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RATIONAL AND ALGEBRAIC Q_2-REPRESENTATIONS OF REAL NUMBERS

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Abstract. We consider rational and algebraic Q_2-representations of the fractional part of a real number generalizing the classic binary representation. We specify probabilistic problems leading to this concept, prove sufficient condition of rationality of a number and reject a hypothesis that a criterion of rationality for rational Q_2-representation of a number is completely analogous to the known criterion of rationality for binary representation. We construct a rational number having a non-periodic Q_2-representation.

Key words and phrases: classic binary expansion, Q_2-representation of numbers, rational Q_2-representation, two-symbol representation.

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1. Introduction

Recently, mathematicians intensively search for new systems of representation of real numbers suitable for construction and study of mathematical objects with complicated local structure, everywhere dense set of peculiarities (fractal sets, singular, continuous nowhere monotonic and non-differentiable functions, probability measures and probability distributions on fractal sets, complicated systems with chaotic dynamics, etc.) and subtly reflecting their local structure.

Now there are many different ways of expansion and representation of real numbers with a finite alphabet $A = \{0, 1, \ldots, s-1\}$, $1 < s \in \mathbb{N}$. Two-symbol representation is a particular case of s-symbol representation, but the case $s = 2$ deserves special attention for various reasons. Admittedly, the two-symbol representation is convenient in the technical sense. It is well known that the binary numeral system takes a special place in theoretical and applied mathematics as well as in computer science.

The classical binary expansion and representation is well-known and widely used representation of numbers $x \in [0,1]$ with two-symbol alphabet $A_2 = \{0,1\}$ by

$$x = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \ldots + \frac{\alpha_n}{2^n} + \ldots \equiv \Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \quad \text{where} \quad \alpha_n \in A_2.$$

In 1986 [6], the first author proposed the generalization depending on one parameter which is called Q_2-representation.

The need of new two-symbol coding systems is stimulated by the objectives of practical modeling and study of mathematical objects with complicated local structure.

Two classic theories of real numbers (Cantor's and Dedekind's theories) are based on the known theory of rational numbers. This is not expected in Weierstrass' and Kolmogorov-Kavun's models of real number. In such theories, there is an important question about necessary and sufficient conditions for rationality of number in terms of representation. These criteria are known for some representations. Similar problems are solved for representation of numbers by regular continued fractions. There exists a hypothesis that known criterion for rationality of number in terms of classic binary representation holds for some its generalizations also. In this paper, we reject this hypothesis for one of these generalizations.

2. Q_2-expansion (representation) of real numbers

Let q_0 be a fixed real number belonging to the interval $(0,1)$, $q_1 = 1-q_0$, $\beta_i = iq_{1-i}$, $i \in A = \{0,1\}$, i.e., $\beta_0 = 0$, $\beta_1 = q_0$.

THEOREM 1 ([7]). *For any $x \in [0,1]$, there exists a sequence (a_n), $a_n \in A$, such that*

$$x = \beta_{a_1} + \sum_{k=2}^{\infty} \left(\beta_{a_k} \prod_{j=1}^{k-1} q_{a_j} \right) = \Delta^{Q_2}_{a_1a_2\ldots a_n\ldots}, \quad a_n \in A. \tag{1}$$

Expression (1) is called the Q_2-expansion of a real number x, and its formal notation $x = \Delta^{Q_2}_{a_1a_2\ldots a_n\ldots}$ is called the Q_2-representation.

Q_2-representation $\Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$ of a number $x \in [0,1]$ is called periodic if there exist $m \in N_0$ and $t \in \mathbb{N}$ such that

$$\alpha_{m+nt+j}(x) = \alpha_{m+j}(x)$$

for any $n, j \in N$. It is denoted briefly by

$$\Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k\ldots(\alpha_{m+1}\alpha_{m+2}\ldots\alpha_{m+t})}.$$

The set of numbers $(\alpha_{m+1}\alpha_{m+2}\ldots\alpha_{m+t})$ is called the period, and the number t is called its length.

Q_2-representation is the two-symbol coding system of fractional part of real numbers with alphabet $A = \{0,1\}$. It has zero redundancy (this means that almost all numbers have a unique representation and only countable subset of the rational numbers has two representations). If number has two representations, then we will use only one, for example, such that it contains period (0). After this agreement, each point has a unique Q_2-representation.

The number is called Q_2-rational if there exists its Q_2-expansion containing period (0). These numbers have the following form $\Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k 1(0)}$. The remaining numbers are called Q_2-irrational.

It is easy to give examples of problems leading to the concept of Q_2-representation of numbers. One of them is the problem of the expression of the distribution function of a random variable with independent identically distributed binary digits. The following statement gives the solution of this problem.

THEOREM 2 ([7]). *Let F_ξ be a nondegenerate distribution function of random variable*

$$\xi = \Delta^2_{\eta_1\eta_2\ldots\eta_k\ldots}$$

with independent identically distributed binary digits η_k with probabilities $P\{\eta_k = 0\} = p_0$, $P\{\eta_k = 1\} = 1 - p_0 = p_1$. Then value of the function F_ξ has a Q_2-representation with $q_0 = p_0$, i.e.,

$$F_\xi(x) = \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty}\left(\beta_{\alpha_k(x)} \prod_{j=1}^{k-1} q_{\alpha_j(x)}\right).$$

3. Rational Q_2-expansion (representation) of real numbers

The Q_2-representation of real numbers is called rational if q_0 is a rational number. We specify expression (1) for rational Q_2-representation of the number.

Let $q_0 = \frac{p}{s}$ be a common irreducible fraction. Then $q_1 = 1 - q_0 = \frac{s-p}{s}$. Denote

$$N_1(x,k) = \alpha_1 + \alpha_2 + \ldots + \alpha_k.$$

Then

$$x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = \beta_{\alpha_1} + p\sum_{k=2}^{\infty}\left(s^{-k+1} bigg(\frac{s-p}{p}\right)^{N_1(x,k-1)}\Bigg), \tag{2}$$

$$x = \alpha_1 q_{1-\alpha_1} + p\sum_{k=2}^{\infty}\left(s^{-k+1}\left(\frac{s-p}{p}\right)^{\sum_{j=1}^{k-1}\alpha_j}\right), \tag{3}$$

$$x = \beta_{\alpha_1} + \sum_{k=2}^{\infty}\left(\beta_{\alpha_k}\prod_{j=1}^{k-1}\left(\frac{p^{1-\alpha_j}(s-p)^{\alpha_j}}{s}\right)\right). \tag{4}$$

EXAMPLE. Let $q_0 = \frac{1}{3}$, $q_1 = \frac{2}{3}$. Then, for corresponding Q_2-representation, we have

$$x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = \frac{\alpha_1}{3} + \sum_{k=2}^{\infty}\left(3^{-k+1}2^{N_1(x,k-1)}\right),$$

$$
\begin{aligned}
x &= \frac{\alpha_1}{3} + \sum_{k=2}^{\infty}\left(3^{-k+1} 2^{\sum_{j=1}^{k-1}\alpha_j}\right),\\
x &= \frac{\alpha_1}{3} + \sum_{k=2}^{\infty}\beta_{\alpha_k}\left(\frac{\alpha_k}{3}\prod_{j=1}^{k-1}\left(\frac{2^{\alpha_j}}{3}\right)\right).
\end{aligned}
$$

Lemma 1. *If the rational Q_2-representation of x is periodic, then this number is rational.*

Proof. Let x have a periodic representation

$$x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k(c_1c_2\ldots c_m)}.$$

Then

$$
\begin{aligned}
x &= \beta_{\alpha_1} + \beta_{\alpha_2}q_{\alpha_1} + \beta_{\alpha_3}q_{\alpha_1}q_{\alpha_2} + \ldots + \beta_{\alpha_k}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_{k-1}}\\
&\quad + \beta_{c_1}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k} + \beta_{c_2}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}q_{c_1} + \ldots\\
&\quad + \beta_{c_m}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}q_{c_1}\cdots q_{c_{m-1}} + \beta_{c_1}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}q_{c_1}\cdots q_{c_{m-1}}q_{c_m}\\
&\quad + \beta_{c_2}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}q^2_{c_1}\cdots q_{c_{m-1}}q_{c_m} + \ldots\\
&\quad + \beta_{c_m}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}q^2_{c_1}\cdots q^2_{c_{m-1}}q_{c_m} + \ldots\\
&= \beta_{\alpha_1} + \beta_{\alpha_2}q_{\alpha_1} + \beta_{\alpha_3}q_{\alpha_1}q_{\alpha_2} + \ldots + \beta_{\alpha_k}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_{k-1}}\\
&\quad + q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}\cdot\frac{\beta_{c_1} + \beta_{c_2}q_{c_1} + \ldots + \beta_{c_m}q_{c_1}\cdots q_{c_{m-1}}}{1 - q_{c_1}q_{c_2}\cdots q_{c_m}}.
\end{aligned}
$$

The obtained expression is the sum of two rational numbers, therefore, x is a rational number, and this completes the proof.

Corollary 1. *Rational Q_2-representation of irrational number is non-periodic.*

Is converse statement correct? The following proposition answers this question.

Theorem 3. *If the number $q_0 = \frac{p}{s}$, $s - p \neq 1$, is an irreducible fraction, then the rational number $\frac{1}{s-p}$ has a non-periodic Q_2-representation.*

Proof. Assume the converse, i.e., the number $\frac{1}{s-p}$ has a periodic Q_2-representation for $q_0 = \frac{p}{s}$. Then, by Lemma 1, it follows that

$$
\begin{aligned}
\frac{1}{s-p} &= \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_k(c_1c_2\ldots c_m)}\\
&= \beta_{\alpha_1} + \beta_{\alpha_2}q_{\alpha_1} + \beta_{\alpha_3}q_{\alpha_1}q_{\alpha_2} + \ldots + \beta_{\alpha_k}q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_{k-1}}\\
&\quad + q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_k}\cdot\frac{\beta_{c_1} + \beta_{c_2}q_{c_1} + \ldots + \beta_{c_m}q_{c_1}\cdots q_{c_{m-1}}}{1 - q_{c_1}q_{c_2}\cdots q_{c_m}}.
\end{aligned}
$$

Suppose digit “one” occurs j times among the numbers $c_1, c_2, \ldots, c_m$. It is clear that the numbers

$$a = s^k(\beta_{\alpha_1} + \beta_{\alpha_2}q_{\alpha_1} + \beta_{\alpha_3}q_{\alpha_1}q_{\alpha_2} + \ldots + \beta_{\alpha_k}q_{\alpha_1}q_{\alpha_2}\cdot\ldots\cdot q_{\alpha_{k-1}}),$$

$$\begin{aligned} b &= s^k q_{\alpha_1} q_{\alpha_2} \cdot \ldots \cdot q_{\alpha_k}, \\ c &= s^m(\beta_{c_1} + \beta_{c_2} q_{c_1} + \ldots + \beta_{c_m} q_{c_1} \cdot \ldots \cdot q_{c_{m-1}}) \end{aligned}$$

are integer. Then

$$\frac{1}{s-p} = \frac{a}{s^k} + \frac{b}{s^k} \cdot \frac{c}{s^m - p^{m-j}(s-p)^j},$$

where

$$\frac{s^k(s^m - p^{m-j}(s-p)^j)}{s-p} = (s^m - p^{m-j}(s-p)^j)a + bc.$$

The right-hand side of the last equality is a positive integer, so the left-hand side is a positive integer, too, that is, $(s^m - p^{m-j}(s-p)^j)|(s-p)$. It is obvious that this does not hold if $j \geqslant 1$. Therefore, $j = 0$, then $c = 0$, $s^k = a(s-p)$, and this is impossible.

Thus, this contradiction proves that the number $\frac{1}{s-p}$ has a non-periodic Q_2-representation for $q_0 = \frac{p}{s}$. The theorem is proved.

This means that the criterion of rationality of a number in terms of binary representation cannot be transferred to Q_2-representation of real numbers.

Let $(c_1, c_2, \ldots, c_m)$ be a fixed ordered set of numbers from $\{0, 1\}$.

DEFINITION 1. Cylinder of rank m with base $(c_1, c_2, \ldots, c_m)$ is the set $\Delta^{Q_2}_{c_1 c_2 \ldots c_m}$ of all numbers $x \in [0, 1]$ having the following Q_2-representation

$$\Delta^{Q_2}_{c_1 c_2 \ldots c_m a_{m+1} a_{m+2} \ldots a_{m+k} \ldots}.$$

THEOREM 4. *The set of rational numbers such that their given rational Q_2-representation is non-periodic is everywhere dense set in* $[0, 1]$.

Proof. Let $q_0 = \frac{p}{s}$. Consider a Q_2-representation of the number $\frac{1}{s-p}$. According to the previous theorem, it is non-periodic

$$\frac{1}{s-p} = \Delta^{Q_2}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots}.$$

Numbers having representations in the form

$$x = \Delta^{Q_2}_{\beta_1 \beta_2 \ldots \beta_m \alpha_1 \alpha_2 \ldots \alpha_n \ldots}, m \in \mathbb{N},$$

are also rational, and their Q_2-representations are non-periodic. This means that, in each cylinder of arbitrary rank m, there exists a rational number having non-periodic Q_2-representation. Therefore, the set of rational numbers having non-periodic Q_2-representation is everywhere dense set in $[0, 1]$, and this completes the proof.

4. Q_2-expansion of rational numbers in the interval $(0, 1)$ for algebraic $q_0 \in (0, 1)$

DEFINITION 2. The number α is called algebraic if there exists a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \tag{5}$$

with integer coefficients $a_n, a_{n-1}, ..., a_1, a_0$ ($a_n \neq 0$) such that $f(\alpha) = 0$.

It is clear that for all algebraic numbers α there exist infinitely many polynomials $f(x)$ with integer coefficients such that $f(\alpha) = 0$. Therefore, it is natural to consider the polynomial which is "minimal" in some sense with respect to the number α. In this regard, consider the following definition.

DEFINITION 3. The number α is called algebraic number of degree n if there exists a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0, \tag{6}$$

such that

1) $a_n \in \mathbb{N}$, $a_{n-1}, ..., a_1, a_0 \in \mathbb{Z}$, $\gcd(a_n, a_{n-1}, ..., a_1, a_0) = 1$;
2) $f(\alpha) = 0$;
3) for any integer numbers $b_k, b_{k-1}, ..., b_0$, $\sum_{j=0}^{k} b_j^2 \neq 0$, $k < n$,

$$b_k x^k + b_{k-1}x^{k-1} + ... + a_1 x + a_0 \neq 0.$$

Polynomial $f(x)$ is called the minimal polynomial of number α.

It is well known [5, p. 25] that, for any algebraic number α, there exists the minimal polynomial $f(x)$ and it is unique. For example, $\sqrt[3]{2}$ is an algebraic number of degree 3, and $f(x) = x^3 - 2$ is its minimal polynomial.

LEMMA 2. *Let α be an algebraic number of degree n. If, for any rational numbers a_k, a_{k-1}, ..., a_0, b_k, b_{k-1}, ..., b_0, $k < n$, the equality*

$$a_k\alpha^k + a_{k-1}\alpha^{k-1} + ... + a_0 = b_k\alpha^k + b_{k-1}\alpha^{k-1} + ... + b_0$$

is satisfied, then $a_j = b_j$ for all $j \in \{0, 1, ..., k\}$.

Proof. Let m be an integer number such that $ma_j, mb_j \in \mathbb{Z}$ for any $j \in \{0, 1, ..., k\}$. Then we have

$$\sum_{j=0}^{k} m(a_j - b_j)\alpha^j = 0.$$

Moreover, $m(a_j - b_j) \in \mathbb{Z}$ for any $j \in \{0, 1, ..., k\}$.

Since degree of the number α is equal to n, the last equality holds only if

$$\sum_{j=0}^{k} m^2(a_j - b_j)^2 = 0,$$

that is,

$$\sum_{j=0}^{k} (a_j - b_j)^2 = 0,$$

where $a_j = b_j$ for any $j \in \{0, 1, ..., k\}$.

LEMMA 3. *Suppose α is an algebraic number of degree n and (6) is its minimal polynomial. Then, for any $k \in \mathbb{Z}_+$, there exists a unique tuple of integer numbers $a_{(n-1)k}, a_{(n-2)k}, ..., a_{0k}$ such that*

$$\alpha^{n+k} = \frac{a_{(n-1)k}\alpha^{n-1} + a_{(n-2)k}\alpha^{n-2} + ... + a_{0k}}{a_n^{k+1}}$$

and

$$\begin{array}{rcl} a_{(n-i)0} & = & -a_i, \\ a_{(n-j)(k+1)} & = & a_n a_{(n-j-1)k} + a_{n-j} a_{(n-1)k}, \\ a_{0(k+1)} & = & -a_0 a_{(n-1)k}, i \in \{0, 1, ..., n\}, \quad j \in \{1, ..., n-1\}. \end{array}$$

Proof. The proof is by induction on k. Let $k = 0$. Then, since

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + ... + a_1\alpha + a_0 = 0,$$

we have

$$\alpha^n = \frac{-a_{n-1}\alpha^{n-1} - ... - a_1\alpha - a_0}{a_n}.$$

Let the statement be correct for $n = k$, i.e.,

$$\alpha^{n+k} = \frac{a_{(n-1)k}\alpha^{n-1} + a_{(n-2)k}\alpha^{n-2} + ... + a_{0k}}{a_n^{k+1}}.$$

Then

$$\begin{array}{rcl} \alpha^{n+k+1} & = & \alpha^{n+k}\alpha = \frac{1}{a_n^{k+1}}(a_{(n-1)k}\alpha^n + a_{(n-1)k}\alpha^{n-2} + ... + a_{0k}\alpha) \\ & = & \frac{1}{a_n^{k+1}}\big(a_{(n-1)k}\frac{-a_{n-1}\alpha^{n-1} - ... - a_1\alpha - a_0}{a_n} + a_{(n-1)k}\alpha^{n-2} + ... \\ & & +a_{0k}\alpha\big) \\ & = & \frac{(a_n a_{(n-2)k} - a_{n-1}a_{(n-1)k})\alpha^{k-1} + (a_n a_{(n-3)k} - a_{n-2}a_{(n-2)k})\alpha^{k-2}}{a_n^{k+2}} \\ & & +... - \frac{a_0 a_{(n-1)k}}{a_n^{k+2}} \\ & = & \frac{a_{(n-1)(k+1)}\alpha^{n-1} + a_{(n-2)(k+1)}\alpha^{n-2} + ... + a_{0(k+1)}}{a_n^{k+2}}. \end{array}$$

Using Lemma 2, we see that the last equality is satisfied if

$$\begin{array}{rcl} a_{(n-i)0} & = & -a_i, \\ a_{(n-j)(k+1)} & = & a_n a_{(n-j-1)k} + a_{n-j} a_{(n-1)k}, \\ a_{0(k+1)} & = & -a_0 a_{(n-1)k}, \quad i \in \{0, 1, ..., n\}, \quad j \in \{1, ..., n-1\}. \end{array}$$

DEFINITION 4. The Q_2-representation of real number is called algebraic if q_0 is an algebraic number.

THEOREM 5. *Let $i \in \{0,1\}$, let $q_i \in (0,1)$ be an irrational algebraic number of degree n with minimal polynomial (6). Suppose there exists a prime number p such that number a_0 is divisible by p but a_n is not. Then the number $\frac{1}{p}$ has a non-periodic algebraic Q_2-expansion.*

Proof. Let $i = 0$. For $i = 1$, the proof is similar. Assume the converse, i.e.,

$$\frac{1}{p} = \Delta^Q_{\alpha_1\alpha_2...\alpha_k(\gamma_1\gamma_2...\gamma_m)}.$$

Then

$$\begin{aligned}\frac{1}{p} &= \beta_{\alpha_1} + \beta_{\alpha_2} q_{\alpha_1} + ... + \beta_{\alpha_k} q_{\alpha_1} ... q_{\alpha_{k-1}} \\ &\quad + q_{\alpha_1} ... q_{\alpha_k} \frac{\beta_{\gamma_1} + \beta_{\gamma_2} q_{\gamma_1} + ... + \beta_{\gamma_m} q_{\gamma_1} ... q_{\gamma_{m-1}}}{1 - q_{\gamma_1} ... q_{\gamma_{m-1}}},\end{aligned}$$

where $\beta_0 = 0$, $\beta_1 = q_0$.

Let us consider two cases.

Case 1. $\gamma_1 = \gamma_2 ... = \gamma_m = 0$. Then we have

$$\frac{1}{p} = \beta_{\alpha_1} + \beta_{\alpha_2} q_{\alpha_1} + ... + \beta_{\alpha_k} q_{\alpha_1} ... q_{\alpha_{k-1}}.$$

Since $q_1 = 1 - q_0$, it is clear that, for any $r \in \{1, ..., k\}$, there exist integer numbers $b_{r,r}, b_{(r-1),r}, ..., b_{0,r}$ such that

$$\beta_{\alpha_r} q_{\alpha_1} ... q_{\alpha_{r-1}} = b_{r,r} q_0^{r-1} + b_{(r-2),r} q_0^{r-2} + ... + b_{0,r}.$$

If $r > n$, then, using Lemma 3, we have

$$\beta_{\alpha_r} q_{\alpha_1} ... q_{\alpha_{r-1}} = \sum_{k=n}^{r} b_{kr} \sum_{j=0}^{n-1} \frac{a_{j(r-n)} q_0^j}{a_n^{r-n+1}} + \sum_{k=0}^{n-1} b_{kr} q_0^k,$$

that is, there exist integer numbers $c_{(n-1)r}, c_{(n-2)r}, ..., c_{0r}$ such that

$$\beta_{\alpha_r} q_{\alpha_1} ... q_{\alpha_{r-1}} = \frac{1}{a_n^{r-n+1}} \sum_{j=0}^{n-1} c_{jr} q_0^j.$$

Let us consider two cases.

I) $k < n$. Then

$$\frac{1}{p} = \sum_{r=0}^{k} b_{r,r} q_0^r + b_{(r-1),r} q_0^{r-1} + ... + b_{0,r} = \sum_{r=0}^{k} d_j q_0^j$$

for some integer numbers $d_0, d_1, ..., d_k$. By Lemma 2, $\frac{1}{p} = d_0$. This contradiction proves the theorem in this case.

II) $k \geq n$. Then

$$\begin{aligned}\frac{1}{p} &= \sum_{r=0}^{n-1}(b_{r,r}q_0^r + b_{(r-1),r}q_0^{r-1} + ... + b_{0,r}) + \sum_{k=n}^{r}\frac{1}{a_n^{r-n+1}}\sum_{j=0}^{n-1}c_{jr}q_0^j \\ &= \frac{l_{n-1}q_0^{n-1} + l_{n-2}q_0^{n-2} + ... + l_0}{a_n^{r-n+1}}\end{aligned}$$

for some integer numbers $l_0, l_1, ..., l_k$. By Lemma 2,

$$\frac{1}{p} = \frac{l_0}{a_n^{r-n+1}}, \quad a_n^{r-n+1} = pl_0.$$

This is impossible because the right-hand side of the latter equality is divisible by p but the left-hand side is not. This contradiction proves the theorem in this case.
Case 2. Let $\sum_{j=1}^{m}\beta_j^2 \neq 0$ and $\sum_{j=1}^{m}(\beta_j - 1)^2 \neq 0$. Recall that the fact that Q_2-representation contains period (1) is equivalent to the existence of the other representation with period (0). We have

$$\begin{aligned}\frac{1 - q_{\gamma_1}q_{\gamma_2}...q_{\gamma_m}}{p} &= (1 - q_{\gamma_1}q_{\gamma_2}...q_{\gamma_m})\sum_{j=1}^{k}\beta_{\alpha_1}q_{\alpha_1}q_{\alpha_2}...q_{\alpha_{j-1}} \\ &\quad + q_{\alpha_1}q_{\alpha_2}...q_{\alpha_k}\sum_{j=1}^{m}\beta_{\gamma_1}q_{\gamma_1}q_{\gamma_2}...q_{\gamma_j}.\end{aligned}$$

Similarly to the Case 1, we conclude that there exist integer numbers $c_0, c_1, ...,$ c_{n-1} and a number $s \in \mathbb{Z}_+$ such that

$$\begin{aligned}&(1 - q_{\gamma_1}q_{\gamma_2}...q_{\gamma_m})\sum_{j=1}^{k}\beta_{\alpha_1}q_{\alpha_1}q_{\alpha_2}...q_{\alpha_{j-1}} + q_{\alpha_1}q_{\alpha_2}...q_{\alpha_k}\sum_{j=1}^{m}\beta_{\gamma_1}q_{\gamma_1}q_{\gamma_2}...q_{\gamma_j} \\ &= \frac{\sum_{j=0}^{n-1}c_jq_0^j}{a_n^s}.\end{aligned}$$

Suppose the digit "one" occurs l times, and the digit "zero" occurs r times among the numbers $\gamma_1, \gamma_2, ..., \gamma_m$. Then $r, l > 0$, $r + l = m$.

Let us consider two cases:

I) $m < n$. Then

$$\frac{1 - q_{\gamma 1}q_{\gamma 2}...q_{\gamma m}}{p} = \frac{1}{p}(1 - q_0^r(1 - q_0)^l),$$

that is,

$$\frac{1}{p}(1 - q_0^r(1 - q_0)^l) = \frac{\sum_{j=0}^{n-1}c_jq_0^j}{a_n^s}.$$

By Lemma 2,

$$\frac{1}{p} = \frac{c_0}{a_n^s}, \quad a_n^s = pc_0.$$

This is impossible because the right-hand side of the latter equality is divisible by p but the left-hand side is not. This contradiction proves the theorem in this case.
II) Let $m \geq n$. We have

$$1 - q_0^r(1-q_0)^l = 1 - q_0^r \sum_{i=0}^{l} (-1)^i C_l^i q_0^i = 1 - \sum_{i=0}^{l} (-1)^i C_l^i q_0^{r+i}.$$

Let us consider two cases.
IIa) Let $r \geq n$. We have

$$1 - q_0^r(1-q_0)^l = 1 - \sum_{i=0}^{l} (-1)^i C_l^i q_0^{r+i} = 1 - \sum_{i=0}^{l} (-1)^i C_l^i \sum_{j=0}^{n-1} \frac{a_{j(r+i-n)} q_0^j}{a_n^{r+i-n+1}}.$$

By Lemma 3, since $a_{0(k+1)} = -a_0 a_{(n-1)k} | a_0$, $a_{0(k+1)} | p$, that is, a_{0j} is divisible by p for all $j \in \mathbb{Z}_+$. Therefore,

$$1 - q_0^r(1-q_0)^l = \frac{a_n^{r+l-n+1} - \sum_{j=0}^{n-1} d_j q_0^j}{a_n^{r+i-n+1}},$$

where $d_0, d_1, ..., d_{n-1}$ are some integer numbers, and d_0 is divisible by p.

We see that
$$\frac{a_n^{r+l-n+1} - \sum_{j=0}^{n-1} d_j q_0^j}{p a_n^{r+l-n+1}} = \frac{\sum_{j=0}^{n-1} c_j q_0^j}{a_n^s}.$$

By Lemma 2,
$$\frac{a_n^{r+l-n+1} - d_0}{p a_n^{r+l-n+1}} = \frac{c_0}{a_n^s},$$

whence
$$a_n^{s+r+l-n+1} = d_0 a_n^s + c_0 p a_n^{r+l-n+1}.$$

This is impossible because the right-hand side of the latter equality is divisible by p but the left-hand side is not. This contradiction proves the theorem in this case.
IIb) Let $r < n$.

Taking into account Lemma 3 and arguments from case IIa), we obtain

$$\begin{aligned} 1 - q_0^r(1-q_0)^l &= 1 - \sum_{i=0}^{l} (-1)^i C_l^i q_0^{r+i} \\ &= 1 - \sum_{i=0}^{n-1-r} (-1)^i C_l^i q_0^{r+i} - \sum_{i=n-1}^{l} (-1)^i C_l^i q_0^{r+i} \\ &= 1 - \sum_{i=0}^{n-1-r} (-1)^i C_l^i q_0^{r+i} - \frac{\sum_{j=0}^{n-1} h_j q_0^j}{a_n^{r+l-n+1}}, \end{aligned}$$

where $h_0, h_1, ..., h_{n-1}$ are some integer numbers, and h_0 is divisible by p. Therefore,

$$\frac{1}{p}\left(1-\sum_{i=0}^{n-1-r}(-1)^i C_l^i q_0^{r+i}-\frac{\sum_{j=0}^{n-1} h_j q_0^j}{a_n^{r+l-n+1}}\right)=\frac{\sum_{j=0}^{n-1} c_j q_0^j}{a_n^s}.$$

By Lemma 2,

$$\frac{1}{p}\left(1-\frac{h_0}{a_n^{m-n+1}}\right)=\frac{c_0}{a_n^s},$$

whence

$$a_n^{m-n+s+1}=h_0 a_n^s+c_0 p a_n^{m-n+1}.$$

This is impossible because the right-hand side of the latter equality is divisible by p but the left-hand side is not. This contradiction proves the theorem.

Lemma 4. *Let α be an irrational algebraic number of degree n with minimal polynomial* (6)*. Then $f(r)\neq 0$ for any rational r.*

Proof. Assume the converse, i.e., $f(\frac{m}{k})=0$ for some $m\in Z$, $n\in\mathbb{N}$. Then, by the Bézout theorem, there exist real numbers $b_0, b_1, ..., b_{n-1}$ such that

$$a_n x^n+a_{n-1}x^{n-1}+...+a_0=a_n\left(x-\frac{m}{k}\right)(b_{n-1}x^{n-1}+b_{n-2}x^{n-2}+...+b_0).$$

Then we have

$$\begin{aligned} b_{n-1} &= 1,\\ \frac{a_{n-1}}{a_n} &= b_{n-2}-\frac{m}{k}b_{n-1},\\ &\dots\\ \frac{a_1}{a_n} &= b_0-\frac{m}{k}b_1,\\ \frac{a_0}{a_n} &= -\frac{m}{k}b_0. \end{aligned}$$

It is clear that $b_j\in Q$ for any $j\in\{0,1,...,n-1\}$, that is, $b_j=\frac{c_j}{d_j}$, where $c_j\in\mathbb{Z}$, $d_j\in\mathbb{N}$ for any $j\in\{0,1,...,n-1\}$. We have

$$\alpha^{n-1}+\frac{c_{n-2}}{d_{n-2}}\alpha^{n-2}+...+\frac{c_1}{d_1}\alpha+\frac{c_0}{d_0}=0,$$

whence

$$h_{n-1}\alpha^{n-1}+...+h_1\alpha+h_0=0,$$

where $h_{n-1}=\prod_{j=0}^{n-2}d_j\in\mathbb{N}$ and $h_j=\frac{c_j}{d_j}\prod_{j=0}^{n-2}d_j\in\mathbb{Z}$ for any $j\in\{0,1,...,n-1\}$. However,the degree of the number α is equal to n. This contradiction proves the lemma.

Lemma 5. *Suppose α is an irrational algebraic number of degree n with minimal polynomial* (6)*, and $b_0, b_1, ..., b_n$ are integer numbers such that*

$$b_n\alpha^n+...+b_1\alpha+b_0=0.$$

Then there exists an integer number λ such that $b_j=\lambda a_j$ for any $j\in\{0,1,...,n\}$.

Proof. If $b_n = 0$, then since n is a degree of the number α, from equality

$$b_{n-1}\alpha^{n-1} + ... + b_1\alpha + b_0 = 0$$

it follows that $b_j = 0$ for any $j \in \{0, 1, ..., n-1\}$, so, we obtain $\lambda = 0$.

We can assume without loos of generality that $b_n > 0$ and $l = \gcd(b_0, b_1, ..., b_n)$. Then $b_j = lc_j$ for any $j \in \{0, 1, ..., n\}$, where $c_j \in \mathbb{Z}$, $\gcd(c_0, c_1, ..., c_n) = 1$ and

$$c_n\alpha^n + ... + c_1\alpha + c_0 = 0.$$

Since the minimal polynomial is unique, we have $a_j = c_j$ for any $j \in \{0, 1, ..., n\}$. Thus, $b_j = \lambda a_j$, where $\lambda = l$.

LEMMA 6. *Let α be an irrational algebraic number of degree n with minimal polynomial*

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0.$$

Then $1 - \alpha$ is an irrational algebraic number of degree n with minimal polynomial

$$\frac{f(1-x)}{\lambda},$$

where λ is some integer number.

Proof. It is clear that $1 - \alpha$ is a zero of the polynomial $g(x) = f(1-x)$ of degree n. Hence, $1 - \alpha$ is an algebraic number of degree $k \leq n$.

Let $\varphi(x) = b_kx^k + b_{k-1}x^{k-1} + ... + b_0$ be a minimal polynomial of the number $1 - \alpha$. Since $1 - \alpha \notin \mathbb{Q}$, by Lemma 4, $\varphi(-1) \neq 0$ and we have $\sum_{j=0}^{k}(-1)^j b_j \neq 0$.

If $k < n$, then the polynomial $\psi(x) = \varphi(1-x)$ satisfies equality

$$0 = \varphi(1-\alpha) = \psi(\alpha).$$

Moreover, the degree of the polynomial $\psi(x)$ is equal to $k < n$. But this contradicts the fact that the degree of the number α is equal to n. Therefore, $k = n$. Because of $\psi(\alpha) = 0$, by Lemma 5, there exists integer number λ such that

$$\psi(x) = \lambda\varphi(x),$$

whence

$$\varphi(x) = \frac{\psi(x)}{\lambda} = \frac{f(1-x)}{\lambda}.$$

THEOREM 6. *Let $i \in \{0, 1\}$, let $q_i \in (0, 1)$ be an irrational algebraic number of degree n with minimal polynomial* (6)*. Suppose there exists prime number p such that $\sum_{j=0}^{n} a_j$ is divisible by p, but a_n is not. Then the number $\frac{1}{p}$ has a non-periodic algebraic Q_2-expansion.*

Proof. Let $j' = 1 - j$. Then $q_{j'} = q_{1-j} = 1 - q_j$. By Lemma 6, there exists a number $0 \neq \lambda \in \mathbb{Z}$ such that $\frac{f(1-x)}{\lambda}$ is a minimal polynomial of the number $q_{j'}$. Let us denote

$$\frac{f(1-x)}{\lambda} = b_n x^n + b_{n-1} x^{n-1} + ... + b_0.$$

Then $a_n = (-1)^n \lambda b_n$ and $\sum_{j=0}^{n} a_j = \lambda b_0$. Since number $\sum_{j=0}^{n} a_j$ is divisible by p, we see that $\lambda | p$ or $b_0 | p$. If $\lambda | p$, then $a_n | p$, and this contradicts the statement of the theorem. So, b_0 is divisible by p but λ and b_n are not. Using Theorem 5 for the number $q_{j'}$, we prove the theorem.

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