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ON ASYMPTOTIC EXPANSION OF A GENERALIZATION OF INTEGRAL TRANSFORM WITH THE TRICOMI FUNCTION AS THE KERNEL

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Abstract. The article is devoted to the study of an asymptotic behavior of the integral transform $(\boldsymbol{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda) = \lambda^{\sigma}\int_0^\infty \Psi(a,c,h\lambda t)t^{\kappa}f(t)\mathrm{d}t$, $\lambda, h > 0$, $a, c, \sigma, \kappa \in \mathbb{C}$, involving the Tricomi function $\Psi(a,c,z)$ in the kernel. It is proved that $(\boldsymbol{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ has power or power-logarithmic asymptotic expansion, as $\lambda \to 0^+$ and $\lambda \to \infty$, provided that $f(t)$ has power asymptotic behavior at infinity and zero, respectively.

Key words and phrases: asymptotic expansions, G-transform, Parseval formula for the Mellin transform, Tricomi function.

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1. Introduction

In this article, we investigate the asymptotic behavior at zero and infinity of the integral transform

$$(\boldsymbol{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda) = \lambda^{\sigma}\int_0^\infty \Psi(a,c,h\lambda t)t^{\kappa}f(t)\mathrm{d}t, \quad \lambda, h > 0,\ a, c, \sigma, \kappa \in \mathbb{C}, \tag{1}$$

involving so-called Tricomi's confluent hypergeometric function $\Psi(a,c,z)$ in the kernel. This function is defined by the formula

$$\Psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}{}_1F_1(a;c;z) + z^{1-c}\frac{\Gamma(c-1)}{\Gamma(a)}{}_1F_1(a-c+1;2-c;z),$$

where ${}_1F_1(a;c;z)$ is the Kummer confluent hypergeometric function

$$ {}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, $$

with $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ being the Pochhammer symbol.

In a series of papers [2]–[4], Handelsman and Lew have developed a theory which yields asymptotic expansions of integrals of the form

$$ I(\lambda) = \int_0^{\infty} g(\xi) f(\lambda\xi) \mathrm{d}\xi \tag{2} $$

based on the Parseval formula for the inverse Mellin transforms

$$ \Omega(\lambda) = \int_0^{\infty} K(\lambda t) H(t) \mathrm{d}t = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \kappa(s) h(1-s) \lambda^{-s} \mathrm{d}s, \tag{3} $$

where $s = \sigma + i\tau$ and

$$ \kappa(s) = \int_0^{\infty} t^{s-1} K(t) \mathrm{d}t, \tag{4} $$

$$ h(s) = \int_0^{\infty} t^{s-1} H(t) \mathrm{d}t \tag{5} $$

are Mellin transforms of the functions $K(t)$ and $H(t)$, respectively. Relation (3) holds for $\lambda > 0$ if there is γ such that the functions $t^{\gamma-1}K(t)$, $t \in [0, +\infty)$, and $h(1-\gamma-i\tau)$, $\tau \in (-\infty, +\infty)$, are absolutely integrable. Existence conditions for Parseval formula for the inverse Mellin transform in the above form are given for instance in [5].

If the asymptotic relation

$$ G(s) = \kappa(s) h(1-s) = o(e^{-\theta|\tau|}), \quad \theta \in (-\pi, \pi], $$

is satisfied, then the function $\Omega(\lambda)$ is analytic for $|arg\lambda| \leqslant \theta - \varepsilon$, $\theta > \varepsilon > 0$, and the Parseval formula holds in that sector.

In order to obtain asymptotic expansions for $\Omega(\lambda)$ as $\lambda \to +\infty$ (as $\lambda \to 0^+$), we have to move the contour of integration to the right (to the left) in formula (3). The desired expansions will be presented as the corresponding residue series. Then, by meromorphic extension of $G(s)$, we obtain an asymptotic series for $\Omega(\lambda)$.

Note that, if we move the contour of integration in a finite distance, then we obtain finite number of asymptotic terms for $\Omega(\lambda)$. By Cauchy's theorem, it is sufficient to have

$$ G(s) = \mathrm{o}(1) $$

as

$$ |\tau| \to \infty, \quad s = \sigma + i\tau, $$

in the corresponding strip.

We note that if

$$\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} |G(s)| \mathrm{d}\tau < M,$$

then

$$\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} G(s)\lambda^{-s} \mathrm{d}s = \mathrm{O}(\lambda^{-\gamma_1}),$$

and, thus, to justify the asymptotic representations, it is enough to get the estimate for the first integral on a displaced contour.

In our study, we use the results by Handelsman and Lew, which can be formulated in the form of two theorems (see, e.g., [6], Theorems 31.1 and 31.3).

THEOREM 1. *Let the following conditions be valid:*

1) *for some* γ *formula* (3) *takes place, where the functions* $\kappa(s)$ *and* $h(s)$ *are determined by* (4) *and* (5)*;*
2) *the function* $G(s) = \kappa(s)h(1-s)$ *can be continued into the half-plane* $\operatorname{Re} s > \gamma$ *as a meromorphic function with poles* s_k*;*
3) $G(s) = \mathrm{o}(1)$ *in each vertical strip* $\gamma < \gamma_1 \leqslant \operatorname{Re} s \leqslant \gamma_2$ *as* $|\tau| \to \infty$*;*
4) $\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} |G(s)| \mathrm{d}\tau < M(\gamma_0)$ *for* $\gamma_0 > \gamma$*.*

Then

$$\Omega(\lambda) = \int_0^\infty K(\lambda t)H(t)\mathrm{d}t \sim -\sum_{k=0}^{\infty} \operatorname{Res}[\lambda^{-s}G(s); s_k]$$

as $\lambda \to +\infty$*. If the function* $G(s)$ *is analytic in the half-plane* $\operatorname{Re} s > \gamma$*, then* $\Omega(\lambda) = \mathrm{o}(\lambda^{-N})$*.*

THEOREM 2. *Let the following conditions be valid:*

1) *for some* γ*, the formula* (3) *holds, where the functions* $\kappa(s)$ *and* $h(s)$ *are defined by* (4) *and* (5)*, respectively;*
2) *function* $G(s) = \kappa(s)h(1-s)$ *can be continued into the half-plane* $\operatorname{Re} s < \gamma$ *as a meromorphic function with poles* s_k*;*
3) $G(s) = \mathrm{o}(1)$ *as* $|\tau| \to \infty$ *in each vertical strip* $\gamma_2 \leqslant \operatorname{Re} s \leqslant \gamma_1 < \gamma$*;*
4) $\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} |G(s)| \mathrm{d}\tau < M(\gamma_0)$ *for* $\gamma_0 < \gamma$*.*

Then

$$\Omega(\lambda) = \int_0^\infty K(\lambda t)H(t)\mathrm{d}t \sim \sum_{k=0}^{\infty} \operatorname{Res}[\lambda^{-s}G(s); s_k]$$

as $\lambda \to 0^+$*. If* $G(s)$ *is analytic for* $\operatorname{Re} s < \gamma$*, then* $\Omega(\lambda) = \mathrm{o}(\lambda^{N})$*.*

2. $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as a convolution of Mellin transforms

Let $\xi = h\lambda t$, $\Lambda = \frac{1}{h\lambda}$ in equation (1). Then we obtain

$$(\mathbf{\Psi_1}f)(\Lambda) = \frac{\Lambda^{k+1-\sigma}}{h^\sigma}\int_0^\infty \xi^k \Psi(a,c,\xi)f(\Lambda\xi)\mathrm{d}\xi, \quad \Lambda > 0, \tag{6}$$

which is similar to (2)

$$(\mathbf{\Psi_1}f)(\Lambda) = \frac{\Lambda^{k+1-\sigma}}{h^\sigma}I(\Lambda) = \frac{\Lambda^{k+1-\sigma}}{h^\sigma}\int_0^\infty g(\xi)f(\Lambda\xi)\mathrm{d}\xi, \quad \Lambda > 0,$$

where $g(\xi) = \xi^k\Psi(a,c,\xi)$.

Applying the Handelsman-Lew method, we obtain the asymptotic expansions for (6) as $\Lambda \to 0^+$ and $\Lambda \to +\infty$ which correspond to asymptotic expansions for (1) as $\lambda \to +\infty$ and $\lambda \to +0$, respectively.

We first note that, under suitable conditions on $f(t)$ [1], the Parseval formula for the inverse Mellin transform applying to (6) has the form

$$(\mathbf{\Psi_1}f)(\Lambda) = \frac{\Lambda^{k+1-\sigma}}{h^\sigma}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Lambda^{-z}\mathcal{M}[f;z]\mathcal{M}[g;1-z]\mathrm{d}z.$$

Calculating the Mellin transform of $g(t)$ evaluated at the point $1-z$ with the contour of integration being a Bromwich contour in the common domain of analyticity of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$, and assuming that $\mathcal{M}[f;z]$ exists in the ordinary sense, we have

$$\begin{aligned}\mathcal{M}[g;1-z] &= \mathcal{M}[\xi^k\Psi(a,c,\xi);1-z]\\ &= \frac{\Gamma(1+k-z)\Gamma(2+k-c-z)\Gamma(z+a-k-1)}{\Gamma(a)\Gamma(1+a-c)}.\end{aligned}$$

Therefore, we have the following representation of $(\mathbf{\Psi_1}f)(\Lambda)$

$$\begin{aligned}&(\mathbf{\Psi_1}f)(\Lambda)\\ &= \frac{\Lambda^{k+1-\sigma}}{2\pi i h^\sigma}\\ &\times\int_{\gamma-i\infty}^{\gamma+i\infty}\Lambda^{-z}\mathcal{M}[f;z]\frac{\Gamma(1+k-z)\Gamma(2+k-c-z)\Gamma(z+a-k-1)}{\Gamma(a)\Gamma(1+a-c)}\mathrm{d}z.\end{aligned} \tag{7}$$

Then we move the contour of integration in (7) to the left and calculate the residues of corresponding orders at poles of the integrand. Here we take into account the coincidence of poles of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$, if it happens. As a result, we obtain the asymptotic expansions for $(\mathbf{\Psi_1}f)(\Lambda)$ as $\Lambda \to 0^+$ which correspond to the asymptotic expansions for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda \to +\infty$.

Next we move the contour of integration in (7) to the right and calculate the residues at the corresponding poles. In this way we obtain the asymptotic expansions for $(\mathbf{\Psi_1}f)(\Lambda)$ as $\Lambda \to +\infty$ which correspond to the asymptotic expansions for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda \to 0^+$.

To justify the corresponding formulas, we have to suppose the fulfilment of conditions of Theorem 2 as $\lambda \to +\infty$ and the fulfilment of conditions of Theorem 1 as $\lambda \to 0^+$.

3. The asymptotic expansion of $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ at infinity

Suppose that

$$f \sim \sum_{m=0}^{\infty} b_m t^{a_m}, \quad \operatorname{Re} a_m \uparrow \infty \quad \text{as} \quad t \to 0^+.$$

Then $\mathcal{M}[f;z]$ has simple poles at the points $z = -a_m$, and its Laurent expansions at these poles have main parts

$$\frac{b_m}{z+a_m}.$$

If the poles of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$ are disjoint, namely $z_n = 1-a+k-n \neq -a_m$ for each pair n,m, then, after computing the corresponding residues, we obtain the asymptotic expansion for $(\mathbf{\Psi_1}f)(\Lambda)$ as $\Lambda \to 0^+$ in the form

$$\begin{aligned}(\mathbf{\Psi_1}f)(\Lambda) &\\ \sim \sum_{m=0}^{\infty} b_m &\frac{\Lambda^{a_m+k+1+\sigma}}{h^\sigma} \frac{\Gamma(1+k+a_m)\Gamma(2+k-c+a_m)\Gamma(a-k-1-a_m)}{\Gamma(a)\Gamma(1+a-c)} \\ + \sum_{n=0}^{\infty} &\frac{(-1)^n}{n!} \frac{\Lambda^{a+n-\sigma}}{h^\sigma} \frac{\mathcal{M}[f;1-a+k-n]\Gamma(a+n)\Gamma(1-c+a+n)}{\Gamma(a)\Gamma(1+a-c)}.\end{aligned}$$

Now we perform the inverse change $\lambda = \frac{1}{h\Lambda}$ and obtain the required asymptotic expansion for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda \to +\infty$ in the form

$$\begin{aligned}(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda) &\\ \sim \sum_{m=0}^{\infty} b_m &\frac{\lambda^{-a_m-k-1+\sigma}}{h^{a_m+k+1}} \frac{\Gamma(1+k+a_m)\Gamma(2+k-c+a_m)\Gamma(a-k-1-a_m)}{\Gamma(a)\Gamma(1+a-c)} \\ + \sum_{n=0}^{\infty} &\frac{(-1)^n}{n!} \frac{\lambda^{-a-n+\sigma}}{h^{a+n}} \frac{\mathcal{M}[f;1-a+k-n]\Gamma(a+n)\Gamma(1-c+a+n)}{\Gamma(a)\Gamma(1+a-c)}.\end{aligned}$$

If some poles of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$ coincide, then they are second-order poles for the integrand in (7), and produce logarithmic terms in asymptotic expansions accounting the derivative of λ^{-z}. For example, let we have $z_l = 1-a+k-l = -a_0$ for $n = l$, but $z_n = 1-a+k-n \neq -a_m$ for each other pairs (n,m). Then we obtain the asymptotic expansion of $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda \to +\infty$ in the form

$$\begin{aligned}(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda) &\\ \sim \sum_{m=1}^{\infty} b_m &\frac{\lambda^{-a_m-k-1+\sigma}}{h^{a_m+k+1}} \frac{\Gamma(1+k+a_m)\Gamma(2+k-c+a_m)\Gamma(a-k-1-a_m)}{\Gamma(a)\Gamma(1+a-c)}\end{aligned}$$

$$+\sum_{n=0,n\neq l}^{\infty}\frac{(-1)^n}{n!}\frac{\lambda^{-a-n+\sigma}}{h^{a+n}}\frac{\mathcal{M}[f;1-a+k-n]\Gamma(a+n)\Gamma(1-c+a+n)}{\Gamma(a)\Gamma(1+a-c)}$$
$$+\frac{(-1)^l}{l!}b_0\frac{\lambda^{-a-l+\sigma}}{h^{a+l}}\ln\lambda\frac{\Gamma(a+l)\Gamma(1-c+a+l)}{\Gamma(a)\Gamma(1+a-c)}.$$

Now we are at the position to write the final form of the expansion by λ after the inverse change.

4. The asymptotic expansion of $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ at zero

Suppose that

$$f\sim \mathrm{e}^{-\alpha t}\sum_{m=0}^{\infty}d_m t^{-r_m}\quad \text{as}\quad t\to+\infty,$$

with $\alpha\geqslant 0$ and $\operatorname{Re} r_m\uparrow\infty$ as $m\to\infty$. We have two situations depending on α.

First, if $\alpha>0$, then $\mathcal{M}[f;z]$ is analytic in the half-plane $\operatorname{Re} z>x_0$ for some x_0, and the analytic continuation of $\mathcal{M}[g;1-z]$ into the right half-plane $\operatorname{Re} z\geqslant \min\{1+k,\operatorname{Re}(2-c+k)\}$ has simple poles at the points $z=2-c+k+n$ and $z=1+k+n$, $n=0,1,2,\ldots$.

Moving the contour of integration arbitrarily to the right and computing residues at the poles of $\mathcal{M}[g;1-z]$, we obtain the asymptotic expansion for $(\mathbf{\Psi_1}f)(\Lambda)$ as $\Lambda\to+\infty$. It gives the required expansion for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda\to 0^+$ after the inverse change of variables $\lambda=\frac{1}{h\Lambda}$ in the form

$$\begin{aligned}(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)&\\ \sim\sum_{n=0}^{\infty}&\frac{(-1)^n}{n!\Gamma(a)\Gamma(1+a-c)}\\ &\times\Bigg[\mathcal{M}[f;2-c+k+n]\Gamma(c-1-n)\Gamma(1-c+a+n)\frac{\lambda^{1-c+n+\sigma}}{h^{c-1-n}}\\ &+\mathcal{M}[f;1+k+n]\Gamma(1-c-n)\Gamma(a+n)\frac{\lambda^{n+\sigma}}{h^{-n}}\Bigg].\end{aligned}\tag{8}$$

In the second case, if $\alpha=0$, then $\mathcal{M}[f;z]$ has simple poles at the points $z=r_m$, and its Laurent expansion at these poles has main part

$$\frac{d_m}{z-r_m}.$$

Here we should consider several possible situations of relative position of the poles of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$.

If all poles of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$ are different, namely, $r_m\neq 2-c+k+n$ and $r_m\neq 1+k+n$ for each pair (n,m), then we obtain the asymptotic expansion for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda\to 0^+$ by adding the series

$$\sum_{m=0}^{\infty}d_m\frac{\lambda^{r_m-k-1+\sigma}}{h^{k+1-r_m}}\frac{\Gamma(1+k-r_m)\Gamma(2+k-c-r_m)\Gamma(a-k-1+r_m)}{\Gamma(a)\Gamma(1+a-c)}$$

to the series from the formula (8). In this case, the required asymptotic expansion takes the form

$$\begin{aligned}
&(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)\\
&\sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(a)\Gamma(1+a-c)}\\
&\qquad \times\Big[\mathcal{M}[f;2-c+k+n]\Gamma(c-1-n)\Gamma(1-c+a+n)\frac{\lambda^{1-c+n+\sigma}}{h^{c-1-n}}\\
&\qquad\quad +\mathcal{M}[f;1+k+n]\Gamma(1-c-n)\Gamma(a+n)\frac{\lambda^{n+\sigma}}{h^{-n}}\Big]\\
&\quad+\sum_{m=0}^{\infty} d_m \frac{\lambda^{r_m-k-1+\sigma}}{h^{k+1-r_m}}\frac{\Gamma(1+k-r_m)\Gamma(2+k-c-r_m)\Gamma(a-k-1+r_m)}{\Gamma(a)\Gamma(1+a-c)}. \qquad (9)
\end{aligned}$$

Now we consider some cases when there are coinciding poles of $\mathcal{M}[f;z]$ and $\mathcal{M}[g;1-z]$. For example, let $r_0 = 2-c+k+l$. Then the corresponding terms in expansion (9) are replaced by the second-order residue at the corresponding pole

$$\frac{(-1)^{l+1}}{l!} d_0 \ln\lambda \frac{\Gamma(c-1-l)\Gamma(1-c+l+a)}{\Gamma(a)\Gamma(1+a-c)}\frac{\lambda^{1-c+l+\sigma}}{h^{c-l-1}}.$$

The formula of the asymptotic expansion for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda\to 0^+$ has the form

$$\begin{aligned}
&(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)\\
&\sim \sum_{\substack{n=0\\ n\neq l}}^{\infty} \frac{(-1)^n}{n!\Gamma(a)\Gamma(1+a-c)}\\
&\qquad \times\Big[\mathcal{M}[f;2-c+k+n]\Gamma(c-1-n)\times\Gamma(1-c+a+n)\frac{\lambda^{1-c+n+\sigma}}{h^{c-1-n}}\\
&\qquad\quad +\mathcal{M}[f;1+k+n]\Gamma(1-c-n)\Gamma(a+n)\frac{\lambda^{n+\sigma}}{h^{-n}}\Big]\\
&\quad+\frac{(-1)^l}{l!\Gamma(a)\Gamma(1+a-c)}\mathcal{M}[f;1+k+l]\Gamma(1-c-l)\Gamma(a+l)\frac{\lambda^{l+\sigma}}{h^{-l}}\\
&\quad+\sum_{m=1}^{\infty} d_m \frac{\lambda^{r_m-k-1+\sigma}}{h^{k+1-r_m}}\frac{\Gamma(1+k-r_m)\Gamma(2+k-c-r_m)\Gamma(a-k-1+r_m)}{\Gamma(a)\Gamma(1+a-c)}\\
&\quad+\frac{(-1)^{l+1}}{l!} d_0 \log\lambda \frac{\Gamma(c-1-l)\Gamma(1-c+l+a)}{\Gamma(a)\Gamma(1+a-c)}\frac{\lambda^{1-c+l+\sigma}}{h^{c-l-1}}.
\end{aligned}$$

Now, if we suppose that, for the pair (l,p), we have $r_0 = 2-c+k+l = 1+k+p$, then the integrand has a third-order pole at this point. In this case, the asymptotic expansion for $(\mathbf{\Psi}_{\sigma,\kappa;1,1,h}f)(\lambda)$ as $\lambda\to 0^+$ has the term

$$\frac{(-1)^{l+p}}{l!p!} d_0 \log^2\lambda \frac{\Gamma(p+a)}{\Gamma(a)\Gamma(1+a-c)}\frac{\lambda^{p+\sigma}}{h^{-p}}.$$

If we assume that all other poles are simple, then the required asymptotic expansion can be presented in the form

$$
\begin{aligned}
&(\boldsymbol{\Psi}_{\sigma,\kappa;1,1,h} f)(\lambda) \\
&\sim \sum_{n=0,n\neq l,p}^{\infty} \frac{(-1)^n}{n!\Gamma(a)\Gamma(1+a-c)} \\
&\qquad \times \Big[\mathcal{M}[f;2-c+k+n]\Gamma(c-1-n)\Gamma(1-c+a+n)\frac{\lambda^{1-c+n+\sigma}}{h^{c-1-n}} \\
&\qquad\quad +\mathcal{M}[f;1+k+n]\Gamma(1-c-n)\Gamma(a+n)\frac{\lambda^{n+\sigma}}{h^{-n}}\Big] \\
&\quad +\frac{(-1)^l}{l!\Gamma(a)\Gamma(1+a-c)}\mathcal{M}[f;1+k+l]\Gamma(1-c-l)\Gamma(a+l)\frac{\lambda^{l+\sigma}}{h^{-l}} \\
&\quad +\sum_{m=1}^{\infty} d_m \frac{\lambda^{r_m-k-1+\sigma}}{h^{k+1-r_m}} \frac{\Gamma(1+k-r_m)\Gamma(2+k-c-r_m)\Gamma(a-k-1+r_m)}{\Gamma(a)\Gamma(1+a-c)} \\
&\quad +\frac{(-1)^p}{p!\Gamma(a)\Gamma(1+a-c)}\mathcal{M}[f;2-c+k+p]\Gamma(c-1-p) \\
&\qquad \times\Gamma(1-c+a+p)\frac{\lambda^{1-c+p+\sigma}}{h^{c-1-p}} \\
&\quad +\frac{(-1)^{l+p}}{l!p!} d_0 \log^2\lambda \frac{\Gamma(p+a)}{\Gamma(a)\Gamma(1+a-c)}\frac{\lambda^{p+\sigma}}{h^{-p}}.
\end{aligned}
$$

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