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DIOPHANTINE APPROXIMATION OF COMPLEX NUMBERS

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Abstract. We give an upper bound for the approximation quality of diophantine approximations by quotients of lattice points in the complex plane. This upper bound depends on a certain lattice invariant. In particular, we generalize a method based on geometrical ideas of Hermann Minkowski and improved by Hilde Gintner. Subsequently we examine the spectrum arising from the infimum of the constants occuring in the upper bound and give a proof of the existence of infinitely many solutions of generalized Pell equations in the complex case.

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1. Introduction and statement of the main results

One can say that the foundation for a diophantine treatment of complex numbers was laid by the brothers Adolf and Julius Hurwitz. In 1888, A. Hurwitz published a first complex continued fraction expansion [7] using Gaussian integers as set of possible partial quotients. Some years later, in 1895, J. Hurwitz developed a similar algorithm with set of partial quotients $(1+i)\mathbb{Z}[i]$ [9]. In the real case, the quality of diophantine approximation is ruled by A. Hurwitz's famous result [8], that for every irrational number z there are infinitely many rationals $\frac{p}{q} \in \mathbb{Q}$ such that

$$\left| z - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}, \tag{1}$$

and if the constant $\frac{1}{\sqrt{5}}$ is replaced by any smaller quantity, there are only finitely many $\frac{p}{q} \in \mathbb{Q}$ approximating $\frac{1}{2}(\sqrt{5}+1)$ and all equivalent numbers z with the

corresponding quality. Referring to this best possible result, there were various attempts in the complex case. A first approach due to Hermann Minkowski [14] (cf. [10]), using geometry of numbers, lead to the upper bound $\frac{\sqrt{6}}{\pi|q|^2}$. In 1925, Lester R. Ford finally found the sharp complex analogue [4]:
If $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ is any complex irrational number, then there exist infinitely many $p, q \in \mathbb{Z}[i]$ such that[§]

$$\left|z - \frac{p}{q}\right| < \frac{1}{\sqrt{3}|q|^2}.$$

This upper bound is as well best possible. Fifty years later, Richard B. Lakein [11] gave a constructive proof using continued fractions in tradition of A. Hurwitz. In 1933, Oskar Perron [17] extended Ford's examinations to more general imaginary quadratic number rings $\mathbb{Z}[i\sqrt{D}]$. Especially for $D = 2$ he published another best possible result:
If $z \in \mathbb{C}\setminus\mathbb{Q}(i\sqrt{2})$ is any complex irrational number, then there exist infinitely many $p, q \in \mathbb{Z}[i\sqrt{2}]$ such that

$$\left|z - \frac{p}{q}\right| < \frac{1}{\sqrt{2}|q|^2};$$

if the constant $\frac{1}{\sqrt{2}}$ is replaced by any smaller quantity, in general there are only infinitely many $\frac{p}{q}$, where $p, q \in \mathbb{Z}[i\sqrt{2}]$. Interestingly, Perron uses, among other tools, again geometrical methods of Minkowski, in particular his lattice point theorem. In 1936, Hilde Gintner [5], generalising results [10] of her thesis advisor Nikolaus Hofreiter, continued this approach, applying additionally Minkowski's theorem on linear forms [14]. She proved that, in any ring $O_{\sqrt{-D}}$ of integers associated with an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$, there exist infinitely many integers p, q such that, for any arbitrary complex number $z \in \mathbb{C}$, one has

$$\left|z - \frac{p}{q}\right| < \frac{\sqrt{6D}}{\pi|q|^2} \quad \text{if} \quad D \not\equiv 3 \pmod 4,$$

respectively

$$\left|z - \frac{p}{q}\right| < \frac{\sqrt{6D}}{2\pi|q|^2} \quad \text{if} \quad D \equiv 3 \pmod 4.$$

Notice that only in the few examples of euclidean rings (for $D = 1, 2, 3, 7, 11$) of integers of imaginary quadratic fields an analogue of A. Hurwitz's continued fraction can be realized in a straightforward manner (see [6]). Although Gintner's result provides better, probably not best possible, upper bounds for arbitrary imaginary quadratic number fields than known before, all considerations are still restricted to certain imaginary quadratic number rings. We follow her geometrical point of view and apply her approach to an arbitrary lattice:

[§] In 1930 Oskar Perron published the same result, since he had forgotten that he even refereed Ford's work. Embedded in a 'historical correction', Perron expressly apologised for his failure in a second version of 1931 [16].

Theorem 1. *Let λ be a lattice in $\mathbb{C}$ of full rank. For any $z \in \mathbb{C} \setminus \mathbb{Q}(\lambda)$, there exist infinitely many lattice points $p, q \in \lambda$ such that*

$$\left| z - \frac{p}{q} \right| < \frac{\sqrt{6} \cdot \Delta(\lambda)}{\pi |q|^2},$$

where $\Delta(\lambda)$ denotes the determinant of the lattice λ.

In view of the above given sharp approximation bounds and some more results by, for example, Schmidt [19], Cassels [1] and Poitou [18], we know first points of the Lagrange spectrum for certain $D \in \mathbb{N}$, where $\mathbb{Z}[i\sqrt{D}]$ forms a factorial ring.[¶] Subsequently, we consider the arising spectrum classifying the upper bound constant $c(\lambda) := c \in \mathbb{R}$ in dependence of a general lattice. Considering points

$$\nu(z) := \inf \left\{ c > 0 : \exists^\infty \ p, q \in \lambda : \left| z - \frac{p}{q} \right| \leqslant \frac{c}{|q|^2} \right\}$$

for arbitrary but fixed $z \in \mathbb{C}$, we define the spectrum

$$\mathbb{L}(\tau) := \{ \nu(z) : z \in \mathbb{C} \},$$

assuming that $\lambda := \mathbb{Z} + \tau\mathbb{Z}$ with $\operatorname{Im} \tau > 0$. If $z \in \mathbb{Q}(\lambda)$, i.e., $z = \frac{P}{Q}$ with $P, Q \in \lambda$, we may take $p = kP$, $q = kQ$, where $k \in \mathbb{N}$, and thus $\nu(z) = 0$. In other words, there is no influence from $\mathbb{Q}(\lambda)$ on the supremum of $\mathbb{L}(\tau)$.

Corollary 2. *The function*

$$\mu(\tau) := \mu(\lambda) := \sup \mathbb{L}(\tau) = \inf \left\{ c > 0 | \ \forall z \in \mathbb{C} \ \exists^\infty p, q \in \lambda : \left| z - \frac{p}{q} \right| < \frac{c}{|q|^2} \right\}$$

(1) *is automorphic,*

(2) *continuous,*

(3) *non-constant and always $\mu(\tau) \geqslant \frac{1}{\sqrt{5}}$.*

It should be noted here that we call a function *automorphic* when it is invariant under the action of $SL_2(\mathbb{Z})$.

Furthermore, we apply Theorem 1 to prove a complex analogue of a classical arithmetical result:

Corollary 3. *Let $d \in \mathbb{Z}[i\sqrt{D}]$ with $D \in \mathbb{N}$ and $\sqrt{d} \notin \mathbb{Q}(i\sqrt{D})$. Then the Pell equation*

$$X^2 - dY^2 = 1$$

has infinitely many non-trivial solutions $x, y \in \mathbb{Z}[i\sqrt{D}]$.

[¶] A wide-ranging collection of results regarding Lagrange and Markov spectra is given in [13].

2. Geometry of numbers

Hilde Gintner [5] mainly refers to two theorems of Minkowksi [14] coming from his famous theory of 'Geometry of Numbers'. Those theorems can also be found in [2], where J.W.S. Cassels so to say translated them to a more modern language of lattices which he considered as “the most important concept in geometry of numbers” ([2], p. 9). Here we want to stay close to the original version of Minkowski.

THEOREM 4 (Gitterpunktsatz, Minkowski, 1889/91). *Let* Λ *be a full lattice in* $\mathbb{R}^n$ *and* $C \in \mathbb{R}^n$ *be a convex and symmetric set with bounded volume* $\mathrm{vol}(C) \geqslant 2^n \det(\Lambda)$. *Then* C *contains at least one lattice point of* Λ *different from the origin.*

THEOREM 5 (Linearformensatz, Minkowski, 1896). *Let* $Y_1, Y_2, \ldots, Y_{2s-1}, Y_{2s}$ *be* s *pairs of linear forms with complex conjugated coefficients* z_{ik} *and another* $r = n-2s$ *linear forms* $Y_{2s+1}, \ldots Y_n$ *with real coefficients. Then there exist integers* $x_1, \ldots, x_n$, *not all zero, such that*

$$|Y_j(x_1, \ldots, x_n)| \leqslant \left(\frac{2}{\pi}\right)^{s/n} |\det(z_{ik})|^{1/n}$$

for $j = 1, \ldots, n$.

Gintner considered integers of an arbitrary imaginary quadratic number ring $O_{\sqrt{-D}}$, $D \in \mathbb{N}$, as lattice points in the complex plane. Using Theorem 5, she showed that there exist infinitely many numbers $p, q \in O_{\sqrt{-D}}$ such that $\frac{p}{q}$ approximates a given arbitrary complex number $z \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-D})$ with a certain accuracy. This upper bound is subsequently improved by Theorem 4.

The restriction to integer rings in former studies providing a unique factorisation of the approximants $\frac{p}{q}$ can generally be overcome. Since for algebraic u_j their norm satisfies $|N(u_j)| \in \mathbb{N}_0$, there can only be finitely many factorisations $q = u_1 \cdot \ldots \cdot u_r$ for $q \in O_{\sqrt{-D}}$. This eventual finite ambiguity of $|q|$ does not change the fact that there exist infinitely many approximations. Of course, this remark also applies to quadratic number fields with slightly different rings of integers.

3. Application to a general lattice

Considering a lattice λ generated by two arbitrary $\mathbb{R}$-linearly independent vectors $\omega_1 = a + ib$, $\omega_2 = c + id$, where $a, b, c, d \in \mathbb{R}$, we obtain

$$\lambda = (a + ib)\mathbb{Z} + (c + id)\mathbb{Z} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$$

with lattice determinant

$$\Delta(\lambda) = \left|\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc| > 0.$$

We define pairs of linear forms

$$Y_1 = \frac{1}{\sqrt{2}}((\omega_1 X_1 + \omega_2 X_2) - (\beta + i\gamma)(\omega_1 X_3 + \omega_2 X_4)), \quad Y_2 = \overline{Y_1}$$

and

$$Y_3 = \frac{1}{\sqrt{2}t^2}(\omega_1 X_3 + \omega_2 X_4), \quad Y_4 = \overline{Y_3},$$

where t is a positive real parameter. The determinant of the coefficient matrix is computed as follows

$$\begin{aligned} \det(z_{ij}) &= \frac{1}{4t^4}\det\begin{pmatrix} \omega_1 & \omega_2 & \star & \star \\ \overline{\omega_1} & \overline{\omega_2} & \star & \star \\ 0 & 0 & \omega_1 & \omega_2 \\ 0 & 0 & \overline{\omega_1} & \overline{\omega_2} \end{pmatrix} = \frac{1}{4t^4}\det\begin{pmatrix} \omega_1 & \omega_2 \\ \overline{\omega_1} & \overline{\omega_2} \end{pmatrix}^2 \\ &= \frac{1}{4t^4}\left((a+ib)(c-id) - (c+id)(a-ib)\right)^2 = \frac{1}{4t^4}(2i(bc-ad))^2 \\ &= -\frac{1}{t^4}(ad-bc)^2. \end{aligned}$$

Application of Theorem 5 provides the simultaneous estimate

$$|Y_j(x_1, \ldots, x_4)| \leqslant \sqrt{\frac{2}{\pi}}\sqrt[4]{\left|\frac{1}{t^4}(ad-bc)^2\right|} = \frac{\sqrt{2}\sqrt{|ad-bc|}}{\sqrt{\pi}t}$$

with a certain lattice point $(x_1, ..., x_4) \in \mathbb{Z}^4 \setminus \{0\}$, and therewith for the product of two linear forms

$$|Y_1| \cdot |Y_3| = \frac{1}{2t^2}|(\omega_1 x_1 + \omega_2 x_2) - (\beta + i\gamma)(\omega_1 x_3 + \omega_2 x_4)||\omega_1 x_3 + \omega_2 x_4| \leqslant \frac{2|ad-bc|}{\pi t^2}.$$

Setting $p = (\omega_1 x_1 + \omega_2 x_2)$, $q = (\omega_1 x_3 + \omega_2 x_4)$ and $z = \beta + i\gamma$, there exist complex numbers $p, q \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ such that

$$\left|z - \frac{p}{q}\right| \leqslant \frac{4|ad-bc|}{\pi |q|^2}. \tag{2}$$

Since $t > 0$ can increase arbitrarily in Y_3, it follows that: *If $z \in \mathbb{C} \setminus \mathbb{Q}(\lambda)$ is an arbitrary non-lattice-point complex number, there exist infinitely many $p, q \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ satisfying*

$$\left|z - \frac{p}{q}\right| \leqslant \frac{4|ad-bc|}{\pi |q|^2}.$$

On the one hand, this already shows the influence of the lattice structure on the approximation quality (at least following this approach); on the other hand, we notice that the obtained bound on the right hand-side is invariant under any change of the basis of the lattice. To improve the approximation quality, we want to apply Theorem 4. Therefore we define the symmetric convex body

$$C := \{x = (x_1, \ldots, x_4) \in \mathbb{R}^4 | \quad |Y_1| + |Y_3| \leqslant M, \ |Y_2| + |Y_4| \leqslant M\},$$

using the above defined linear forms. For calculating the volume

$$\mathrm{vol}(C) = \iiiint \mathbb{1}_C \,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4,$$

we divide the above stated linear forms into real and imaginary parts as

$$Y_1 = \overline{Y_2} = \frac{1}{\sqrt{2}}(\varphi_1 + i\psi_1) \quad \text{and} \quad Y_3 = \overline{Y_4} = \frac{1}{\sqrt{2}}(\varphi_2 + i\psi_2),$$

where the functional determinant of this transformation is

$$\det(Df(\varphi_1,\psi_1)\cdot Df(\varphi_2,\psi_2)) = \left(\frac{1}{\sqrt{2}}\right)^4 \det\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^2 = \frac{1}{4}(-2i)^2 = -1.$$

The transformation formula provides

$$\mathrm{vol}(C) = |\det(Df)| \iiiint \mathbb{1}_{\sqrt{\phi_1^2+\psi_1^2}+\sqrt{\varphi_2^2+\psi_2^2}\leqslant\sqrt{2}M} \,\mathrm{d}\varphi_1\mathrm{d}\varphi_2\mathrm{d}\psi_1\mathrm{d}\psi_2,$$

and by applying polar coordinates $\varphi_j = Mr_j\cos z_j, \psi_j = Mr_j \sin z_j$, we get

$$\mathrm{vol}(C) = M^4 \iint_{[0,2\pi)^2} \mathrm{d}z_1\mathrm{d}z_2 \iint_{r_1+r_2\leqslant\sqrt{2},r_j\geqslant 0} r_1r_2\mathrm{d}r_1\mathrm{d}r_2 = \frac{2\pi^2}{3}M^4.$$

Following Theorem 5, there exists a lattice point $x = (x_1, ..., x_4) \in \mathbb{Z}^4 \setminus \{0\}$ in C whenever

$$\mathrm{vol}(C) \geqslant 2^4 \left|\frac{1}{t^4}(ad-bc)^2\right|.$$

This leads to the restriction

$$M \geqslant \frac{\sqrt{2}}{t\sqrt{\pi}}\sqrt[4]{6|ad-bc|^2},$$

giving

$$|Y_1| + |Y_3| \leqslant \frac{\sqrt{2}}{t\sqrt{\pi}}\sqrt[4]{6|ad-bc|^2}$$

and

$$|Y_1|\cdot|Y_3| \leqslant (\frac{1}{2}(|Y_1|+|Y_3|))^2 \leqslant \frac{\sqrt{6}|ad-bc|}{2t^2\pi}.$$

Applying the same argument as above to receive (2) this proves Theorem 1.

4. An automorphic function

It is a natural question to ask how good an approximation of a complex number z by quotients of lattice points in the sense of Theorem 1 can be. Therefore, we want to prove some characteristics of the quantity

$$\mu(\tau) = \mu(\lambda) = \inf\left\{c > 0 | \forall z \in \mathbb{C}\ \exists^\infty p, q \in \lambda : |z - \frac{p}{q}| < \frac{c}{|q|^2}\right\}$$

as a function of τ, where $\lambda := \mathbb{Z} + \tau\mathbb{Z}$.

1. The existence of the automorphic form is a direct consequence of Theorem 1, where we found an explicit, probably not optimal, upper bound.
   For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have
   $$M\tau = \frac{a\tau + b}{c\tau + d} = \frac{(a\tau + b)(c\overline{\tau} + d)}{|c\tau + d|^2}.$$
   Since $\tau = x + iy$, $y > 0$ and $\det M = ad - bc = +1$, it follows that
   $$M\tau = \frac{x(ad + bc) + iy(ad - bc)}{|c\tau + d|^2} \in \mathbb{H},$$
   where $\mathbb{H}$ is the upper half-plane. Since a short computation shows
   $$\frac{p}{q} = \frac{p_1\tau + p_2}{q_1\tau + q_2} = \frac{P_1 M\tau + P_2}{Q_1 M\tau + Q_2} = \frac{P}{Q}$$
   with
   $$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
   the best approximations to a given $z \in \mathbb{C}$ by lattice points from $\lambda = \mathbb{Z} + \tau\mathbb{Z}$ and from $\mathbb{Z} + M\tau\mathbb{Z}$ equal one another; hence,
   $$\mu(\tau) = \mu(M\tau)$$
   for any $M \in SL_2(\mathbb{Z})$, which proves that μ is an automorphic function.
2. In order to prove continuity, we consider a second lattice $\lambda' := \mathbb{Z} + \tau'\mathbb{Z}$ for which we have
   $$|\tau - \tau'| < \delta$$
   with an arbitrary small $\delta > 0$. According to $p = a + b\tau$ and $q = c + d\tau$, we define lattice points
   $$p' = a + b\tau' = a + b\tau + b(\tau' - \tau) = p + b\delta$$
   and
   $$q' = c + d\tau' = c + d\tau + d(\tau' - \tau) = q + d\delta$$
   with $\delta = \tau - \tau'$. It follows that
   $$\begin{aligned} \left|z - \frac{p'}{q'}\right| &= \left|z - \frac{p}{q} + \frac{p}{q} - \frac{p'}{q'}\right| \\ &\leqslant \left|z - \frac{p}{q}\right| + \frac{|pq' - p'q|}{|qq'|} < \frac{c + \epsilon}{|q|^2} + \frac{|p(q + d\delta) - (p + b\delta)q|}{|q(q + d\delta)|} \\ &= \frac{c + \epsilon}{|q|^2} + \frac{|(a + b\tau)d - (c + d\tau)b||\delta|}{|q|^2|1 + \frac{d\delta}{q}|}. \end{aligned}$$

We simplify the numerator:

$$|ad + bd\tau - bc - bd\tau| = |(ad - bc)||\delta| = |\delta|,$$

since we may assume p and q to be coprime (respectively $ad - bc = \pm 1$). For the denominator we get (by geometrical series expanion)

$$\frac{1}{|q|^2|1 + \frac{d\delta}{q}|} = \frac{1}{|q|^2}\left(1 + \mathrm{O}\left(\frac{|\delta|}{|q|}\right)\right).$$

Hence, we receive

$$\left|z - \frac{p'}{q'}\right| < \frac{c + \epsilon^* + |\delta|}{|q|^2},$$

where ϵ^* can be made arbitrarily small depending only on ϵ. If δ is sufficiently small, this leads to

$$\left|z - \frac{p'}{q'}\right| < \frac{c + \epsilon'}{|q|^2}$$

with ϵ' as small as we ask, which implies the continuity.

3. That μ is non-constant is a direct consequence of existing results (see page 2). We know, for example, that for $\tau = i$, the bounding constant is $\mu(\lambda) = \frac{1}{\sqrt{3}}$, whereas for $\tau = i\sqrt{2}$, we have $\mu(\lambda) = \frac{1}{\sqrt{2}}$. Furthermore, Hurwitz's approximation result (1) provides a lower bound for c. We have

$$\left|z - \frac{p}{q}\right| < \frac{c + \epsilon}{|q|^2}$$

and, since

$$\left|z - \frac{p}{q}\right| \geqslant \left|\mathrm{Re}\, z - \mathrm{Re}\, \frac{p}{q}\right|,$$

it follows $c \geqslant \frac{1}{\sqrt{5}}$.

Remark. Unfortunately, μ is not analytic, since $\mu(\tau) \in \mathbb{R}$ for all $\tau \in \mathbb{H}$.

5. Arithmetical application

A very nice arithmetical application of Hilde Gintner's result and Theorem 1 can be found by transferring the classical real case theorem that the Pell equation $X^2 - dY^2 = 1$ has a solution $x, y \in \mathbb{N}$ to the complex case:

Corollary 3 can be proved easily by the classical application of complex continued fractions. However, the constraint is not necessary. Let D be an arbitrary squarefree positive integer. In Theorem 1, we showed that there are infinitely many $x, y \in \mathbb{Z}[i\sqrt{D}]$ satisfying

$$\left|\sqrt{d} - \frac{x}{y}\right| < \frac{c(D)}{|y|^2}$$

with a certain positive constant $c(D) := \mu(i\sqrt{D})$, depending on $\mathbb{Z}[i\sqrt{D}]$. In view of Ford's result [4], we have

$$c(D) = \frac{\sqrt{6D}}{\pi} \geqslant c(1) = \frac{1}{\sqrt{3}}.$$

Due to

$$\left|\sqrt{d} + \frac{x}{y}\right| = \left|\frac{x}{y} - \sqrt{d} + 2\sqrt{d}\right| < \frac{c(D)}{|y|^2} + 2\sqrt{|d|},$$

it follows

$$\begin{aligned} |x^2 - y^2 d| &= |x - y\sqrt{d}| \cdot |x + y\sqrt{d}| < \frac{c(D)}{|y|}\left(\frac{c(D)}{|y|} + 2|y|\sqrt{|d|}\right) \\ &= \left(\frac{c(D)}{|y|}\right)^2 + 2c(D)\sqrt{|d|}. \end{aligned}$$

Consequently, for all forms $X^2 - dY^2$ exists a $k \in \mathbb{Z}[i\sqrt{D}]$ satisfying

$$-\left(\left(\frac{c(D)}{|y|}\right)^2 + 2c(D)\sqrt{|d|}\right) < k < \left(\left(\frac{c(D)}{|y|}\right)^2 + 2c(D)\sqrt{|d|}\right)$$

such that, for infinitely many $x, y \in \mathbb{Z}[i\sqrt{D}]$, we have

$$x^2 - y^2 d = k.$$

Since $\sqrt{d} \notin \mathbb{Q}(i\sqrt{D})$, it follows that $k \neq 0$. If $k = 1$ we are finished.

Otherwise, if $k \neq 1$, we define equivalence classes in the set of such pairs x, y as follows. Let $N(k)$ denote the norm of the ideal generated by k in the ring $\mathbb{Z}[i\sqrt{D}]/k\mathbb{Z}[i\sqrt{D}]$. Two pairs x_1, y_1 and x_2, y_2 belong to the same class if and only if

$$x_1 \equiv x_2 \pmod{k} \qquad \text{and} \quad y_1 \equiv y_2 \pmod{k}.$$

Since there are only $N(k)^2 < \infty$ classes but infinitely many pairs, some class contains at least two of such pairs where $x_1 \neq \pm x_2$ and $y_1 \neq \pm y_2 \neq 0$. Furthermore, we define

$$x_0 = \frac{x_1 x_2 - y_1 y_2 d}{k} \quad \text{and} \quad y_0 = \frac{x_1 y_2 - x_2 y_1}{k}.$$

Obviously, the numerators of x_0 and y_0 are $\equiv 0 \bmod k$ and $x_0, y_0 \in \mathbb{Z}[i\sqrt{D}]$. Moreover, we observe

$$x_0^2 - y_0^2 d = \frac{1}{k^2}(x_1^2 - y_1^2 d)(x_2^2 - y_2^2 d) = 1,$$

which shows that there is always a solution $x_0, y_0 \in \mathbb{Z}[i\sqrt{D}]$ of the Pell equation. Here we can exclude the trivial case $y_0 = 0$, where $x_1 = \frac{x_2 y_1}{y_2}$. In view of

$$k = x_1^2 - y_1^2 d = \left(\frac{x_2 y_1}{y_2}\right)^2 - y_1^2 d = \left(\frac{y_1^2}{y_2^2}\right)(x_2^2 - y_2^2 d) = \left(\frac{y_1^2}{y_2^2}\right) k,$$

it follows that $y_1^2 = y_2^2$, however, since $y_1 \neq \pm y_2$, this is a contradiction to the assumption of the chosen pairs.

Furthermore, we can show that there are even infinitely many solutions. Starting with two (not necessarily distinct) non-trivial solutions $x_j + y_j\sqrt{d}$ to $X^2 - dY^2 = 1$, we find further solutions by

$$(x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d}) = (a + b\sqrt{d}).$$

Since $a = x_1x_2 + y_1dy_2$ and $b = x_1y_2 + x_2y_1$, we have $a, b \in \mathbb{Z}[i\sqrt{D}]$ and

$$a^2 - b^2d = (x_1^2 - y_1^2d)(x_2^2 - y_2^2d) = 1.$$

This proves that the product of two solutions provides again a solution and, furthermore, we obtain infinitely many solutions by raising them to the nth power, $n \in \mathbb{N}$, of a non-trivial one:

$$(x_0 + y_0\sqrt{d})^n = a_n + b_n\sqrt{d}.$$

Corollary 3 can be illustrated by a concrete example. Therefore, we apply a real case method of Arturas Dubickas and Jörn Steuding [3] giving a generalization of Melvyn B. Nathanson's result [15] to find families of solutions for Pell equations. They showed that for the polynomial equation

$$P(X)^2 - (X^2 + 1)Q(X)^2 = 1$$

a family of solutions can be generated by the sequence of polynomials

$$P_n(X) := (2X^2 + 1)P_{n-1}(X) + 2X(X^2 + 1)Q_{n-1}(X)$$

and

$$Q_n(X) := 2XP_{n-1}(X) + (2X^2 + 1)Q_{n-1}(X),$$

where $P_0(X) = 1$ and $Q_0(X) = 0$. We easily calculate $P_1 = 2X^2 + 1, Q_1 = 2X$ and receive the (since Euclid known) equation

$$(2X^2 + 1)^2 - (X^2 + 1)(2X)^2 = 1.$$

Now we can chose an arbitrary X. For $x = 1 + i\sqrt{5}$, respectively $x^2 = -4 + 2i\sqrt{5}$, we get

$$(4i\sqrt{5} - 7)^2 - (2i\sqrt{5} - 3)(2 + 2i\sqrt{5})^2 = 1,$$

which provides an example for $d = 2i\sqrt{5} - 3$.

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