, we seek solutions of IVP (1) by getting the fixed point of T in $X \times Y$, and T is well defined and is completely continuous.

It is easy to show that

$$\begin{aligned} D_{0^+}^q \Phi_0(t) &= -\int_0^t \frac{(t-s)^{\sigma_n-q-1}}{\Gamma(\sigma_n-q)}\phi(s)\Phi(s)\mathrm{d}s + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1}-q)} t^{\sigma_{j+1}-q-1}, \\ D_{0^+}^p \Psi_0(t) &= -\int_0^t \frac{(t-s)^{\tau_m-p-1}}{\Gamma(\tau_m-p)}\psi(s)\Psi(s)\mathrm{d}s + \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1}-p)} t^{\tau_{j+1}-p-1}. \end{aligned} \quad (12)$$

It is easy to show that $\Phi_0 \in X$, $\Psi_0 \in Y$. Let $r_1, r_2 > 0$, and define

$$\overline{\Omega}_{r_1,r_2} = \{(x,y) \in Z : \|x - \Phi_0\| \leqslant r_1,\ \|y - \Psi_0\| \leqslant r_2\}.$$

For $(x,y) \in \overline{\Omega}_{r_1,r_2}$, we have $\|x - \Phi_0\| \leqslant r_1$ and $\|y - \Psi_0\| \leqslant r_2$. Then

$$\begin{aligned} ||x|| &\leqslant ||x - \Phi_0|| + ||\Phi_0|| \leqslant r_1 + ||\Phi_0||, \\ ||y|| &\leqslant ||y - \Psi_0|| + ||\psi_0|| \leqslant r_2 + ||\Psi_0||. \end{aligned}$$

Using (H) and (8), (9), (10) and (12), we find

$$\begin{aligned} &\frac{t^{1-\alpha_1}}{1+t^\sigma}|(T_1y)(t) - \Phi_0(t)| \\ &\leqslant \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}\phi(s) \\ &\quad\times \left| f\left(s, \frac{1+s^\tau}{t^{1-\beta_1}}\frac{t^{1-\beta_1}}{1+s^\tau}y(s), \frac{1+s^\tau}{t^{1+p-\beta_1}}\frac{t^{1+p-\beta_1}}{1+s^\tau}D_{0^+}^p y(s)\right) - \Phi(s)\right| \mathrm{d}s \\ &\leqslant \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}\phi(s) \\ &\quad\times \left[\sum_{j=1}^{s} A_j \left|\frac{t^{1-\beta_1}}{1+s^\tau}y(s)\right|^{\mu_j} + \sum_{j=1}^{s} B_j \left|\frac{t^{1+p-\beta_1}}{1+s^\tau}D_{0^+}^p y(s)\right|^{\mu_j}\right] \mathrm{d}s \\ &\leqslant \frac{1}{\Gamma(\sigma_n)}\frac{t^{1-\alpha_1}}{1+t^\sigma}\int_0^t (t-s)^{\sigma_n-1}s^{k_1}\left[\sum_{j=1}^{s}[A_j + B_j]||y||^{\mu_j}\right]\mathrm{d}s \\ &\leqslant \frac{\mathbf{B}(\sigma_n, k_1+1)}{\Gamma(\sigma_n)}\frac{t^{\sigma_n+k_1+1-\alpha_1}}{1+t^\sigma}\left[\sum_{j=1}^{s}[A_j + B_j]||y||^{\mu_j}\right] \\ &\leqslant \frac{\mathbf{B}(\sigma_n, k_1+1)}{\Gamma(\sigma_n)}M_{\sigma,\sigma_n+k_1+1-\alpha_1}\left[\sum_{j=1}^{s}[A_j + B_j]||y||^{\mu_j}\right]. \end{aligned}$$

Similarly, we have

$$\frac{t^{1+q-\alpha_1}}{1+t^\sigma}|D_{0^+}^q (T_1y)(t) - D_{0^+}^q \Phi_0(t)|$$

$$\leqslant \frac{\mathbf{B}(\sigma_n - q, k_1 + 1)}{\Gamma(\sigma_n - q)} M_{\sigma,\sigma_n+k_1+1-\alpha_1} \left[\sum_{j=1}^{s} [A_j + B_j] ||y||^{\mu_j} \right].$$

It follows that

$$\begin{aligned} ||T_1 y - \Phi_0|| &\leqslant M_0 \left[\sum_{j=1}^{s} [A_j + B_j] ||y||^{\mu_j} \right] \leqslant M_0 \left[\sum_{j=1}^{s} [A_j + B_j][r_2 + ||\Psi_0||]^{\mu_j} \right] \\ &\leqslant M_0 [r_2 + ||\Psi_0||]^{\mu_s} \left[\sum_{j=1}^{s-1} [A_j + B_j] ||\Psi_0||^{\mu_j - \mu_s} + [A_s + B_s] \right]. \end{aligned}$$

Hence,

$$||T_1 y - \Phi_0|| \leqslant a[r_2 + ||\Psi_0||]^{\mu_s}. \tag{13}$$

Similarly, we can show that

$$||T_2 x - \Psi_0|| \leqslant b[r_1 + ||\Phi_0||]^{\delta_r}. \tag{14}$$

We will prove that the following inequality system

$$a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1, \quad b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2$$

has positive solution $R_1, r_2 > 0$, i.e., the system

$$b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2 \leqslant \left(\frac{r_1}{a} \right)^{1/\mu_s} - ||\Psi_0||, \tag{15}$$

or

$$a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1 \leqslant \left(\frac{r_2}{b} \right)^{1/\delta_r} - ||\Phi_0|| \tag{16}$$

has positive solution $r_1, r_2 > 0$.

Case (i). $\mu_s \delta_r > 1$.

It is easy to show that $e^l + f^l \leqslant (e + f)^l$ for all $e, f > 0$ and $l > 1$. If $\delta_r > 1$, choose

$$r_1 = \frac{1}{\delta_r \mu_s - 1} \left[||\Phi_0|| + \left(\frac{||\Psi_0||}{b} \right)^{1/\delta_r} \right].$$

Then we get from

$$\frac{(\delta_r \mu_s)^{\delta_r \mu_s}}{(\delta_r \mu_s - 1)^{\delta_r \mu_s - 1}} \left[||\Phi_0|| + \left(\frac{||\Psi_0||}{b} \right)^{\frac{1}{\delta_r}} \right]^{\delta_r \mu_s - 1} \leqslant \frac{1}{a b^{\mu_s}}$$

that

$$\frac{\left[r_1 + ||\Phi_0|| + \left(\frac{||\Psi_0||}{b} \right)^{\frac{1}{\delta_r}} \right]^{\delta_r \mu_s}}{r_1} \leqslant \frac{1}{a b^{\mu_s}}.$$

Since

$$b[r_1 + ||\Phi_0||]^{\delta_r} + ||\Psi_0|| \leqslant b\left[r_1 + ||\Phi_0|| + \left(\frac{||\Psi_0||}{b}\right)^{1/\delta_r}\right]^{\delta_r},$$

we get

$$lb[r_1 + ||\Phi_0||]^{\delta_r} \leqslant \left(\frac{r_1}{a}\right)^{1/\mu_s} - ||\Psi_0||.$$

Choose r_2 such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2 \leqslant \left(\frac{r_1}{a}\right)^{1/\mu_s} - ||\Psi_0||. \tag{17}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$, using (17), we have

$$||T_1 y - \Phi_0|| \leqslant a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1, \quad ||T_2 x - \Psi_0|| \leqslant b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2.$$

Then $T(x, y) = (T_1 y, T_2 x) \in \overline{\Omega}_{r_1,r_2}$. Then the Schauder fixed point theorem implies that T has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1).

If $\mu_s > 1$, choose

$$r_2 = \frac{1}{\delta_r\mu_s - 1}\left[||\Psi_0|| + \left(\frac{||\Phi_0||}{a}\right)^{1/\mu_s}\right].$$

Then we get from

$$\frac{(\delta_r\mu_s)^{\delta_r\mu_s}}{(\delta_r\mu_s - 1)^{\delta_r\mu_s - 1}}\left[||\Psi_0|| + \left(\frac{||\Phi_0||}{a}\right)^{\frac{1}{\mu_s}}\right]^{\delta_r\mu_s - 1} \leqslant \frac{1}{ba^{\delta_r}}$$

that

$$\frac{\left[r_2 + ||\Psi_0|| + \left(\frac{||\Phi_0||}{a}\right)^{\frac{1}{\mu_s}}\right]^{\delta_r\mu_s}}{r_2} \leqslant \frac{1}{ba^{\delta_r}}.$$

Since

$$a[r_2 + ||\Psi_0||]^{\mu_s} + ||\Phi_0|| \leqslant a\left[r_2 + ||\Psi_0|| + \left(\frac{||\Phi_0||}{a}\right)^{1/\mu_s}\right]^{\mu_s},$$

we get

$$a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant \left(\frac{r_2}{b}\right)^{1/\delta_r} - ||\Phi_0||.$$

Choose r_1 such that

$$a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1 \leqslant \left(\frac{r_2}{b}\right)^{1/\delta_r} - ||\Phi_0||. \tag{18}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$, using (18), we have

$$||T_1 y - \Phi_0|| \leqslant a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1, \quad ||T_2 x - \Psi_0|| \leqslant b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2.$$

Then $T(x,y) = (T_1y, T_2x) \in \overline{\Omega}_{r_1,r_2}$.

Then the Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1).

Case (ii). $\mu_s\delta_r = 1$.

For $a < \left(\frac{1}{b}\right)^{1/\delta_r}$, since

$$\lim_{r_1\to+\infty} \frac{a[r_2 + ||\Psi_0||]^{\mu_s}}{\left(\frac{r_2}{b}\right)^{1/\delta_r} - ||\Phi_0||} = \frac{a}{\left(\frac{1}{b}\right)^{1/\delta_r}} < 1,$$

we can choose $r_2 > 0$ sufficiently large such that

$$a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant \left(\frac{r_2}{b}\right)^{1/\delta_r} - ||\Phi_0||.$$

Then we can choose r_1 such that

$$a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1 \leqslant \left(\frac{r_2}{b}\right)^{1/\delta_r} - ||\Phi_0||. \tag{19}$$

Then, for $(x,y) \in \overline{\Omega}_{r_1,r_2}$, using (19), we have

$$\begin{aligned} ||T_1y - \Phi_0|| &\leqslant Ma[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1, \\ ||T_2x - \Psi_0|| &\leqslant b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2. \end{aligned}$$

Then $T(x,y) = (T_1y, T_2x) \in \overline{\Omega}_{r_1,r_2}$. Then the Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1).

For $b < \left(\frac{1}{a}\right)^{1/\mu_s}$, since

$$\lim_{r_1\to+\infty} \frac{b[r_1 + ||\Phi_0||]^{\delta_r}}{\left(\frac{r_1}{a}\right)^{\frac{1}{\mu_s}} - ||\Psi_0||} = \frac{b}{\left(\frac{1}{a}\right)^{1/\mu_s}} < 1,$$

we can choose $r_1 > 0$ sufficiently large such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant \left(\frac{r_1}{a}\right)^{1/\mu_s} - ||\Psi_0||.$$

Then we can choose r_2 such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2 \leqslant \left(\frac{r_1}{a}\right)^{1/\mu_s} - ||\Psi_0||. \tag{20}$$

Then, for $(x,y) \in \overline{\Omega}_{r_1,r_2}$, using (20), we have

$$\begin{aligned} ||T_1y - \Phi_0|| &\leqslant Ma[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1, \\ ||T_2x - \Psi_0|| &\leqslant b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2. \end{aligned}$$

Then $T(x,y) = (T_1y, T_2x) \in \overline{\Omega}_{r_1,r_2}$. Then the Schauder fixed point theorem implies that T has a fixed point $(x,y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1).

Case (iii). $\mu_s\delta_r < 1$.

It follows from (15) that there exists $r_1 > 0$ sufficiently large such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant \left(\frac{r_1}{a}\right)^{1/\mu_s} - ||\Psi_0||.$$

This allows us to choose r_2 such that

$$b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2 \leqslant \left(\frac{r_1}{a}\right)^{1/\mu_s} - ||\Psi_0||. \tag{21}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$ using (21), we have

$$||T_1y - \Phi_0|| \leqslant a[r_2 + ||\Psi_0||]^{\mu_s} \leqslant r_1, \quad ||T_2x - \Psi_0|| \leqslant b[r_1 + ||\Phi_0||]^{\delta_r} \leqslant r_2.$$

Then $T(x, y) = (T_1y, T_2x) \in \overline{\Omega}_{r_1,r_2}$. Then the Schauder fixed point theorem implies that T has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of IVP (1).

The proof is complete.

4. An example

In this section, we given an example to illustrate Theorem 3.1.

EXAMPLE 4.1. Consider the following problem

$$\begin{cases} D_0^{1/2}D_0^{1/3}x(t) + t^{-1/2}f(t, y(t), D_{0^+}^{1/5}y(t)) = 0, \quad t \in (0, \infty), \\ D_0^{1/4}D_0^{1/8}y(t) + t^{-1/2}g(t, x(t), D_{0^+}^{1/6}x(t)) = 0, \quad t \in (0, \infty), \\ \lim\limits_{t\to 0} t^{2/3}x(t) = x_0, \quad \lim\limits_{t\to 0} t^{7/8}y(t) = y_0, \\ \lim\limits_{t\to 0} t^{1/2}D^{1/3}x(t) = x_1, \quad \lim\limits_{t\to 0} t^{3/4}D^{1/8}x(t) = y_1, \end{cases} \tag{22}$$

where $x_0, x_1, y_0, y_1 \in \mathbb{R}$, $\phi(t) = \psi(t) = t^{-1/2}$, and

$$\begin{aligned} f(t, u, v) &= 1 + A\left(\frac{t^{7/8}}{1+t}u\right)^{\mu} + B\left(\frac{t^{43/40}}{1+t}v\right)^{\mu}, \\ g(t, u, v) &= 1 + C\left(\frac{t^{2/5}}{1+t^{3/2}}u\right)^{\delta} + D\left(\frac{t^{5/6}}{1+t^{3/2}}v\right)^{\delta} \end{aligned}$$

with $A, B, C, D \geqslant 0, \delta, \mu > 0$. Then IVP (22) has at least one solution for all sufficiently small $A, B, C, D, |x_0|, |y_0|$.

Proof. Corresponding to (1), we have $m = n = 2$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{2}$ and $\beta_1 = \frac{1}{8}$ and $\beta_2 = \frac{1}{4}$, $p = \frac{1}{5}$ and $q = \frac{1}{6}$.

It is easy to see that

- $x_i \in \mathbb{R}$, $i = 0, 1$, $y_i \in \mathbb{R}$, $i = 0, 1$, $p, q \in (0, 1)$ with $q < \sigma_2 = \alpha_1 + \alpha_2 = \frac{5}{6}$ and $p < \tau_2 = \beta_1 + \beta_2 = \frac{3}{8}$,
- $\alpha_i \in (0, 1)$, $i = 1, 2$, $\beta_i \in (0, 1)$, $i = 1, 2$,

- $\phi, \psi : (0,\infty) \to [0,\infty)$ satisfy that
$$\phi(t) \leqslant t^{k_1}, \quad \psi(t) \leqslant t^{k_2}, \quad t \in (0,\infty),$$
with $k_1 = k_2 = -\frac{1}{2}$.

Choose $\tau = 1$, $\sigma = \frac{3}{2}$. Then $\tau > \tau_2 + k_2 + 1$ and $\sigma > \sigma_2 + k_1 + 1$. By computation, one sees that

$$\begin{aligned} f\left(t, \frac{1+t^\tau}{t^{1-\beta_1}}u, \frac{1+t^\tau}{t^{1+p-\beta_1}}v\right) &= 1 + Au^\mu + Bv^\mu, \\ g\left(t, \frac{1+t^\sigma}{t^{1-\alpha_1}}u, \frac{1+t^\sigma}{t^{1+q-\alpha_1}}v\right) &= 1 + Cu^\delta + Dv^\delta. \end{aligned}$$

So, $f, g : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, and f is a τ-Caratheodory function, and g is a σ-Caratheodory function.

It is easy to see that **(H).** f is a a τ-Caratheodory function and g is a σ-Caratheodory function satisfying the following assumptions: there exist non-zero functions $\Phi(t) = \Psi(t) = 1$ and numbers $A, B \geqslant 0$, $C, D \geqslant 0$, $\mu > 0, \delta > 0$, such that

$$\begin{aligned} \left|f\left(t, \frac{1+t^\tau}{t^{1-\beta_1}}u, \frac{1+t^\tau}{t^{1+p-\beta_1}}v\right) - \Phi(t)\right| &\leqslant A|u|^\mu + B|v|^\mu, \\ \left|g\left(t, \frac{1+t^\sigma}{t^{1-\alpha_1}}u, \frac{1+t^\sigma}{t^{1+q-\alpha_1}}v\right) - \Psi(t)\right| &\leqslant C|u|^\delta + D|v|^\delta \end{aligned}$$

hold for all $t \in (0,\infty)$, $u, v \in \mathbb{R}$ with $r = s = 1$ and $\mu = \mu_1$, $\delta = \delta_1$.

We have

$$\begin{aligned} \Phi_0(t) &= -\frac{1}{\Gamma(\sigma_n)}\int_0^t (t-s)^{\sigma_n - 1}\phi(s)\Phi(s)\mathrm{d}s + \sum_{j=0}^{n-1} x_j \frac{\Gamma(\alpha_{j+1})}{\Gamma(\sigma_{j+1})} t^{\sigma_{j+1}-1} \\ &= -\frac{1}{\Gamma(5/6)}\int_0^t (t-s)^{-1/6} s^{-1/2}\mathrm{d}s + x_0 t^{-2/3} \\ &= -\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)} t^{1/3} + x_0 t^{-2/3} \\ \Psi_0(t) &= -\frac{1}{\Gamma(\tau_m)}\int_0^t (t-s)^{\tau_m - 1}\psi(s)\Psi(s)\mathrm{d}s + \sum_{j=0}^{m-1} y_j \frac{\Gamma(\beta_{j+1})}{\Gamma(\tau_{j+1})} t^{\tau_{j+1}-1} \\ &= -\frac{1}{\Gamma(3/8)}\int_0^t (t-s)^{-5/8} s^{-1/2}\mathrm{d}s + y_0 t^{-7/8} \\ &= -\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)} t^{-1/8} + y_0 t^{-7/8}, \\ D_{0^+}^q \Phi_0(t) &= D_{0^+}^{1/6}\left(-\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)} t^{1/3} + x_0 t^{-2/3}\right) \\ &= -\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}\frac{\Gamma(4/3)}{\Gamma(7/6)} t^{1/6} + x_0 \frac{\Gamma(1/3)}{\Gamma(1/6)} t^{-5/6} \\ D_{0^+}^p \Psi_0(t) &= D_{0^+}^{1/5}\left(-\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)} t^{-1/8} + y_0 t^{-7/8}\right) \end{aligned}$$

$$= -\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}\frac{\Gamma(7/8)}{\Gamma(27/40)}t^{-13/40}+y_0\frac{\Gamma(1/8)}{\Gamma(3/40)}t^{-43/40}.$$

Then

$$\begin{aligned}
&||\Phi_0||\\
&= \max\left\{\sup_{t\in(0,\infty)}\frac{t^{2/3}}{1+t}|\Phi_0(t)|,\ \sup_{t\in(0,\infty)}\frac{t^{5/6}}{1+t}|D_{0^+}^{1/6}\Phi_0(t)|\right\}\\
&\leqslant \max\left\{\sup_{t\in(0,\infty)}\frac{(\mathbf{B}(5/6,1/2)/\Gamma(5/6))t+|x_0|}{1+t},\right.\\
&\qquad\left.\sup_{t\in(0,\infty)}\frac{(\mathbf{B}(5/6,1/2)/\Gamma(5/6))(\Gamma(4/3)/\Gamma(7/6))t+|x_0|(\Gamma(1/3)/\Gamma(1/6))}{1+t}\right\}\\
&\leqslant \max\left\{\frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}M_{1,1}+|x_0|,\ \frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}\frac{\Gamma(4/3)}{\Gamma(7/6)}M_{1,1}+|x_0|\frac{\Gamma(1/3)}{\Gamma(1/6)}\right\}
\end{aligned}$$

and

$$\begin{aligned}
||\Psi_0|| &= \max\left\{\sup_{t\in(0,\infty)}\frac{t^{7/8}}{1+t^{3/2}}|\Phi_0(t)|,\ \sup_{t\in(0,\infty)}\frac{t^{43/40}}{1+^{3/2}}|D_{0^+}^{1/5}\Psi_0(t)|\right\}\\
&\leqslant \max\left\{\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}M_{3/2,3/4}+|y_0|,\right.\\
&\qquad\left.\frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}\frac{\Gamma(7/8)}{\Gamma(27/40)}M_{3/2,3/4}+|y_0|\frac{\Gamma(1/8)}{\Gamma(3/40)}\right\}.
\end{aligned}$$

By direct computation, we get

$$\begin{aligned}
M_{1,1} &= \sup_{t\in(0,+\infty)}\frac{t}{1+t}\leqslant 1,\quad M_{3/2,3/4}=\sup_{t\in(0,+\infty)}\frac{t^{3/4}}{1+t^{3/2}}\leqslant 1,\\
M_{3/2,1} &= \sup_{t\in(0,+\infty)}\frac{t}{1+t^{3/2}}\leqslant 1,\quad M_{1,3/4}=\sup_{t\in(0,+\infty)}\frac{t^{3/4}}{1+t}\leqslant 1,\\
||\Phi_0|| &\leqslant \max\{1.9849+|x_0|,\ 1.9105+|x_0|0.4813\}\leqslant 1.9849+|x_0|,\\
||\Psi_0|| &\leqslant \max\{1.6266+|y_0|,\ 1.3233+|y_0|0.5874\}\leqslant 1.6266+|y_0|,\\
M_0 &\leqslant \max\left\{\frac{\mathbf{B}(2/3,1/2)}{\Gamma(2/3)},\ \frac{\mathbf{B}(5/6,1/2)}{\Gamma(5/6)}\right\}\approx\max\{1.9105,1.9849\}=1.9849,\\
N_0 &= \max\left\{\frac{\mathbf{B}(7/40,1/2)}{\Gamma(7/40)},\ \frac{\mathbf{B}(3/8,1/2)}{\Gamma(3/8)}\right\}\approx\max\{1.3233,1.6266\}=1.6266,\\
a &\leqslant 1.9849(A+B),\quad b=1.6266(C+D).
\end{aligned}$$

From Theorem 3.1, we have that IVP (22) has at least one solution $(x,y)\in Z$ if (i) $\mu\delta>1$ with

$$1.9849(A+B)[1.6266(C+D)]^{\mu}\left[1.9849+|x_0|\ +\left(\frac{1.6266+|y_0|}{1.6266(C+D)}\right)^{\frac{1}{\delta}}\right]^{\delta\mu-1}$$

$$\leqslant \frac{(\delta\mu-1)^{\delta\mu-1}}{(\delta\mu)^{\delta\mu}} \text{ for } \delta > 1,$$

$$1.6266(C+D)[1.9849(A+B)]^{\delta}\left[1.6266+|y_0|+\left(\frac{1.9849+|x_0|}{1.9849(A+B)}\right)^{\frac{1}{\mu}}\right]^{\delta\mu-1}$$
$$\leqslant \frac{(\delta\mu-1)^{\delta\mu-1}}{(\delta\mu)^{\delta\mu}} \quad \text{for} \quad \mu > 1, \tag{23}$$

or (ii) $\mu\delta = 1$ with either

$$1.9849(A+B)[1.6266(C+D)]^{\frac{1}{\delta}} < 1,$$

or

$$1.6266(C+D)[1.9849(A+B)]^{\frac{1}{\mu}} < 1,$$

or (iii) $\mu\delta < 1$.

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References

[1] R.P Agarwal, M. Benchohra, N. Hamidi, S. Pinelas, Boundary value problems for differential equations involving Riemann-Liouville fractional derivative on the half-line, *Dynam. Contin., Discr. Impul. Syst.*, **18**, 235–244 (2011).

[2] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point bounndary conditions, *Comput. Math. Appl.*, **58**, 1838–1843 (2009).

[3] A. Arara, M. Bencjohra, N. Hammidi, J.J. Nieto, Fractional differential equations on an unbounded domain, *Nonl. Anal.*, **72** 580–586 (2010).

[4] R. Caponetto, G. Dongola, L. Fortuna, *Frational order Systems Modeling and Control Applications*, World Scientific Series on Nonlinear Science, Ser. A, Vol. 72, World Scientific, Publishing Co. Pte. Ltd, Singapore, 2010.

[5] Y. Chen, X. Tang, Positive solutions of fractional differential equations at resonance on the half-line, *Bound. Value Probl.*, **1**, (2012).

[6] V. Daftardar-Gejji, A. Babakhani, Analysis of a system of fractional differential equations, *J. Math. Anal. Appl.*, **293**, 511–522 (2004).

[7] W.H. Deng, C.P. Li, Chaos synchronization of the fractional Lu system, *Phys. A*, **353**, 61–72 (2005).

[8] K. Diethelm, Multi-term fractional differential equations, multi-order fractional differential systems and their numerical solution, *J. Eur. Syst. Autom.*, **42**, 665–676 (2008).

[9] C.S. Goodrich, Existence of a positive solution to systems of differential equations of fractional order, *Comput. Math. Appl.*, **62**, 1251–1268 (2011).

[10] L.J. Guo, Chaotic dynamics and synchronization of fractional-order Genesio-Tesi systems, *Chinese Phys.*, **14**, 1517–1521 (2005).

[11] N. Heymans, I. Podlubny, Physical interpretation of initial cinditions for fractional differential equations involving Riemann-Liouville fractional derivatives on the half-line, *Rheolog. Acta*, **45**, 765–771 (2006).

[12] C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, *Nonlinear Anal.*, **74**, 5975—5986 (2011).

[13] V. Lakshmikantham, S. Leela, J.V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[14] C. Li, G. Chen, Chaos and hyperchaos in the fractional-order Rossler equations, *Phys. A*, **341**, 55–61 (2004).

[15] S. Liang, J. Zhang, Existence of three positive solutions of m-point boundary value problems for some nonlinear fractional differential equations on an infinite interval, *Comput. Math. Appl.*, **61**, 3343–3354 (2011).

[16] Y. Liu, Existence and uniqueness of solutions for initial value problems of multi-order fractional differential equations on the half-lines, *Sci. Sin. Math.*, **42**(7), 735–756 (2012) (in Chinese).

[17] X. Liu, M. Jia, Multiple solutions of nonlocal boundary value problems for fractional differential equations on half-line, *Electr. J. Qual. Theory Differ. Eq.*, **56**, 1–14 (2011).

[18] I. Petras, Fractional-order feedback control of a DC motor, *J. Electr. Engin.*, **60**, 117–128 (2009).

[19] I. Petras, Chaos in the fractional-order Volta's system: modeling and simulation, *Nonlinear Dyn.*, **57**, 157–170 (2009).

[20] I. Podludny, *Fractional Differential Equations*, Mathematics in Science and Engineering Vol. 198, Academic Press, San Diegoand ect., 1999.

[21] I. Podlubny, Geometric and physical interpretation of fractional integral and fractional differentiation, *Fract. Calc. Appl. Anal.*, **5**, 367–386 (2002).

[22] D. Qian, C. Li, R.P. Agarwal, P.J.Y. Wong, Stability analysis of fractional differential system with Riemann-Liouville derivative, *Math. Comput. Model.*, **52**, 862–874 (2010).

[23] M. Rehman, R. Ali Khan, Positive solutions to coupled system of fractional differential equations, *Int. J. Nonlinear Sci.*, **10**, 96–104 (2010) .

[24] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integral and Derivative. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[25] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.*, **22**, 64–69 (2009).

[26] X. Su, S. Zhang, Unbounded solutions to a boundary value problem of fractional order on the half-line, *Comput. Math. Appl.*, **61**, 1079–1087 (2011).

[27] X. Su, S. Zhang, Unbounded solutions to a boundary value problem of fractional order on the half-line, *Comput. Math. Appl.*, **61**, 580–586 (2011).

[28] T. Trif, Existence of solutions to initial value problems for nonlinear fractional differential equations on the semi-axis, *Fract. Calc. Appl. Anal.*, **3**(16), 595–612 (2013).

[29] Z. Wei, Q. Li, J. Che, Initial value problems for fractional differential equations involving Riemann-CLiouville sequential fractional derivative, *J. Math. Anal. Appl.*, **367**(1), 260–272 (2010).

[30] B.J. West, M. Bologna, P. Grigolini, *Physics of Fractal Operators*, Springer, New York, 2003.

[31] C. Yuan, Multiple positive solutions for $(n-1,1)$-type semipositone conjugate boundary value problems for coupled systems of nonlinear fractional differential equations, *Electr. J. Qual. Theory Diff. Eq.*, **13**, 1–12 (2011).

[32] C. Yuan, D. Jiang, D. O' Regan, P.R. Agawarl, Multiple positive to solutions systems of nonlinear semipositone fractional differential equations with coupled boundary conditions, *Electr. J. Qual. Theory Diff. Eq.*, **13**, 1–17 (2012).

[33] X. Zhao, W. Ge, Unbounded solutions for fractional bounded value problems on the infinite interval, *Acta Appl. Math.*, **109**, 495–505 (2010).

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