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# THE LAPLACE TRANSFORM FOR A PAIR OF HECKE $Z$-FUNCTIONS

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**Abstract.** For the sum of pairwise products of values $Z_m\big(s; \frac{\delta_1}{\gamma}, 0\big) Z_m\big(s; 0, \frac{\delta_2}{\gamma}\big)$ of the Hecke zeta-functions over the ring of Gaussian integers, we obtain a formula for the Laplace transform on the half-line.

*Key words and phrases:* Dirichlet character, Hecke zeta-function, Kloosterman sum, Laplace transform.

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## 1. Introduction

The Laplace transform $L_F(s)$ of the function $F(x)$, $0 \leqslant x < \infty$, is defined by the formula

$$L_F(s) := \int_0^\infty F(x) \mathrm{e}^{-sx} \mathrm{d}x, \quad s = \sigma + it \in \mathbb{C},$$

provided the integral exists for $\sigma_1 < \sigma < \sigma_2$ with some $\sigma_1$ and $\sigma_2$.

The function $L_F(s)$ often applies for investigation of the mean value $M_T(F) = \int_0^T F(x)\mathrm{d}x$ as $T \to \infty$. In the theory of zeta-functions, the Laplace transforms are used to construct the asymptotic formulas for the moments of such functions (see, for example, [4, 11]). In the works of A. Ivić [4, 5], A. Ivić, M. Jutila and Y. Motohashi [6], A. Laurinčikas [8], M. Lukkarinen [9], A. Balčiūnas and A. Laurinčikas [1], the applications of the Laplace transforms for studying the modified Mellin transforms of powers of the Riemann zeta-function $\zeta(s)$ or Dirichlet $L$-functions have been applied.

Our aim is to study the Laplace transform of the sum of products of two Hecke $Z$-functions $Z_m\big(s, \frac{\alpha_1}{\gamma}, \frac{\alpha_2}{\gamma}\big)$ and $Z_m\big(s, \frac{\beta_1}{\gamma}, \frac{\beta_2}{\gamma}\big)$ on the line $\sigma = \frac{1}{2}$. More precisely, we investigate the Laplace transform for the function

$$F_m(s;\delta,\gamma)=\sum_{\substack{\delta_1,\delta_2\in\mathbb{Z}[i]\\ \delta_1\delta_2\equiv q\,(\mathrm{mod}\,\gamma)}} Z_m\left(s;\frac{\delta_1}{\gamma},0\right)Z_m\left(\overline{s};0,\frac{\delta_2}{\gamma}\right).$$

The main result of this paper is the following statement.

Theorem 1. *Let $\delta$ and $\gamma$ be the Gaussian integers, $(\delta,\gamma)=1$. Then*

$$\begin{aligned}&L_{F_m}(s;\delta,\gamma)\\ &\quad=4\pi^3\mathrm{e}^{(i\pi-s)/(2)}\\ &\quad\times\left[\pi\prod_{\mathfrak{p}|\gamma}\left(1-\frac{1}{N(\mathfrak{p})}\right)^{-1}\left(\log N(\mathfrak{p})+\frac{\varphi(\gamma)}{\pi}b_0(\gamma)\right)-\frac{i(\pi-s)}{2}\right]+\lambda_0(s,\delta,m),\end{aligned}$$

*where*

$$b_0(\gamma)=\pi\prod_{\mathfrak{p}|\gamma}\left(1-\frac{1}{N(\mathfrak{p})}\right)^{-1}\left(E+\frac{L'(1,\chi_4)}{L(1,\chi_4)}+\sum_{\mathfrak{p}|\gamma}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p}-1)}\right),$$

*$E$ is the Euler constant, $L(s,\chi_4)$ is the Dirichlet $L$-function with non-trivial character* mod 4. *Moreover, the function $\lambda_0(s,\delta,m)$ is analytic for $|\sigma|<\frac{\pi}{2}$, and the estimate $|\lambda_0(s,\delta,m)|\ll(1+|s|)^{-1}$ holds for $|\sigma|\leqslant\theta$, $0<\theta<\frac{\pi}{2}$.*

In order to obtain the above statement for $L_{F_m}(s;\delta,\gamma)$, we need some notation.

## 2. Notation

Let $G$ denote the ring of Gaussian integers, i.e., $G:=\{a+bi:\ a,b\in\mathbb{Z},\ i\in\mathbb{C},\ i^2=-1\}$. $G_\gamma$ (respectively, $G_\gamma^*$) stands for the ring of the residue classes (respectively, reduced residue) modulo $\gamma$. For convenience, a summing over $\delta_1,\delta_2 \pmod{\gamma}$ under condition $\delta_1\delta_2\equiv 1 \pmod{\gamma}$ will be denoted by $(\delta)$. As usual, $(\alpha,\beta)$ stands for the greatest common divisor of $\alpha$ and $\beta$, $\alpha,\beta\in G$.

## 3. Auxiliary results

For $\sigma>1$, we consider the pair of Hecke zeta-functions

$$Z_m(s;\delta_1,0)=\sum_{\omega\in G}\frac{\mathrm{e}^{4mi\arg(\omega+\delta_1)}}{N(\omega+\delta_1)^s},\quad \delta_1\in\mathbb{Q}(i),$$

and

$$Z_m(s;0,\delta_2)=\sum_{\omega\in G}\frac{\mathrm{e}^{4mi\arg(\omega)}\mathrm{e}^{2\pi i\mathrm{Re}\,(\delta_2\omega)}}{N(\omega)^s},\quad \delta_2\in\mathbb{Q}(i).$$

These functions are the special cases of the Hecke $Z$-function defined by

$$Z_m(s;\delta_1,\delta_2)=\sum_{\substack{\omega\in G\\ \omega\neq-\delta_1}}\frac{\mathrm{e}^{4mi\arg(\omega+\delta_1)}\mathrm{e}^{2\pi i\mathrm{Re}\,\delta_2\omega}}{N(\omega+\delta_1)^s},\quad \sigma>1.$$

From the work of Hecke [3], we infer the following statement.

LEMMA 1 (Functional Equation). *The function $Z_m(s;\delta_1,\delta_2)$ has an analytic continuation to the whole $s$-plane, and it is an entire function if $m\neq 0$. For $m=0$, $Z_0(s;\delta_1,\delta_2)$ is entire if $\delta_2$ is not a Gaussian integer. For $m=0$, $\delta_2\in G$, the function $Z_0(s;\delta_1,\delta_2)$ is holomorphic, except for a simple pole at the point $s=1$ with residue $\pi$. Moreover, the functional equation*

$$\begin{aligned}&\pi^{-s}\Gamma(2|m|+s)Z_m(s;\delta_1,\delta_2)\\&\quad=\pi^{-(1-s)}\Gamma(2|m|+1-s)Z_{-m}(1-s;-\delta_2,\delta_1)\mathrm{e}^{2\pi i\mathrm{Re}\,(\delta_1\delta_2)}\end{aligned}\tag{1}$$

*holds in both the cases.*

REMARK 1. For $\delta_1=0$ or $\delta_2\equiv 0\,(\mathrm{mod}\,\gamma)$, we have

$$Z_m\left(s;\frac{\delta_1}{\gamma},\frac{\delta_2}{\gamma}\right)=Z_{-m}\left(s;\frac{\delta_1}{\gamma},\frac{\delta_2}{\gamma}\right).$$

REMARK 2. By the Phragmen-Lindelöf theorem [10], Lemma 1 and Kaufman's estimate of the Hecke $Z$-function on the line $\sigma=\frac{1}{2}$ [7], we obtain that, for $\frac{1}{2}\leqslant\sigma\leqslant 1$, $|t|\geqslant 3$,

$$Z_m\left(s;\frac{\delta_1}{\gamma},\frac{\delta_2}{\gamma}\right)\ll N(\gamma)(m^2+t^2)^{(1-\sigma)/2}\log^3(m^2+t^2).$$

This remark and the integral Cauchy formula imply an estimate

$$Z_m'\left(s;\frac{\delta_1}{\gamma},\frac{\delta_2}{\gamma}\right)\ll N(\gamma)(m^2+t^2)^{(1-\sigma)/3}\log^3(m^2+t^2).$$

LEMMA 2. *For every sufficiently small $\varepsilon>0$, in the region $-\varepsilon<\sigma\leqslant 1+\varepsilon$, $|t|\geqslant 3$, we have*

$$\sum_{(\delta)}Z_m\left(s;\frac{\delta_1}{\gamma},0\right)Z_m\left(s;\frac{\delta_2}{\gamma},0\right)\ll N(\gamma)^{(1+\sigma)/2+\varepsilon}(m^2+t^2)^{(1-\sigma)/2+\varepsilon}$$

*with constant in symbol "$\ll$" which depends only on $\varepsilon$.*

*Proof.* The result follows essentially from the fact that, by the functional equation (1), the estimate

$$\left|\sum_{(\delta)}Z_m\left(-\varepsilon+it;\frac{\delta_1}{\gamma},0\right)Z_m\left(-\varepsilon+it;\frac{\delta_2}{\gamma},0\right)\right|\ll$$

$$\ll \left|\frac{\Gamma(2|m|+1+\varepsilon-it)}{\Gamma(2|m|-\varepsilon+it)}\right|^2 \sum_{0\neq\omega\in G}\frac{1}{N(\omega)^{1+\varepsilon}}\sum_{\omega_1,\omega_2=\omega}\left|\sum_{(\delta)} \mathrm{e}^{2\pi i \mathrm{Re}\,\left((\omega_1\delta_1+\omega_2\delta_2)/\gamma\right)}\right|$$

holds. For the Kloosterman sum over $G$, the estimate

$$\sum_{(\delta)} \mathrm{e}^{2\pi i \mathrm{Re}\,((\omega_1\delta_1+\omega_2\delta_2)/\gamma)} \ll N((\omega_1,\omega_2,\gamma))^{1/2}N(\gamma)^{1/2}\tau(\gamma)$$

is valid. Thus,

$$\sum_{(\delta)} Z_m\Big(-\varepsilon+it;\frac{\delta_1}{\gamma},0\Big)Z_m\Big(-\varepsilon+it;\frac{\delta_2}{\gamma},0\Big) \ll N(\gamma)^{1/2+\varepsilon}(m^2+t^2)^{1+\varepsilon}.$$

Moreover,

$$\sum_{(\delta)} Z_m\Big(1+\varepsilon+it;\frac{\delta_1}{\gamma},0\Big)Z_m\Big(1+\varepsilon+it;\frac{\delta_2}{\gamma},0\Big) \ll \sum_{\omega\equiv\ell\,(\mathrm{mod}\ \gamma)}\frac{N(\gamma)^{1+\varepsilon}\tau(\omega)}{N(\omega)^{1+\varepsilon}} \ll N(\gamma)^{1+2\varepsilon}.$$

These two estimates together with the Phragmen-Lindelöf theorem give the assertion of Lemma 2.

LEMMA 3. *Let* $\lambda\in\mathbb{C}$, $|\arg\lambda|\leqslant\frac{\pi}{2}-\varepsilon$, $f(x),S(x)\in C^\infty$ *for* $x$ *in the neighbourhood of* $x_0$, $0<x_0<\infty$. *Suppose that* $x_0$ *is an unique maximum point for* $S(x)$, $S'(x_0)=0$, $S''(x_0)\neq 0$. *Then, for* $\lambda\to\infty$, *the asymptotic expansion*

$$\int_0^\infty f(x)\mathrm{e}^{\lambda S(x)}dx \sim \mathrm{e}^{-\lambda S(x_0)}\sum_{k=0}^\infty c_k\lambda^{-k-1/2},$$

*where*

$$c_k=\frac{\Gamma\left(k+1/2\right)}{(2k)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\left[f(x)\Big(\frac{-2\lambda(S(x_0)-S(x))}{(x-x_0)^2}\Big)^{-k-1/2}\right]\Bigg|_{x=x_0},$$

*holds. This expansion can be differentiated any number of times. In particular, for* $m\in\mathbb{Z}$, *we have*

$$\int_0^\infty \mathrm{e}^{-\lambda(x+1/x)}x^m\mathrm{d}x\sim\sqrt{\frac{\pi\lambda}{2}}\mathrm{e}^{-2\lambda}$$

*as* $\mathrm{Re}\,\lambda\to+\infty$.

This statement is a corollary of the general theorem on the Laplace integral, see, for example, [2], Chapter 2, Theorem 1.3.

For $x \in [0, +\infty)$, consider the function

$$F_m(x) := \sum_{(\delta)} Z_m\left(\frac{1}{2} + ix; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(\frac{1}{2} - ix; 0, \frac{\delta_2}{\gamma}\right).$$

It is clear that $F_m(x) = F_{-m}(x)$.

Consider the Laplace transform

$$L_{F_m}(s; \delta, \gamma) := \int_0^\infty F_m(x) \mathrm{e}^{-sx} \mathrm{d}x.$$

Define

$$I_m(s; \delta, \gamma) = \frac{\mathrm{e}^{-is/2}}{2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \sum_{(\delta)} Z_m\left(z; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(1 - z; 0, \frac{\delta_2}{\gamma}\right) \frac{\mathrm{e}^{-iz(\pi - s)}}{\sin \pi z} \mathrm{d}z. \quad (2)$$

LEMMA 4. *There exists the function $\lambda_0(s; \delta, m)$ analytic in the strip $\{s \in \mathbb{C} : |\sigma| \leqslant \pi\}$ such that*

$$L_{F_m}(s; \delta, m) = I_m(s, \delta, \gamma) + \lambda_0(s; \delta, m).$$

*Moreover, for $|\sigma| \leqslant \theta$, $0 < \theta < \pi$, the estimate*

$$\lambda_0(s; \delta, m) \ll (1 + (m^2 + t^2))^{-1/2}$$

*holds.*

*Proof.* From the definition of $I_m(s, \delta, \gamma)$, we have

$$\begin{aligned}
&I_m(s; \delta, \gamma) \\
&= -\mathrm{e}^{is/2} \int_{-\infty}^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2} + ix; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(\frac{1}{2} - ix; 0, \frac{\delta_2}{\gamma}\right) \frac{\mathrm{e}^{\pi i/2 + \pi x + is/2 - sx}}{\mathrm{e}^{\pi i/2 - \pi x} - \mathrm{e}^{-\pi i 2 + \pi x}} \mathrm{d}x \\
&= -\int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2} + ix; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(\frac{1}{2} - ix; 0, \frac{\delta_2}{\gamma}\right) \frac{\mathrm{e}^{-\pi i/2 + \pi x - sx}}{\mathrm{e}^{\pi i/2 - \pi x} - \mathrm{e}^{-\pi i/2 + \pi x}} \mathrm{d}x \\
&\quad - \int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2} - ix; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(\frac{1}{2} + ix; 0, \frac{\delta_2}{\gamma}\right) \frac{\mathrm{e}^{-\pi i/2 - \pi x + sx}}{\mathrm{e}^{\pi i/2 + \pi x} - e^{-\pi i/2 - \pi x}} \mathrm{d}x.
\end{aligned}$$

By the equality

$$1 = \frac{\mathrm{e}^{\pi i/2 - \pi x}}{\mathrm{e}^{\pi i/2 - \pi x} - \mathrm{e}^{-\pi i/2 + \pi x}} - \frac{\mathrm{e}^{-\pi i/2 + \pi x}}{\mathrm{e}^{\pi i/2 - \pi x} - \mathrm{e}^{-\pi i/2 + \pi x}},$$

we infer

$$\begin{aligned}
&I_m(s; \delta, \gamma) \\
&\quad = L_{F_m}(s; \delta, \gamma) -
\end{aligned}$$

$$-\int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2}+ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}-ix;0,\frac{\delta_2}{\gamma}\right)\frac{\mathrm{e}^{\pi i/2-\pi x-sx}}{\mathrm{e}^{\pi i/2-\pi x}-\mathrm{e}^{-\pi i/2+\pi x}}\mathrm{d}x$$
$$-\int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2}-ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}+ix;0,\frac{\delta_2}{\gamma}\right)\frac{\mathrm{e}^{-\pi i/2-\pi x+sx}}{\mathrm{e}^{\pi i/2+\pi x}-\mathrm{e}^{-\pi i/2-\pi x}}\mathrm{d}x.$$

Hence,

$$\begin{aligned}\lambda_0(s;\delta,\gamma) &= \int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2}+ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}-ix;0,\frac{\delta_2}{\gamma}\right)\frac{\mathrm{e}^{\pi i/2-\pi x-sx}}{\mathrm{e}^{\pi i/2-\pi x}-\mathrm{e}^{-\pi i/2+\pi x}}\mathrm{d}x\\ &+\int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2}-ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}+ix;0,\frac{\delta_2}{\gamma}\right)\frac{\mathrm{e}^{-\pi i/2-\pi x+sx}}{\mathrm{e}^{\pi i/2+\pi x}-\mathrm{e}^{-\pi i/2-\pi x}}\mathrm{d}x. \quad (3)\end{aligned}$$

The integrals in (3) converge uniformly in any compact subset of the strip $|\sigma|<2\pi$, consequently, in this strip $\lambda_0(s;\delta,\gamma)$ is an analytic function.

Furthermore, for $|t|\leqslant C$, we have $\lambda_0(s;\delta,\gamma)\ll 1$. In remaining part of the strip $|\sigma|<2\pi$, we apply integrating by parts and have

$$\begin{aligned}&\int_0^\infty \sum_{(\delta)} Z_m\left(\frac{1}{2}\pm ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}\mp ix;0,\frac{\delta_2}{\gamma}\right)\frac{\mathrm{e}^{\pi i/2-\pi x\mp sx}}{\mathrm{e}^{\pi i/2\mp\pi x}-\mathrm{e}^{-\pi i/2\pm\pi x}}\mathrm{d}x\\ &=\Bigg[\sum_{(\delta)} Z_m\left(\frac{1}{2}\pm ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}\mp ix;0,\frac{\delta_2}{\gamma}\right)\\ &\quad\times\frac{1}{\mathrm{e}^{\pi i/2\mp\pi x}-\mathrm{e}^{-\pi i/2\pm\pi x}}\mathrm{e}^{\mp sx}\left(\frac{1}{\mp s}\right)\Bigg]_0^\infty\\ &\quad+\left(\frac{\pm 1}{s}\right)\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\Bigg(\sum_{(\delta)} Z_m\left(\frac{1}{2}\pm ix;\frac{\delta_1}{\gamma},0\right)Z_m\left(\frac{1}{2}\mp ix;0,\frac{\delta_2}{\gamma}\right)\\ &\quad\times\frac{1}{\mathrm{e}^{\pi i/2\mp\pi x}-\mathrm{e}^{-\pi i/2\pm\pi x}}\Bigg)\mathrm{e}^{\mp sx}\mathrm{d}x.\end{aligned}$$

Thus, Lemma 2 shows that the second part is also true.

From Lemma 3, it follows that, for the derivation of formula for $L_{F_m}(s;\delta,\gamma)$, it is sufficient to calculate the integral $I_m(s;\delta,\gamma)$.

## 4. Proof of the main theorem

Applying the functional equation (1) for $Z_m(s;0,\frac{\delta_2}{\gamma})$, we obtain from (2)

$$\begin{aligned}&I_m(s;\delta,\gamma)\\ &\quad=\mathrm{e}^{-is/2}\int_{(1/2)}\sum_{(\delta)} Z_m\left(z;\frac{\delta_1}{\gamma},0\right)Z_m\left(z;\frac{\delta_2}{\gamma},0\right)\pi^{2z-1}\end{aligned}$$

$$\times \frac{\Gamma(2|m|+z)}{\Gamma(2|m|+1-z)} \frac{e^{-iz(\pi-s)}}{\sin \pi z} dz. \tag{4}$$

In virtue of the equality $\frac{1}{\Gamma(2|m|+z)\Gamma(2|m|+1-z)\sin \pi z} = \pi\Gamma^2(2|m|+z)$, we see that, for $m = 0$, the integrand in (4) has a double pole at the point $z = 1$, while, for $m \neq 0$, it is a regular function in the strip $\left\{\frac{1}{2} \leqslant \operatorname{Re} z \leqslant 2\right\}$. Therefore,

$$\begin{aligned} &I_m(s, \delta; \gamma) \\ &= e^{-is/2} \\ &\times \left\{ 2\pi i \varepsilon_m + \int_{(2)} \sum_{(\delta)} Z_m\left(z; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(z; \frac{\delta_2}{\gamma}, 0\right) \Gamma^2(2|m|+z) \left(\pi^{2e^{-i(\pi-s)}}\right)^z dz \right\}, \end{aligned} \tag{5}$$

where $\varepsilon_m = 0$ if $m \neq 0$, and $\varepsilon_0$ is a residue of the integrand at the point $z = 1$.

We shall calculate $\varepsilon_0$. For $\operatorname{Re} z > 1$, we have

$$\begin{aligned} Z_0\left(z; \frac{\delta}{\gamma}, 0\right) &= N(\gamma)^z \sum_{\substack{\omega \\ \omega \equiv \delta \pmod{\gamma}}} N(\omega)^{-z} = N(\gamma)^z \frac{1}{\varphi(\gamma)} \sum_{(\chi)} \chi(\delta^{-1}) \sum_{\omega} \frac{\chi(\omega)}{N(\omega)^z} \\ &= \frac{N(\gamma)^s}{\varphi(\gamma)} \sum_{(\chi)} \chi(\delta^{-1}) Z_0(s, \chi), \end{aligned}$$

where $Z_0(s, \chi)$ is the Hecke $Z$-function, $\varphi(\gamma)$ denotes the Euler function over $G$, a sign $(\chi)$ shows that the summing runs over all characters of the group $G_\gamma^*$, and $\delta^{-1}$ is a multiplicative inverse for $\delta \pmod{\gamma}$.

The Hecke $Z$-function $Z_0(s, \chi)$ is an entire function for a non-principal character $\chi$, and, for $\chi = \chi_0$, $\chi_0$ is the principal character, we have

$$Z_0(z, \chi_0) = 4\zeta(z) L(z, \chi_4) \prod_{\mathfrak{p} | \gamma} \left(1 - \frac{1}{N(\mathfrak{p})^z}\right),$$

where $\zeta(z)$ and $L(z, \chi_4)$ are the Riemann zeta-function and Dirichlet $L$-function with non-principal Dirichlet character $\chi_4 \pmod 4$, respectively. Thus, $Z_0(z, \chi_0)$ has the Laurent expansion

$$Z_0(z, \chi_0) = \frac{\pi\varphi(\gamma)}{N(\gamma)} \cdot \frac{1}{z-1} + b_0(\gamma) + a_1(z-1) + a_2(z-1)^2 + \dots,$$

where

$$b_0(\gamma) = \frac{\pi\varphi(\gamma)}{N(\gamma)} \left[ E + \frac{L'(1, \chi_4)}{L(1, \chi_4)} + \sum_{\mathfrak{p}|\gamma} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p}) - 1} \right].$$

For arbitrary character $\chi \pmod{\gamma}$, we write

$$Z_0(z, \chi) = b_0(\chi) + b_1(\chi)(z-1) + \dots.$$

From the above remarks, we find that

$$\varepsilon_0 = \operatorname*{Res}_{z=1} \left\{ \sum_{(\delta)} Z_m\left(z; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(z; \frac{\delta_2}{\gamma}, 0\right) (\pi^2 e^{-i(\pi-s)})^z \Gamma^2(z) \right\}$$

$$= \quad \mathrm{e}^{-i(\pi-s)} \sum_{(\delta)} \left[ \frac{2\pi^2 N(\gamma)}{\varphi(\gamma)} \log N(\gamma) + \sum_{(\chi)} b_1(\chi)(\chi(\delta_1^{-1}) + \chi(\delta_2^{-1})) - i\pi^2(\pi - s) \right]. \tag{6}$$

The condition of summation $(\delta)$ gives the congruence $\delta_1\delta_2 \equiv \delta \,(\mathrm{mod}\ \gamma)$, where $(\delta, \gamma) = 1$. Note that every $\delta_1$, $(\delta_1, \gamma) = 1$, uniquely determines a value $\delta_2$, moreover, when $\delta_1$ runs the reduced residue system modulo $\gamma$, then $\delta_2$ runs the same system as well. Consequently,

$$\sum_{(\chi)} b_0(\chi) \sum_{(\delta)} (\chi(\delta_1^{-1}) + \chi(\delta_2^{-1})) = \sum_{(\chi)} b_0(\chi) \sum_{(\delta_1)} 2\chi(\delta_1) = 2\varphi(\gamma) b_0(\chi_0),$$

and, by (6), we have

$$\varepsilon_0 = 2\pi^2 \left( \frac{\pi N(\gamma)}{\varphi(\gamma)} \left( \log N(\gamma) + \frac{1}{\pi} \varphi(\gamma) b_0(\gamma) \right) - \frac{i}{2}(\pi - s) \right). \tag{7}$$

Now we calculate the integral in the right-hand side of equality (5). For $\mathrm{Re}\, z > 1$, we have

$$\sum_{(\delta)} Z_m\left(z; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(z; \frac{\delta_2}{\gamma}, 0\right)$$
$$= \sum_{(\delta)} \sum_{\omega_1, \omega_2} \frac{\mathrm{e}^{4mi\arg((\omega_1 + \delta_1/\gamma)(\omega_2 + \delta_2/\gamma))}}{N\left((\omega_1 + \delta_1/\gamma)(\omega_2 + \delta_2/\gamma)\right)^z} = N(\gamma)^{2z} \mathrm{e}^{8i\arg\gamma} \sum_{\substack{\omega \in G \\ \omega \equiv \delta \,(\mathrm{mod}\ \gamma)}} \frac{\tau^{(m)}(\omega)}{N(\omega)^z},$$

where $\tau^{(m)}(\omega) = \mathrm{e}^{4mi\arg\omega}\tau(\omega)$. Hence, the integral in (5) admits a term-by-term integration, and we have

$$\int_{(2)} \sum_{(\delta)} Z_m\left(z; \frac{\delta_1}{\gamma}, 0\right) Z_m\left(z; \frac{\delta_2}{\gamma}, 0\right) \pi^{2z} \Gamma^2(z) \mathrm{e}^{-iz(\pi-s)} \mathrm{d}z$$
$$= \mathrm{e}^{8mi\arg\gamma} \sum_{\omega \equiv \delta \,(\mathrm{mod}\ \gamma)} \tau^{(m)}(\omega) \int_{(2)} \Gamma^2(2|m| + z) y^{-z} \mathrm{d}z, \tag{8}$$

where $y = N(\omega)\pi^{-2}\mathrm{e}^{i(\pi-s)}$. Further, from the definition of the gamma-function, we obtain

$$\int_{(2)} \Gamma^2(2|m| + z) y^{-z} \mathrm{d}z$$
$$= \int_{(2)} \Gamma(2|m| + z) y^{-z} \left( \int_0^\infty \mathrm{e}^{-x} x^{2|m|+z-1} \mathrm{d}x \right) \mathrm{d}z$$
$$= \int_0^\infty \mathrm{e}^{-x} x^{2|m|-1} \left( \int_{(2)} \Gamma(2|m| + z) \left( \frac{y}{x} \right)^{-z} \mathrm{d}z \right) \mathrm{d}x$$

$$= y^{2|m|}2\pi \int_0^\infty \mathrm{e}^{-x-y/x} x^{2|m|-1}\mathrm{d}x$$
$$= 2y^{3|m|} \int_0^\infty \mathrm{e}^{-\sqrt{y}(x+1/x)} x^{2|m|-1}\mathrm{d}x. \tag{9}$$

The last integral in (9) can be calculated using Lemma 3 for $0 < |\sigma| < \frac{\pi}{2}$. We obtain with $\lambda = \frac{1}{\pi}\sqrt{N(\omega)}\mathrm{e}^{i\pi - s/2}$ that

$$\int_{(2)} \Gamma^2(2|m|+z)y^{-z}\mathrm{d}z$$
$$= \sqrt{2\pi N(\omega)}\mathrm{e}^{i(\pi-s)/4}\mathrm{e}^{-2/\pi\sqrt{N(\omega)}\mathrm{e}^{i(\pi-s)/2}}(1+R(N(\omega),s)), \tag{10}$$

where $R(N(\omega),s)$ is an analytic function in the strip $|\sigma| < \frac{\pi}{2}$ in view of the estimate $|R(N(\omega),s)| \ll N(\omega)^{-1/2}|\mathrm{e}^{i(\pi-s)/2}|$. Now, from (5), (7), (8) and (10), we get

$$I_m(s;\delta,\gamma)$$
$$= 4\pi^3\mathrm{e}^{i(\pi-s)/2}\Big[\pi\prod_{\mathfrak{p}|\gamma}\big(1-\frac{1}{N(\mathfrak{p})}\big)^{-1}\big(\log N(\mathfrak{p}) + \frac{\varphi(\gamma)}{\phi}b_0(\gamma)\big) - \frac{i(\pi-s)}{2}\Big]$$
$$+\mathrm{e}^{-i(\pi-s)/4}\mathrm{e}^{8mi\arg\gamma}\sum_{\substack{\omega\in G\\ \omega\equiv\delta\,(\mathrm{mod}\,\gamma)}}\tau^{(m)}(\omega)\Big(\frac{\pi^3}{N(\omega)}\Big)^{1/4}\mathrm{e}^{-2/\pi N(\omega)^{1/2}\mathrm{e}^{i(\pi-s)/4}}$$
$$\times(1+R(N(\omega),s)). \tag{11}$$

Taking into account that the series in (11) is uniformly convergent in the strip $|\sigma| < \frac{\pi}{2}$, we obtain, by Lemma 4, the statement of Theorem 1 with

$$\lambda_0(s;\delta,m) = \mathrm{e}^{-i(\pi-s)/4}\mathrm{e}^{8i\arg\gamma}$$
$$\times\sum_{\omega\equiv\ell\,(\mathrm{mod}\,\gamma)}\tau^{(m)}(\omega)\Big(\frac{\pi^3}{N(\omega)}\Big)^{1/4}\mathrm{e}^{-2/\pi N(\omega)^{1/2}\mathrm{e}^{i(\pi-s)/2}}$$
$$\times\sum_{k=1}^\infty c_k\Big(\frac{\mathrm{e}^{i(\pi-s)/2}\pi}{N(\omega)^{1/2}}\Big)^{-k} + \lambda(s;\delta,m),$$

where $\lambda(s;\delta,m)$ is defined in Lemma 4, $c_k$, $k=1,2,\ldots$, are defined in Lemma 3 with $f(x) = x^{m-1}$, $S(x) = x + \frac{1}{x}$.

In the case $N(\gamma)=1$, the theorem established gives the Laplace transform of $|Z_m(s)|^2$, where $Z_m(s) = \sum_{\omega\in G}\mathrm{e}^{4mi\arg\omega}N(\omega)^{-s}$, $\sigma > 1$.

COROLLARY 1. *For $|\sigma| < \frac{\pi}{2}$, there exists an analytic function $\lambda_0(s,m)$, $\lambda_0(s,m) \ll (1+(t^2+m^2))^{-1/2}$, such that*

$$L_{F_m}(s) = \int_0^\infty \Big|Z_m\Big(\frac{1}{2}+ix\Big)\Big|^2\mathrm{e}^{-sx}\mathrm{d}x$$
$$= 4\pi^2 i\mathrm{e}^{-is/2}(\pi E + 4L'(1,\chi_4))$$
$$+\pi^{3/4}\mathrm{e}^{-(\pi+s)/4}\sum_\omega\tau^{(m)}(\omega)N(\omega)^{-1/4}\mathrm{e}^{-2/\pi N(\omega)^{1/2}\mathrm{e}^{i(\pi-s)/4}} + \lambda_0(s,m),$$

*where $\lambda_0(s,m)$ for $|\sigma| < \frac{\pi}{2}$ has the same properties as $\lambda_0(s;\delta,m)$.*

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