ON THE DISCRETE UNIVERSALITY OF THE PERIODIC HURWITZ ZETA-FUNCTION

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Abstract. In this paper, a theorem on the approximation of analytic functions by discrete shifts of the periodic Hurwitz zeta-function is obtained. For this, a linear independence condition connecting the parameter of the zeta-function and the step of the arithmetic progression is used.

Key words and phrases: discrete universality, periodic Hurwitz zeta function, zeta-function, universality.

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1. Introduction

Let $s = \sigma + it$ be a complex variable, $\alpha$ be a fixed parameter, $0 < \alpha \leqslant 1$, and let $a = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; a)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$
\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.
$$

The periodicity of the sequence $a$ implies, for $\sigma > 1$, the equality

$$
\zeta(s, \alpha; a) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{l+\alpha}{q}\right),
$$

(1.1)

where $\zeta(s, \alpha) = \zeta(s, \alpha; a \equiv \{1\}) = \zeta(s, \alpha; 1)$ is the classical Hurwitz zeta-function. It is well known that the function $\zeta(s, \alpha)$ is regular on the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Therefore, equality
(1.1) gives an analytic continuation for the function \( \zeta(s, \alpha; a) \) to the whole complex plane, except for a possible pole at the point \( s = 1 \) with residue

\[
a \overset{\text{def}}{=} \frac{1}{q} \sum_{i=0}^{q-1} a_i.
\]

If \( a = 0 \), then the function \( \zeta(s, \alpha; a) \) is entire.

The function \( \zeta(s, \alpha; a) \), as the majority of other zeta and \( L \)-functions, is universal in the Voronin sense. Roughly speaking, this means that any analytic function uniformly on compact subsets of some region can be approximated by shifts \( \zeta(s + it, \alpha; a) \), \( t \in \mathbb{R} \). For the precise statement, we need some notation. Let \( D = \{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \} \). Denote by \( \mathcal{K} \) the class of compact subsets of the strip \( D \) with connected complements, and by \( H(K) \), \( K \in \mathcal{K} \), the class of continuous functions on \( K \) which are analytic in the interior of \( K \). Then, in [6], the following universality theorem has been obtained.

**Theorem 1.** Suppose that \( \alpha \) is a transcendental number. Let \( K \in \mathcal{K} \) and \( f(s) \in H(K) \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} \left| \zeta(s + it, \alpha; a) - f(s) \right| < \varepsilon \right\} > 0.
\]

Here \( \text{meas} \) denotes the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). The inequality of the theorem shows that the set of shifts \( \zeta(s, \alpha; a) \) approximating a given analytic function is infinite and even has a positive lower density.

In [8], the requirement on the transcendence for the parameter \( \alpha \) was replaced by a weaker hypothesis that the set

\[
L(\alpha) = \{ \log(m + \alpha) : m \in \mathbb{N}_0 \}
\]

is linearly independent over the field of rational numbers \( \mathbb{Q} \). For example, we can take \( \alpha = \frac{1}{e} \).

In Theorem 1, \( \tau \) in shifts \( \zeta(s + it, \alpha; a) \) can take arbitrary real values. This type of universality is called continuous. One can consider the so-called discrete universality when \( \tau \) takes values from the set \( \{ kh : k \in \mathbb{N}_0 \} \), where \( h > 0 \) is a fixed parameter. In this direction, the following result is known [7].

**Theorem 2.** Suppose that \( \alpha \) is a transcendental number, and that \( h > 0 \) is such that the number \( \exp \left( \frac{2\pi}{h} \right) \) is rational. Let \( K \in \mathcal{K} \) and \( f(s) \in H(K) \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} \left| \zeta(s + ikh, \alpha; a) - f(s) \right| < \varepsilon \right\} > 0.
\]

For example, the assertion of Theorem 2 is valid with \( \alpha = e^{-1} \) and \( h = 2\pi (\log 2)^{-1} \).
The aim of this note is to prove Theorem 2 under new hypotheses on the parameters $\alpha$ and $h$. Let
\[
L(\alpha, h, \pi) = \left\{ \log(m + \alpha) : m \in \mathbb{N}_0 \right\}^{\frac{\pi}{h}}.
\]

**Theorem 3.** Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{X}$ and $f(s) \in H(K)$. Then the assertion of Theorem 2 is true.

Clearly, the set $L(\alpha)$ is not linearly independent over $\mathbb{Q}$ with rational $\alpha$, and is independent with transcendental $\alpha$. J.W.S. Cassels proved [3] that at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linearly independent over $\mathbb{Q}$ with algebraic irrational $\alpha$. Thus, it can happen that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with algebraic irrational $\alpha$. Unfortunately, at the moment such an example of $\alpha$ is not known. Moreover, the linear independence of the set is closely related to a very important but complicated problem of the algebraic independence. It is not difficult to see that if the numbers $\alpha$ and $\exp \left\{ \frac{x}{h} \right\}$ are algebraically independent over $\mathbb{Q}$, then the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. By the Nesterenko theorem [10], the numbers $\pi$ and $e^\pi$ are algebraically independent over $\mathbb{Q}$. Therefore, we can take $\alpha = \frac{1}{n}$ and rational $h$ in Theorem 3.

Discrete universality theorems for zeta-functions are more convenient for practical applications. For example, in [2], a discrete universality theorem for the Riemann zeta-function has been applied for solving quantum mechanics problems.

For the proof of Theorem 3, we will apply a probabilistic approach based on a limit theorem in the space of analytic functions.

### 2. Limit theorem

Denote by $H(D)$ the space of analytic functions on $D$ equipped with the topology of uniform convergence on compacta. It is well known that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of $D$ such that $\bigcup_{l=1}^{\infty} K_l, K_l \subset K_{l+1}$ for $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(D)$, define
\[
\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.
\]

Then $\varrho$ is a metric on $H(D)$ inducing its topology of uniform convergence on compacta.

Let $\mathcal{B}(X)$ stand for the Borel $\sigma$-field of the space $X$. In this section, we consider the weak convergence of
\[
P_N(A) \overset{\text{def}}{=} \frac{1}{N+1} \# \{0 \leq k \leq N : \zeta(s + ik\alpha, \alpha) \in A \}, \quad A \in \mathcal{B}(H(D)),
\]
as $N \to \infty$. To state a limit theorem, we need some notation. Let $\gamma$ be the unit circle on the complex plane, and
\[
\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,
\]
where \( \gamma_m = \gamma \) for all \( m \in \mathbb{N}_0 \). With the product topology and pointwise multiplication, the torus \( \Omega \) is a compact topological Abelian group. Therefore, on \( (\Omega, B(\Omega)) \), the probability Haar measure \( m_H \) can be defined, and this leads to the probability space \( (\Omega, B(\Omega), m_H) \). Denote by \( \omega(m) \) the projection of \( \omega \in \Omega \) to the coordinate space \( \gamma_m \), \( m \in \mathbb{N}_0 \), and, on the probability space \( (\Omega, B(\Omega), m_H) \), define the \( H(D) \)-valued random element \( \zeta(s, \alpha, \omega; a) \) by the formula

\[
\zeta_n(s, \alpha, \omega; a) = \sum_{m=0}^{\infty} a_m \omega(m) \frac{1}{(m+\alpha)^n}.
\]

Let \( P_\xi \) be the distribution of \( \zeta_n(s, \alpha, \omega; a) \), i.e.,

\[
P_\xi(A) = m_H(\omega \in \Omega : (s, \alpha, \omega; a) \in A), \quad A \in B(H(D)).
\]

**Theorem 4.** Suppose that the set \( L(\alpha, h) \) is linearly independent over \( \mathbb{Q} \). Then \( P_N \) converges weakly to \( P_\xi \) as \( N \to \infty \). Moreover, the support of \( P_\xi \) is the whole of \( H(D) \).

For the proof of Theorem 4, the standard Fourier transform method is applied. Therefore, we will omit the details which are not connected to new hypotheses on the numbers \( \alpha \) and \( h \). Let, for \( A \in B(\Omega) \),

\[
Q_N(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : ((m+\alpha)^{-ikh} : m \in \mathbb{N}_0) \in A \right\}.
\]

The next lemma is a key in the proof of Theorem 4.

**Lemma 5.** Suppose that the set \( L(\alpha, h) \) is linearly independent over \( \mathbb{Q} \). Then \( Q_N \) converges weakly to \( m_H \) as \( N \to \infty \).

**Proof.** It is well known that a character \( \chi \) of the group \( \Omega \) is of the form

\[
\chi(\omega) = \prod_{m \in \mathbb{N}_0} \omega^{km}(m),
\]

where only a finite number of integers \( k_m \) are distinct from zero. Let \( \mathbf{k} = (k_0, k_1, \ldots) \). Then

\[
g_N(\mathbf{k}) = \int_{\Omega} \prod_{m \in \mathbb{N}_0} \omega^{km}(m) dQ_N,
\]

is the Fourier transform of \( Q_N \). Thus,

\[
g_N(\mathbf{k}) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{m \in \mathbb{N}_0} (m+\alpha)^{-ik_m h}
\]

\[
= \frac{1}{N+1} \sum_{k=0}^{N} \exp \left\{ -ikh \sum_{m \in \mathbb{N}_0} k_m \log(m+\alpha) \right\},
\]

(2.3)
where, as above, only a finite number of integers \( k_m \) are distinct from zero. The linear independence of the set \( L(\alpha, h, \pi) \) shows that

\[
\sum_{m \in \mathbb{N}_0} k_m \log(m + \alpha) = 0 \tag{2.4}
\]

if and only if \( k = 0 \). Moreover, we observe that, for \( k \neq 0 \),

\[
\exp \left\{ -ih \sum_{m \in \mathbb{N}_0} k_m \log(m + \alpha) \right\} \neq 1 \tag{2.5}
\]

because otherwise we obtain that, for some \( l \in \mathbb{Z} \),

\[
\sum_{m \in \mathbb{N}_0} k_m \log(m + \alpha) = \frac{2\pi l}{h},
\]

and this contradicts the linear independence of the set \( L(\alpha, h, \pi) \). Now, in view of (2.3)–(2.5), we find that

\[
g(k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{1 - \exp \left\{ -i(N+1)h \sum_{m \in \mathbb{N}_0} k_m \log(m + \alpha) \right\}}{(N+1)(1 - \exp \left\{ -ih \sum_{m \in \mathbb{N}_0} k_m \log(m + \alpha) \right\})} & \text{if } k \neq 0. \end{cases}
\]

Obviously, this implies

\[
\lim_{N \to \infty} g_N(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}
\]

This together with a continuity theorem on compact groups, Theorem 1.4.2 of [4], proves the lemma.

Let \( \theta > \frac{1}{2} \) be a fixed number, and let, for \( m \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \),

\[
v_n(m, \alpha) = \exp \left\{ - \left( \frac{m + \alpha}{n + \alpha} \right)^\theta \right\},
\]

\[
\zeta_n(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m v_n(m, \alpha)}{(m + \alpha)^s},
\]

and, for \( \omega \in \Omega \),

\[
\zeta_n(s, \alpha, \omega; a) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m, \alpha)}{(m + \alpha)^s}.
\]

Then the series for \( \zeta_n(s, \alpha; a) \) and \( \zeta_n(s, \alpha, \omega; a) \) are absolutely convergent for \( \sigma > \frac{1}{2} \) [5]. For \( A \in B(H(D)) \), define

\[
P_{N,n}(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta_n(s + ikh, \alpha; a) \in A \}
\]

and

\[
\hat{P}_{N,n}(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta_n(s + ikh, \alpha, \omega; a) \in A \}.
\]
Lemma 6. Suppose that the set \( L(\alpha, h, \pi) \) is linearly independent over \( \mathbb{Q} \). Then \( P_{N,n} \) and \( \hat{P}_{N,n} \) both converge weakly to the same probability measure \( P_n \) on \( (H(D), \mathcal{B}(H(D))) \) as \( N \to \infty \).

Proof. The lemma is a result of Lemma 5, Theorem 5.1 of [1], and of the invariance of the Haar measure \( m_H \).

We remind that the metric \( \rho \) is defined by (2.1).

Lemma 7. The equality
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(\zeta(s + ikh, \alpha; a), \zeta_n(s + ikh, \alpha; a)) = 0
\]
holds.

Proof. The lemma is Theorem 4.1 from [7].

Lemma 8. Suppose that the set \( L(\alpha, h, \pi) \) is linearly independent over \( \mathbb{Q} \). Then, for almost all \( \omega \in \Omega \),
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(\zeta(s + ikh, \alpha, \omega; a), \zeta_n(s + ikh, \alpha, \omega; a)) = 0.
\]

Proof. In [7], the equality of the lemma was proved for transcendental \( \alpha \). However, the transcendence of \( \alpha \) is used only for linear independence of the set \( L(\alpha) \). Since the linear independence of \( L(\alpha, h, \pi) \) implies that of \( L(\alpha) \), the proof of the lemma runs in the same way as of Theorem 4.4 from [7].

Lemmas 7 and 8 allow to pass from the function \( \zeta_n(s, \alpha; a) \) to \( \zeta(s, \alpha; a) \), and from \( \zeta_n(s, \alpha, \omega; a) \) to \( \zeta(s, \alpha, \omega; a) \). On \( (H(D), \mathcal{B}(H(D))) \), define
\[
\hat{P}_N(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta(s + ikh, \alpha, \omega; a) \in A \}.
\]

Lemma 9. Suppose that the set \( L(\alpha, h, \pi) \) is linearly independent over \( \mathbb{Q} \). Then \( P_N \) and \( \hat{P}_N \) both converge weakly to the same probability measure \( P \) on \( (H(D), \mathcal{B}(H(D))) \) as \( N \to \infty \).

Proof. We apply Lemmas 6–8 and use analogous arguments to those of the proof of Theorem 6.1 from [7].

Proof of Theorem 4. In virtue of Lemma 9, it remains to show that the limit measure \( P \) in Lemma 9 coincides with \( P_C \). For this, we once more use the linear independence of the set \( L(\alpha, h, \pi) \). Let \( a_{\alpha, \omega} = \{ (m + \alpha)^{-ik} : m \in \mathbb{N}_0 \} \). Clearly, \( a_{\alpha, \omega} \) is an element of the torus \( \Omega \). Let, for \( \omega \in \Omega, \phi_{\alpha, \omega}(\omega) = a_{\alpha, \omega} \omega \). Then \( \phi_{\alpha, \omega} \) is a measurable measure preserving transformation of \( \Omega \) defined on the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \). Using the linear independence of the set \( L(\alpha, h, \pi) \), we will
prove that the transformation \( \phi_{\alpha,h} \) is ergodic. In view of (2.1), we have that, for a non-trivial character \( \chi \) of \( \Omega \),

\[
\chi(a_{\alpha,h}) = \prod_{m \in \mathbb{N}_0} (m + \alpha)^{-ik_m} = \exp \left\{ -i\hbar \sum_{m \in \mathbb{N}_0} k_m \log(m + \alpha) \right\},
\]

where only a finite number of integers \( k_m \) are distinct from zero. Since the set \( L(\alpha, h, \pi) \) is linearly independent, inequality (2.5) is true, therefore,

\[
\chi(a_{\alpha,h}) \neq 1. \tag{2.6}
\]

Denote by \( I_A \) the indicator function of the set \( A \). Then, for an invariant set \( A \) of the transformation \( \phi_{\alpha,h} \), we have that, for almost all \( \omega \in \Omega \),

\[
I_A(a_{\alpha,h}\omega) = I_A(\omega).
\]

This, the invariance of the Haar measure and the multiplicativity of \( \chi \) show that

\[
\hat{I}_A(\chi) = \int_{\Omega} \chi(\omega) I_A(\omega) m_H(d\omega) = \chi(a_{\alpha,h}) \hat{I}_A(\chi),
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \). Hence, by (2.6),

\[
\hat{I}_A(\chi) = 0. \tag{2.7}
\]

Now let \( \hat{I}_A(\chi_0) = u \), where \( \chi_0 \) is the trivial character of \( \Omega \). Then (2.7) and the orthogonality of characters, for any character \( \chi \) of \( \Omega \), lead to

\[
\hat{I}_A(\chi) = u \int_{\Omega} \chi(\omega) m_H(d\omega) = u \hat{1}(\chi) = \hat{u}(\chi).
\]

From this, it follows that \( I_A(\omega) = u \) for almost all \( \omega \in \Omega \). Since \( u = 1 \) or \( 0 \), we obtain that \( m_H(A) = 1 \) or \( m_H(A) = 0 \), i.e., the transformation \( \phi_{\alpha,h} \) is ergodic.

Now let \( A \) be a continuity set the measure \( P \) in Lemma 9. Then we have that

\[
\lim_{N \to \infty} \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta(s + ikh, \alpha, \omega; a) \in A \} = P(A). \tag{2.8}
\]

On \( (\Omega, B(\Omega), m_H) \), define a random variable \( \xi \) by

\[
\xi(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(s, \alpha, \omega; a) \in A, \\ 0 & \text{otherwise}. \end{cases}
\]

Then the expectation

\[
\mathbb{E}\xi = \int_{\Omega} \xi m_H(d\omega) = m_H \{ \omega \in \Omega : \zeta(s, \alpha, \omega; a) \in A \} = P_\zeta(A). \tag{2.9}
\]
The ergodicity of $\phi_{a,h}$ and the Birkhoff-Khinchine theorem imply that, for almost all $\omega \in \Omega$,

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \xi(\phi_{a,h}^k(\omega)) = \mathbb{E}\xi.$$  \hspace{1cm} (2.10)

However, by the definitions of $\xi$ and $\phi_{a,h}$,

$$\frac{1}{N+1} \sum_{k=0}^{N} \xi(\phi_{a,h}^k(\omega)) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta(s + ikh, \alpha, \omega; a) \in A \}.$$ 

Thus, taking into account (2.9) and (2.10), we find that, for almost all $\omega \in \Omega$,

$$\lim_{N \to \infty} \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta(s + ikh, \alpha, \omega; a) \in A \} = P_\zeta(A).$$

This and (2.8) give the equality $P(A) = P_\zeta(A)$ for every continuity set $A$ of $P$. Hence, $P = P_\zeta$.

The measure $P_\zeta$ is independent on the parameter $h$. Therefore, we can use Lemma 5 of [5] with obvious modifications which gives that the support of $P_\zeta$ is the whole of $H(D)$.

3. Proof of Theorem 3

Theorem 3 is a direct corollary of Theorem 4 and the Mergelyan theorem on the approximation of analytic functions by polynomials [9], see also [11]. We note that the requirement $K \in \mathcal{K}$ comes from the Mergelyan theorem.

Proof of Theorem 3. We apply standard arguments. In virtue of the mentioned above Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} \left| f(s) - p(s) \right| < \frac{\varepsilon}{2}.$$ \hspace{1cm} (3.11)

Define the set

$$G = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - p(s) \right| < \frac{\varepsilon}{2} \right\}.$$ \hspace{1cm} (3.12)

By Theorem 4, $G$ is an open neighbourhood of $p(s)$ which belongs to the support of $P_\zeta$. Therefore, $P_\zeta(G) > 0$. Hence, again by Theorem 4,

$$\lim_{N \to \infty} \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta(s + ikh, \alpha; a) \in G \} > 0.$$ 

This and (3.12) show that

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \{ 0 \leq k \leq N : \sup_{s \in K} \left| \zeta(s + ikh, \alpha; a) - p(s) \right| < \frac{\varepsilon}{2} \} > 0.$$ 

Combining this with (3.11) proves the theorem.
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References


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