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ON THE DISCRETE UNIVERSALITY OF THE PERIODIC HURWITZ ZETA-FUNCTION

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Abstract. In this paper, a theorem on the approximation of analytic functions by discrete shifts of the periodic Hurwitz zeta-function is obtained. For this, a linear independence condition connecting the parameter of the zeta-function and the step of the arithmetic progression is used.

Key words and phrases: discrete universality, periodic Hurwitz zeta function, zeta-function, universality.

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1. Introduction

Let $s = \sigma + it$ be a complex variable, α be a fixed parameter, $0 < \alpha \leqslant 1$, and let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{k-1} a_l \zeta\Big(s, \frac{l+\alpha}{q}\Big), \tag{1.1}$$

where $\zeta(s, \alpha) = \zeta(s, \alpha; \mathfrak{a} \equiv \{1\}) = \zeta(s, \alpha; 1)$ is the classical Hurwitz zeta-function. It is well known that the function $\zeta(s, \alpha)$ is regular on the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Therefore, equality

(1.1) gives an analytic continuation for the function $\zeta(s,\alpha;\mathfrak{a})$ to the whole complex plane, except for a possible pole at the point $s=1$ with residue

$$a \overset{\text{def}}{=} \frac{1}{q}\sum_{l=0}^{q-1} a_l.$$

If $a=0$, then the function $\zeta(s,\alpha;\mathfrak{a})$ is entire.

The function $\zeta(s,\alpha;\mathfrak{a})$, as the majority of other zeta and L-functions, is universal in the Voronin sense. Roughly speaking, this means that any analytic function uniformly on compact subsets of some region can be approximated by shifts $\zeta(s+i\tau,\alpha;\mathfrak{a}), \tau\in\mathbb{R}$. For the precise statement, we need some notation. Let $D=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H(K)$, $K\in\mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Then, in [6], the following universality theorem has been obtained.

Theorem 1. *Suppose that α is a transcendental number. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}\Big|\zeta(s+i\tau,\alpha;\mathfrak{a})-f(s)\Big|<\varepsilon\Big\}>0.$$

Here meas$A$ denotes the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. The inequality of the theorem shows that the set of shifts $\zeta(s,\alpha;\mathfrak{a})$ approximating a given analytic function is infinite and even has a positive lower density.

In [8], the requirement on the transcendence for the parameter α was replaced by a weaker hypothesis that the set

$$L(\alpha)=\{\log(m+\alpha):m\in\mathbb{N}_0\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$. For example, we can take $\alpha=\frac{1}{\pi}$.

In Theorem 1, τ in shifts $\zeta(s+i\tau,\alpha;\mathfrak{a})$ can take arbitrary real values. This type of universality is called continuous. One can consider the so-called discrete universality when τ takes values from the set $\{kh:k\in\mathbb{N}_0\}$, where $h>0$ is a fixed parameter. In this direction, the following result is known [7].

Theorem 2. *Suppose that α is a transcendental number, and that $h>0$ is such that the number $\exp\{\frac{2\pi}{h}\}$ is rational. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\leqslant k\leqslant N:\sup_{s\in K}\Big|\zeta(s+ikh,\alpha;\mathfrak{a})-f(s)\Big|<\varepsilon\Big\}>0.$$

For example, the assertion of Theorem 2 is valid with $\alpha=\mathrm{e}^{-1}$ and $h=2\pi(\log 2)^{-1}$.

The aim of this note is to prove Theorem 2 under new hypotheses on the parameters α and h. Let

$$L(\alpha, h, \pi) = \Big\{ \Big(\log(m+\alpha) : m \in \mathbb{N}_0 \Big), \frac{\pi}{h} \Big\}.$$

THEOREM 3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of* Theorem 2 *is true.*

Clearly, the set $L(\alpha)$ is not linearly independent over $\mathbb{Q}$ with rational α, and is independent with transcendental α. J.W.S. Cassels proved [3] that at least 51 percent of elements of the set $L(\alpha)$ in the sense of density are linearly independent over $\mathbb{Q}$ with algebraic irrational α. Thus, it can happen that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with algebraic irrational α. Unfortunately, at the moment such an example of α is not known. Moreover, the linear independence of the set is closely related to a very important but complicated problem of the algebraic independence. It is not difficult to see that if the numbers α and $\exp\left\{\frac{\pi}{h}\right\}$ are algebraically independent over $\mathbb{Q}$, then the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. By the Nesterenko theorem [10], the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. Therefore, we can take $\alpha = \frac{1}{\pi}$ and rational h in Theorem 3.

Discrete universality theorems for zeta-functions are more convenient for practical applications. For example, in [2], a discrete universality theorem for the Riemann zeta-function has been applied for solving quantum mechanics problems.

For the proof of Theorem 3, we will apply a probabilistic approach based on a limit theorem in the space of analytic functions.

2. Limit theorem

Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. It is well known that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of D such that $\bigcup_{l=1}^{\infty} K_l, K_l \subset K_{l+1}$ for $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(D)$, define

$$\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} \big|g_1(s) - g_2(s)\big|}{1 + \sup_{s \in K_l} \big|g_1(s) - g_2(s)\big|}. \tag{2.1}$$

Then ϱ is a metric on $H(D)$ inducing its topology of uniform convergence on compacta.

Let $\mathcal{B}(X)$ stand for the Borel σ-field of the space X. In this section, we consider the weak convergence of

$$P_N(A) \overset{\text{def}}{=} \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \zeta(s+ikh, \alpha; \mathfrak{a}) \in A\big\}, \quad A \in \mathcal{B}(H(D)),$$

as $N \to \infty$. To state a limit theorem, we need some notation. Let γ be the unit circle on the complex plane, and

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}_0$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \alpha, \omega; \mathfrak{a})$ by the formula

$$\zeta_n(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

Let P_ζ be the distribution of $\zeta_n(s, \alpha, \omega; \mathfrak{a})$, i.e.,

$$P_\zeta(A) = m_H\big(\omega \in \Omega : \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\big), \quad A \in \mathcal{B}(H(D)).$$

Theorem 4. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$.*

For the proof of Theorem 4, the standard Fourier transform method is applied. Therefore, we will omit the details which are not connected to new hypotheses on the numbers α and h. Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_N(A) = \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \Big((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\Big) \in A\Big\}.$$

The next lemma is a key in the proof of Theorem 4.

Lemma 5. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to m_H as $N \to \infty$.*

Proof. It is well known that a character χ of the group Ω is of the form

$$\chi(\omega) = \prod_{m \in \mathbb{N}_0} \omega^{k_m}(m), \tag{2.2}$$

where only a finite number of integers k_m are distinct from zero. Let $\underline{k} = (k_0, k_1, ...)$. Then

$$g_N(\underline{k}) = \int_\Omega \prod_{m \in \mathbb{N}_0} \omega^{k_m}(m) \mathrm{d}Q_N,$$

is the Fourier transform of Q_N. Thus,

$$\begin{aligned} g_N(\underline{k}) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{m \in \mathbb{N}_0} (m+\alpha)^{-ikk_m h} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ikh \sum_{m \in \mathbb{N}_0} k_m \log(m+\alpha)\Big\}, \end{aligned} \tag{2.3}$$

where, as above, only a finite number of integers k_m are distinct from zero. The linear independence of the set $L(\alpha, h, \pi)$ shows that

$$\sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha) = 0 \tag{2.4}$$

if and only if $\underline{k} = \underline{0}$. Moreover, we observe that, for $\underline{k} \neq \underline{0}$,

$$\exp\Big\{ -ih \sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\Big\} \neq 1 \tag{2.5}$$

because otherwise we obtain that, for some $l \in \mathbb{Z}$,

$$\sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha) = \frac{2\pi l}{h},$$

and this contradicts the linear independence of the set $L(\alpha, h, \pi)$. Now, in view of (2.3)–(2.5), we find that

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \ \underline{k} = \underline{0}, \\ \frac{1-\exp\left\{-i(N+1)h\sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\right\}}{(N+1)\left(1-\exp\left\{-ih\sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\right\}\right)} & \text{if } \ \underline{k} \neq \underline{0}. \end{cases}$$

Obviously, this implies

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \ \underline{k} = \underline{0}, \\ 0 & \text{if } \ \underline{k} \neq \underline{0}. \end{cases}$$

This together with a continuity theorem on compact groups, Theorem 1.4.2 of [4], proves the lemma.

Let $\theta > \frac{1}{2}$ be a fixed number, and let, for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$\begin{aligned} v_n(m,\alpha) &= \exp\Big\{ -\Big(\frac{m+\alpha}{n+\alpha}\Big)^{\theta}\Big\}, \\ \zeta_n(s,\alpha;\mathfrak{a}) &= \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^s}, \end{aligned}$$

and, for $\omega \in \Omega$,

$$\zeta_n(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then the series for $\zeta_n(s,\alpha;\mathfrak{a})$ and $\zeta_n(s,\alpha,\omega;\mathfrak{a})$ are absolutely convergent for $\sigma > \frac{1}{2}$ [5]. For $A \in \mathcal{B}(H(D))$, define

$$P_{N,n}(A) = \frac{1}{N+1}\#\big\{0 \leqslant k \leqslant N : \zeta_n(s+ikh,\alpha;\mathfrak{a}) \in A\big\}$$

and

$$\hat{P}_{N,n}(A) = \frac{1}{N+1}\#\big\{0 \leqslant k \leqslant N : \zeta_n(s+ikh,\alpha,\omega;\mathfrak{a}) \in A\big\}.$$

Lemma 6. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n}$ and $\hat{P}_{N,n}$ both converge weakly to the same probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$.*

Proof. The lemma is a result of Lemma 5, Theorem 5.1 of [1], and of the invariance of the Haar measure m_H.

We remind that the metric ρ is defined by (2.1).

Lemma 7. *The equality*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varrho\big(\zeta(s+ikh, \alpha; \mathfrak{a}), \zeta_n(s+ikh, \alpha; \mathfrak{a})\big) = 0$$

holds.

Proof. The lemma is Theorem 4.1 from [7].

Lemma 8. *Suppose that the set $L(\alpha, h, \pi)$is linearly independent over $\mathbb{Q}$. Then, for almost all $\omega \in \Omega$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varrho\Big(\zeta(s+ikh, \alpha, \omega; \mathfrak{a}), \zeta_n(s+ikh, \alpha, \omega; \mathfrak{a})\Big) = 0.$$

Proof. In [7], the equality of the lemma was proved for transcendental α. However, the transcendence of α is used only for linear independence of the set $L(\alpha)$. Since the linear independence of $L(\alpha, h, \pi)$ implies that of $L(\alpha)$, the proof of the lemma runs in the same way as of Theorem 4.4 from [7].

Lemmas 7 and 8 allow to pass from the function $\zeta_n(s, \alpha; \mathfrak{a})$ to $\zeta(s, \alpha; \mathfrak{a})$, and from $\zeta_n(s, \alpha, \omega; \mathfrak{a})$ to $\zeta(s, \alpha, \omega; \mathfrak{a})$. On $(H(D), \mathcal{B}(H(D)))$, define

$$\hat{P}_N(A) = \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \zeta(s+ikh, \alpha, \omega; \mathfrak{a}) \in A\big\}.$$

Lemma 9. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then P_N and $\hat{P}_N$ both converge weakly to the same probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$.*

Proof. We apply Lemmas 6–8 and use analogical arguments to those of the proof of Theorem 6.1 from [7].

Proof of Theorem 4. In virtue of Lemma 9, it remains to show that the limit measure P in Lemma 9 coincides with P_ζ. For this, we once more use the linear independence of the set $L(\alpha, h, \pi)$. Let $a_{\alpha,h} = \big\{(m+\alpha)^{-ih} : m \in \mathbb{N}_0\big\}$. Clearly, $a_{\alpha,h}$ is an element of the torus Ω. Let, for $\omega \in \Omega, \phi_{\alpha,h}(\omega) = a_{\alpha,h}\omega$. Then $\phi_{\alpha,h}$ is a measurable measure preserving transformation of Ω defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Using the linear independence of the set $L(\alpha, h, \pi)$, we will prove that the transformation $\phi_{\alpha,h}$ is ergodic. In view of (2.1), we have that, for a non-trivial character χ of Ω,

$$\chi(a_{\alpha,h}) = \prod_{m\in\mathbb{N}_0} (m+\alpha)^{-ik_m h} = \exp\Big\{-ih \sum_{m\in\mathbb{N}_0} k_m \log(m+\alpha)\Big\},$$

where only a finite number of integers k_m are distinct from zero. Since the set $L(\alpha, h, \pi)$ is linearly independent, inequality (2.5) is true, therefore,

$$\chi(a_{\alpha,h}) \neq 1. \tag{2.6}$$

Denote by I_A the indicator function of the set A. Then, for an invariant set A of the transformation $\phi_{\alpha,h}$, we have that, for almost all $\omega \in \Omega$,

$$I_A(a_{\alpha,h}\omega) = I_A(\omega).$$

This, the invariance of the Haar measure and the multiplicavity of χ show that

$$\hat{I}_A(\chi) = \int_\Omega \chi(\omega) I_A(\omega) m_H(\mathrm{d}\omega) = \chi(a_{\alpha,h})\hat{I}_A(\chi),$$

where $\hat{f}$ denotes the Fourier transform of f. Hence, by (2.6),

$$\hat{I}_A(\chi) = 0. \tag{2.7}$$

Now let $\hat{I}_A(\chi_0) = u$, where χ_0 is the trivial character of Ω. Then (2.7) and the orthogonality of characters, for any character χ of Ω, lead to

$$\hat{I}_A(\chi) = u \int_\Omega \chi(\omega) m_H(\mathrm{d}\omega) = u\hat{1}(\chi) = \hat{u}(\chi).$$

From this, it follows that $I_A(\omega) = u$ for almost all $\omega \in \Omega$. Since $u = 1$ or 0, we obtain that $m_H(A) = 1$ or $m_H(A) = 0$, i.e., the transformation $\phi_{\alpha,h}$ is ergodic.

Now let A be a continuity set the measure P in Lemma 9. Then we have that

$$\lim_{N\to\infty} \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \zeta(s+ikh, \alpha, \omega; \mathfrak{a}) \in A\big\} = P(A). \tag{2.8}$$

On $(\Omega, \mathcal{B}(\Omega), m_H)$, define a random variable ξ by

$$\xi(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(s, \alpha, \omega; \mathfrak{a}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then the expectation

$$\mathbb{E}\xi = \int_\Omega \xi m_H(\mathrm{d}\omega) = m_H\big\{\omega \in \Omega : \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\big\} = P_\zeta(A). \tag{2.9}$$

The ergodicity of $\phi_{\alpha,h}$ and the Birkhoff-Khinchine theorem imply that, for almost all $\omega \in \Omega$,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \xi\bigl(\phi^k_{\alpha,h}(\omega)\bigr) = \mathbb{E}\xi. \tag{2.10}$$

However, by the definitions of ξ and $\phi_{\alpha,h}$,

$$\frac{1}{N+1} \sum_{k=0}^{N} \xi\bigl(\phi^k_{\alpha,h}(\omega)\bigr) = \frac{1}{N+1} \#\bigl\{0 \leqslant k \leqslant N : \zeta(s+ikh,\alpha,\omega;\mathfrak{a}) \in A\bigr\}.$$

Thus, taking into account (2.9) and (2.10), we find that, for almost all $\omega \in \Omega$,

$$\lim_{N\to\infty} \frac{1}{N+1} \#\bigl\{0 \leqslant k \leqslant N : \zeta(s+ikh,\alpha,\omega;\mathfrak{a}) \in A\bigr\} = P_\zeta(A).$$

This and (2.8) give the equality $P(A) = P_\zeta(A)$ for every continuity set A of P. Hence, $P = P_\zeta$.

The measure P_ζ is independent on the parameter h. Therefore, we can use Lemma 5 of [5] with obvious modifications which gives that the support of P_ζ is the whole of $H(D)$.

3. Proof of Theorem 3

Theorem 3 is a direct corollary of Theorem 4 and the Mergelyan theorem on the approximation of analytic functions by polynomials [9], see also [11]. We note that the requirement $K \in \mathcal{K}$ comes from the Mergelyan theorem.

Proof of Theorem 3. We apply standard arguments. In virtue of the mentioned above Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} \bigl|f(s) - p(s)\bigr| < \frac{\varepsilon}{2}. \tag{3.11}$$

Define the set

$$G = \Bigl\{g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\Bigr\}. \tag{3.12}$$

By Theorem 4, G is an open neighbourhood of $p(s)$ which belongs to the support of P_ζ. Therefore, $P_\zeta(G) > 0$. Hence, again by Theorem 4,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\bigl\{0 \leqslant k \leqslant N : \zeta(s+ikh,\alpha,\omega;\mathfrak{a}) \in G\bigr\} > 0.$$

This and (3.12) show that

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Bigl\{0 \leqslant k \leqslant N : \sup_{s\in K} \bigl|\zeta(s+ikh,\alpha;\mathfrak{a}) - p(s)\bigr| < \frac{\varepsilon}{2}\Bigr\} > 0.$$

Combining this with (3.11) proves the theorem.

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